Stable and Negative Edges of $K_{m,m}$ and tC_4

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Abstract

In this paper we discuss how the addition of a new edge changes the irregularity strength in $K_{m,m}$ and tC_4.

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1 Introduction

In this paper we consider simple undirected graphs with no K_2 components and at most one isolated vertex. Let $G = (V, E)$ be a graph. A network $G(f)$ consists of the graph together with an assignment $f : E(G) \to Z^+$. The sum of the labels of the edges incident with a vertex is called the weight of that vertex. If all the weights are pairwise distinct, $G(f)$ is called an irregular network. The strength of the network $G(f)$ is defined by $s(G(f)) = \max\limits_{e \in E} \{f(e)\}$. The irregularity strength $s(G)$ of G is defined as $s(G) = \min \{s(G(f))/G(f) \textit{ is irregular}\}$.

The problem of finding irregularity strength of graphs was proposed by Chartrand et al., [2] and has been proved to be difficult, in general. There are not many graphs for which the irregularity strength is known. The readers may refer to the survey of Lehel [9] and the papers [1, 6, 10, 11]. R.J. Faudree, M.S. Jacobson, J. Lehel and R.H. Schelp studied the irregularity strength of tK_3 in [3]. A. Gyárfás [4] determined the irregularity strength of $K_n - mK_2$. Stanislav Jendroĺ and Michal Tkáč [11] studied the irregularity strength of the union of t copies of the complete graph K_p.

Definition 1.1. [7] *Let G be any graph which is not complete, e be any edge of $\overline{G}$, then e is called a positive edge of G if $s(G+e) > s(G)$, e is called a negative edge of G if $s(G+e) < s(G)$ and e is called a stable edge of G if $s(G+e) = s(G)$.*

Definition 1.2. [7] *Let G be any graph which is not complete. If all the edges of $\overline{G}$ are positive (negative, stable) edges of G, then G is called a positive (negative, stable) graph. Otherwise, G is called a mixed graph.*

Example 1.3. [7] *Star graphs $K_{1,n}$ are negative graphs for $n \geq 3$.*

Example 1.4. [8] *P_3 is a positive graph.*

Example 1.5. *The following graph G (Figure 1) is a stable graph.*

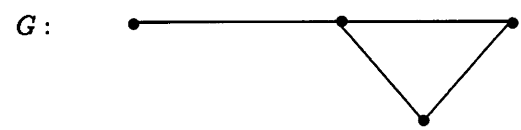

Figure 1

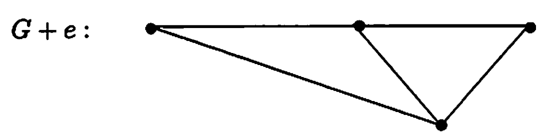

Figure 2

It is easy to verify that $s(G) = 2$ and $s(G+e) = 2$. Hence G is a stable graph.

Example 1.6. *Consider P_4. It is easy to verify that $s(P_4) = 2$. Let v_1, v_2, v_3 and v_4 be the consecutive vertices of P_4. For any $e \in \overline{P_4}$, $P_4 + e$ is isomorphic to either G_1 or G_2 (Figure 3 or 4).*

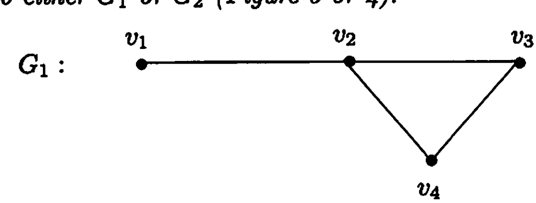

Figure 3

$G_2:$

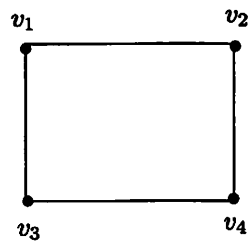

Figure 4

Also $s(G_1) = 2$ and $s(G_2) = 3$. Hence v_2v_4 is a stable edge of P_4 and v_1v_4 is a positive edge of P_4. Thus P_4 is a mixed graph.

In [7], we proved that $P_n, n \geq 4$ is a mixed graph and the cycle C_n is a negative graph for any $n \geq 4$.

In this paper we discuss positive, negative and stable edges of t copies of C_4 and complete bipartite graph $K_{m,m}$.

2 Stable and negative graphs

Gyárfás [5] proved that the irregularity strength of $K_{m,m}$ is 4 for odd m. The following theorem proves that $K_{m,m}$ is a negative graph for odd m.

Theorem 2.1. *For any odd $m \geq 3$, $K_{m,m}$ is a negative graph.*

Proof. Consider a complete bipartite graph $K_{m,m}$ of order $2m$ where m is odd. Let $u_1, u_2, u_3, \ldots, u_m$ and $v_1, v_2, v_3, \ldots, v_m$ be the vertices of the two partite sets of $K_{m,m}$. For any $e \in \overline{K_{m,m}}$, $K_{m,m} + e$ is isomorphic to $K_{m,m} + v_{m-1}v_m$. So, it is enough to prove that $v_{m-1}v_m$ is a negative edge of $K_{m,m}$. Define the edge labeling $f : E(K_{m,m} + v_{m-1}v_m) \to Z^+$ as follows.

$$f(u_1v_j) = 1, 1 \leq j \leq m.$$

$$f(u_2v_j) = \begin{cases} 1, & 1 \leq j \leq m-1 \\ 2, & j = m. \end{cases}$$

$$\text{For } 3 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, f(u_iv_j) = \begin{cases} 1, & 1 \leq j \leq m-i \\ 2, & j = m+1-i, m+2-i \\ 3, & m+3-i \leq j \leq m. \end{cases}$$

$$\text{For } \left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m-1, f(u_iv_j) = \begin{cases} 1, & 1 \leq j \leq m-i \\ 2, & j = m+1-i \\ 3, & m+2-i \leq j \leq m. \end{cases}$$

$$f(u_mv_j) = 3, 1 \leq j \leq m.$$

$$f(v_{m-1}v_m) = 1.$$

The weights of the vertices of $K_{m,m}+v_{m-1}v_m$ are $m, m+1, m+2, \dots, 3m-2, 3m$. All are pairwise distinct. By the above labeling, we have an irregular network $K_{m,m}$ with maximum label 3. Hence, $s(K_{m,m}+v_{m-1}v_m) \leq 3 < s(K_{m,m})$.

Therefore, $v_{m-1}v_m$ is a negative edge of $K_{m,m}$.

Thus $K_{m,m}$ is a negative graph for odd $m \geq 3$. □

Chartrand et al. [2] proved that $s(K_{m,m}) = 3$ when m is even. The following theorem proves that $K_{m,m}$ is a stable graph for even $m \geq 4$ and $K_{2,2}$ is negative a graph.

Theorem 2.2. *If m is even, then $K_{m,m}$ is a stable graph where, $m \geq 4$ and $K_{2,2}$ is a negative graph.*

Proof. It is easy to verify that $K_{2,2} \cong C_4$.

In [7], we proved that cycles are negative graph for any $n \geq 4$ and hence $K_{2,2}$ is a negative graph.

Consider a complete bipartite graph $K_{m,m}$ of order $2m (m \geq 4)$ where m is even. Let $u_1, u_2, u_3, \dots, u_m$ and $v_1, v_2, v_3, \dots, v_m$ be the vertices of the two partite sets of $K_{m,m}$. For any e, $e \in \overline{K_{m,m}}$, $K_{m,m}+e$ is isomorphic to $K_{m,m}+v_{m-1}v_m$. So, it is enough to prove that $v_{m-1}v_m$ is a stable edge of $K_{m,m}$.

Define the edge labeling $f : E(K_{m,m}+v_{m-1}v_m) \rightarrow Z^+$ as follows.

$$f(u_1v_j) = 1, 1 \leq j \leq m.$$

$$\text{For } 2 \leq i \leq \frac{m}{2}, f(u_iv_j) = \begin{cases} 1, & 1 \leq j \leq m+1-i \\ 3, & m+2-i \leq j \leq m. \end{cases}$$

$$\text{For } \frac{m}{2}+1 \leq i \leq m-2, f(u_iv_i) = \begin{cases} 1, & 1 \leq j \leq m-i \\ 2, & j = m+1-i \\ 3, & m+2-i \leq j \leq m. \end{cases}$$

$$f(u_{m-1}v_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2, m \\ 3, & 3 \leq j \leq m-1. \end{cases}$$

$$f(u_mv_j) = \begin{cases} 2, & j = 1, m-1 \\ 3, & \text{otherwise.} \end{cases}$$

$$f(v_{m-1}v_m) = 2.$$

The weights of the vertices of $K_{m,m}+v_{m-1}v_m$ are $m, m+1, m+2, \dots, 3m-1$. All are pairwise distinct. By the above labeling, we have an irregular network $K_{m,m}+v_{m-1}v_m$ with maximum label 3. Hence

$$s(K_{m,m}+v_{m-1}v_m) \leq 3. \tag{1}$$

The minimum possible weights of $K_{m,m}+v_{m-1}v_m$ are $m, m+1, m+2, \ldots, 3m-1$. Since, the maximum degree of $K_{m,m}+v_{m-1}v-m$ is $m+1$, it is not possible to obtain the weight $3m-1$ by using the label 2 and fewer than 2 to the edges of $K_{m,m}+v_{m-1}v_m$. Hence,

$$s(K_{m,m}+v_{m-1}v_m)\geq 3. \tag{2}$$

From (1) and (2), $s(K_{m,m}+v_{m-1}v_m)=3=s(K_{m,m})$. Therefore, $v_{m-1}v_m$ is a stable edge of $K_{m,m}$. Thus, $K_{m,m}$ is a stable graph for even $m\geq 4$. □

In [3], R.J. Faudree et al. proved that any 2-regular graph G with $4p$ vertices with no triangle components has strength $2p+1$.

By the above theorem we observe the following.

Observation 2.3. *Irregularity strength of tC_4 is $2t+1$.*

Theorem 2.4. *For $t\geq 1$, tC_4 is a negative graph.*

Proof. Consider disjoint union of t copies of C_4.

Let v_{i1}, v_{i2}, v_{i3} and v_{i4} be the consecutive vertices of the i^{th} copy of C_4 for $1\leq i\leq t$. For any edge $e\in\overline{tC_4}$, tC_4+e is isomorphic to either $tC_4+v_{(t-1)4}v_{t4}$ or $tC_4+v_{t2}v_{t4}$.

Case 1. Consider the graph $tC_4+v_{(t-1)4}v_{t4}$.

Define the edge labeling $f: E(tC_4+v_{(t-1)4}v_{t4})\to Z^+$ by

$$\begin{aligned}
f(v_{i1}v_{i2}) &= f(v_{i1}v_{i4}) = 2i-1, 1\leq i\leq t-2,\\
f(v_{i2}v_{i3}) &= 2i, 1\leq i\leq t-2,\\
f(v_{i3}v_{i4}) &= 2i+1, 1\leq i\leq t-2,\\
f(v_{(t-1)1}v_{(t-1)2}) &= f(v_{(t-1)1}v_{(t-1)4}) = 2t-3,\\
f(v_{(t-1)2}v_{(t-1)3}) &= f(v_{(t-1)3}v_{(t-1)4}) = 2t-2,\\
f(v_{(t-1)4}v_{t4}) &= 2,\\
f(v_{t1}v_{t2}) &= f(v_{t1}v_{t4}) = 2t-1 \text{ and}\\
f(v_{t2}v_{t3}) &= f(v_{t3}v_{t4}) = 2t.
\end{aligned}$$

The weights of the vertices of $tC_4+v_{(t-1)4}v_{t4}$ are $2,3,4,\ldots,4t,4t+1$. All are pairwise distinct. By the above labeling, we have an irregular network $tC_4+v_{(t-1)4}v_{t4}$ with maximum label $2t$, hence $s(tC_4+v_{(t-1)4}v_{t4})\leq 2t < s(tC_4)$.

Hence, the edge $v_{(t-1)4}v_{t4}$ is a negative of tC_4.

Case 2. Consider the graph $tC_4+v_{t2}v_{t4}$.

Define the edge labeling $f: E(tC_4+v_{t2}v_{t4})\to Z^+$ by

$$f(v_{i1}v_{i2}) = f(v_{i1}v_{i4}) = 2i-1, 1 \le i \le t-1,$$
$$f(v_{i2}v_{i3}) = 2i, 1 \le i \le t-1,$$
$$f(v_{i3}v_{i4}) = 2i+1, 1 \le i \le t-1,$$
$$f(v_{t1}v_{t2}) = 2t-2,$$
$$f(v_{t2}v_{t3}) = f(v_{t3}v_{t4}) = f(v_{t1}v_{t4}) = 2t \text{ and}$$
$$f(v_{t2}v_{t4}) = 1.$$

The weights of the vertices of $tC_4 + v_{t2}v_{t4}$ are $2, 3, 4, \ldots, 4t, 4t+1$. All are pairwise distinct. By the above labeling, we have an irregular network $tC_4 + v_{t2}v_{t4}$ with maximum label $2t$, hence $s(tC_4 + v_{t2}v_{t4}) \le 2t < s(tC_4)$.
Hence $v_{t2}v_{t4}$ is a negative of tC_4. Thus, tC_4 is a negative graph. □

From our verification for small values of m and n we propose the following conjecture.

Conjecture 2.5.

1. *Every complete bipartite graph $K_{m,n}$ with $m \neq n$ is either a stable or a negative graph.*

2. *Disjoint union of cycles is a negative graph.*

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