$\times$-Line Signed Graphs

MUKTI ACHARYA
Department of Applied Mathematics
Delhi College of Engineering
Bawana Road, Delhi - 110 042, INDIA.
e-mail: *mukti1948@yahoo.com*

Abstract

In this paper, a definition of a variation of the standard notion of the *line signed graph* of a given signed graph is recalled from [14] and some fundamental results linking it to the notions of *jump signed graphs* [6] and *adjacency signed graphs* [21], especially with regard to their states of balance, consistency and compatibility are obtained.

Keywords. Line signed graph; balance complement; jump signed graph; matrix signed graph; adjacency signed graph; tensor product; consistency, compatibility.
2000 Mathematics Subject Classification: 05C22.

1 Introduction

For all standard terminology and notation in graph theory, we refer the reader to Harary [15]; the nonstandard ones will be given in this note as and when required. We will treat only finite simple graphs without self-loops and isolates.

A *signed graph* (or *signed graph* in short; *see* [8, 11]) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$ called the *underlying graph* of S, and $\sigma : E \to \{+, -\}$ is a function, called a *signing.* We let $E^+(S) = \{e \in E(G) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(G) : \sigma(e) = -\}$. Then $E(S) = E^+(S) \cup E^-(S)$ and the elements of $E^+(S)(E^-(S))$ are called *positive* (*negative*) edges in S. Two vertices $u, v \in V(S) = V(S^u) = V$ are said to be *adjacent* in S whenever they are adjacent in S^u (*i.e.*, whenever $uv \in E(S^u)$). Thus, graphs may be regarded as signed graphs in which all the edges are positive; hence we regard graphs as *all-positive* signed graphs (*all-negative* signed graphs are defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

Behzad and Chartrand [8] have given a definition of the *line signed graph* $L(S)$ of a given signed graph S as follows: The vertices of $L(S)$ correspond one-to-one with the edges of S, $e_ie_j \in E(L(S))$, if and only if the edges of S corresponding to the vertices e_i and e_j of $L(S)$ have a vertex in common in S, and for any $e_ie_j \in E(L(S))$ one has $e_ie_j \in E^-(L(S))$, if and only if the edges of S corresponding to e_i and e_j are both negative in S. A signed graph H is a *line signed graph* if there exists a signed graph S such that $H \cong L(S)$; S is then called a *line root* of H [25].

In [14], the author introduced the following variation of the above standard notion of line signed graph $L(S)$ of a given signed graph S as follows: It is a signed graph denoted $L_\times(S)$ and defined on the line graph $L(S^u)$ of the graph S^u by assigning to each edge ee' of $L(S^u)$ the product of the signs of the adjacent edges e and e' of S; we shall call $L_\times(S)$ the $\times$*-line signed graph* of S. The purpose of this note is to initiate a study of this notion.

2 Preliminary results

A signed graph S is *balanced* if every cycle in S has an even number of negative edges (*cf.* [16, 17]). In [14], it was observed that for any signed graph S on the cycle C_n, $n \geq 3$, $L_\times(S)$ is balanced. More generally, we can prove the following result.

Theorem 2.1. *For any signed graph S, its $\times$-line signed graph $L_\times(S)$ is a balanced signed graph.*

Proof. Let E^+ and E^- denote respectively the set of positive edges of S and the set of negative edges of S. Then, by the definition of $L_\times(S)$ it may be easily verified that the partition $\{E^+, E^-\}$ of the vertex set of $L_\times(S)$ has the property that every positive edge of $L_\times(S)$ joins two vertices lying within one of the sets E^+ and E^- whereas every negative edge of $L_\times(S)$ joins a vertex of E^+ with one of E^-. Hence, by the well known 'Partition Criterion' for balance due to Harary [16], it follows that $L_\times(S)$ is balanced. □

Let σ be the signing specifying S and let $\mu : V(S) \to \{-1, +1\}$ be any function, called a *marking* of S; accordingly, we shall denote by S_μ the *marked signed graph* which is defined as the signed graph S together with the marking μ. In particular, if E_u denotes the set of the edges x incident at u, then μ is called a *canonical* marking of S if it is obtained by defining $\mu(u)$, for each vertex u, as the product $\prod_{x \in E_u} \sigma(x)$. Then, define the operator c that transforms S into the signed graph $c_\mu(S)$ which has the same vertex set as that of S with two vertices defined adjacent in $c_\mu(S)$ whenever the vertices are not adjacent in S^u and each edge uv in $c_\mu(S)$ signed $\mu(u)\mu(v)$. Clearly, $c_\mu(S)$ so defined is a signed graph whose underlying graph is the usual graph *complement* of S^u and it is also a balanced signed graph due to the Harary's Partition Criterion for Structural Balance (HPCSB) mentioned above. Therefore, we call $c_\mu(S)$ the *μ-balanced complement* of the signed graph S; in particular, if μ is given to be the canonical marking μ_σ of S then it has been called the *balanced complement* of S and is specifically denoted $\overline{S}$ [25]. Apart from the basic purpose for which it has been defined, a study of μ-balanced complements of a given signed graph appears to be of independent theoretical interest, not only in the mathematical theory of signed graphs (*cf.*: [24]) but also possibly in the theory of cognitive balance in social psychology (*cf.*: [2]). These studies also appear to have interesting connections with discovering a proper way of defining the notion of 'complement' S^c of a given signed digraph S, a longstanding open problem in social psychology (*see* [25, 2, 3, 7]).

The complement of the line graph of a given graph G has been called the *jump graph* of G, denoted $J(G)$ [12]. Next, the *jump signed graph* $J(S)$ of a given signed graph S has been defined [25] as a signed graph such that $(J(S))^u \cong J(S^u)$ and two vertices of $J(S)$ are joined by a negative edge if and only if the corresponding edges in S are of opposite signs. Clearly, as noted in [6], for any signed graph S, $J(S)$ so defined is a balanced signed graph.

The idea of *switching* a signed graph was introduced in [1] and may be formally stated as follows: Given a marking μ of a signed graph S, switching S with respect to μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs in S_μ. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called the *μ-switched* signed graph or just switched signed graph when the marking is clear from the context. Further, a signed graph S_1 *switches* to signed graph S_2, written as $S_1 \sim S_2$, whenever, there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Two signed graphs S_1 and S_2 are said to be *weakly isomorphic* (*cf.*: [28]) or *cycle-isomorphic* (*cf.*: [26]) if there exists an isomorphism $f : (S_1)^u \to (S_2)^u$ such that the sign of every cycle Z in S_1 equals the sign of $f(Z)$ in S_2, where the *sign of a set M of edges, denoted $sgn(M)$*, in a signed graph is defined as the product of the signs of the edges in it. The following theorem will be useful in our further investigation, where $\Psi(G)$ denotes the set of all signed graphs whose underlying graph is G.

Lemma 2.2. [28, 26] *Given a graph G, any two signed graphs in $\Psi(G)$ are switching equivalent if and only if they are cycle-isomorphic.*

The following result is easy to verify using Lemma 2.2.

Theorem 2.3. *For any signed graph* $S = (G, \sigma)$, $c_\sigma(L_\times(S)) \sim J(S)$.

3 Consistency and cycle-compatibility

In analogy with Harary's *balance principle* in social psychology [16], Beineke and Harary [9] defined a marked graph G_μ as being *consistent* if every cycle in G_μ contains an even number of negative vertices.

Beineke and Harary [9, 10] were the first to pose the problem of characterizing consistent marked graphs, which was independently settled by Acharya [4, 5], Rao [22] and Hoede [19]. Recently, new characterizations of consistent marked graphs have been obtained by Roberts and Xu [23].

In general, the *mark* $\mu(S')$ of a nonempty subsi(di)graph S' of S_μ is defined as the product of the marks of the vertices in S'. A cycle Z in S_μ is said to be *consistent* (respectively, *compatible*) if $\mu(Z) = +1$ ($\mu(Z) = sgn(Z)$); otherwise, it is said to be *inconsistent* (*incompatible*). We shall call S *consistent* (*cycle-compatible*) if every cycle in it is consistent (compatible).

Definition 3.1. [20] *A given signed graph* $\Gamma = (H, \xi)$ *is* $(S, \mathcal{R})$-marked *if there exists a signed graph* $S = (G, \sigma)$ *(called a* marker *of* Γ*), a bijection* $\varphi : E(S) \to V(\Gamma)$, *a binary relation* $\mathcal{R}$ *on* $E(S)$ *and a marking* $\mu : V(\Gamma) \to \{-, +\}$ *of* Γ *satisfying the following* compatibility conditions,

(CC1): $\mu(u) = \sigma(\varphi^{-1}(u))\ \forall u \in V(\Gamma)$

(CC2): $uv \in E(\Gamma) \Leftrightarrow \{\varphi^{-1}(u), \varphi^{-1}(v)\} \in \mathcal{R}$.

The case when $\mathcal{R}$ is defined by the condition that $\varphi^{-1}(u) \cap \varphi^{-1}(v) \neq \emptyset$ has been dealt in [25] in respect of signed graph equations involving line signed graphs.

Clearly, if Γ_μ is consistent then the subgraph of any of its markers S must be balanced.

Towards studying the properties of structurally evolving social networks, perhaps the simplest model for study could be the si(di)graphs evolving through the unary operator L of taking the line si(di)graph $L(S)$ of a given si(di)graph S. We might then be interested to answer questions like: Precisely which si(di)graphs S are line roots of Γ so that Γ is consistent. Characterizations of signed graphs M whose line signed graphs and iterated line signed graphs $L^k(M)$ are consistent have been obtained [25]; towards attempting to answer such questions, the following result has been obtained, where for any vertex v, $d^+(v)$ and $d^-(v)$ denote respectively the number of positive edges incident at v (called the 'positive degree' of v) and the number of negative edges incident at v (called the 'negative degree' of v) and $d(v)$ denotes the total degree $d(v) = d^-(v) + d^+(v)$.

Theorem 3.2. [25] *For any isolate-free signed graph S of order p, $L(S)$ is consistent if and only if the following conditions hold in S:*

(i) S *is balanced;*

(ii) for every vertex v_i, $1 \leq i \leq p$, *in* S *having total degree greater than or equal to three,*

- *(a) if* $d(v_i) > 3$ *then* $d^-(v_i) = 0$;
- *(b) if* $d(v_i) = 3$ *then either* $d^-(v_i) = 0$ *or* $d^-(v_i) = 2$;
- *(c) if* $d^-(v_i) = 2$ *and* v_i *lies on a cycle of* S *then the negative degree of* v_i *is due to the negative edges of the cycle.*

Observation 3.3. *The validity of the statement of Theorem 3.2 remains unaltered by replacing $L(S)$ by $L_\times(S)$ since the vertices of both $L(S)$ and $L_\times(S)$ are the edges of S along with their signs as marks and the fact that $(L(S))^u \cong (L_\times(S))^u \cong L(S^u)$.*

Next, we have the following result the proof of which is not difficult to see.

Theorem 3.4. *For any isolate-free signed graph S of order p, $L_\times(S)$ is cycle-compatible if and only if $L_\times(S)$ is consistent.*

4 Switching equivalence of jump signed graphs and ×-line signed graphs

Towards searching for an ideal notion of the *complement* of a given signed graph, one is naturally lead to look for the analogue of the graph equation,

$$J(G) \cong L(G) \tag{1}$$

for the case of jump signed graphs. Since $J(G) = \overline{L(G)}$, the solutions to (1) would be graphs whose line graphs are self-complementary; these graphs are determined already.

Theorem 4.1. [27] *The graph equation $J(G) \cong L(G)$ has only six solutions; namely K_2, P_5, C_5, $P_3 o K_1$, $K_{3,3} - e$, $K_{3,3}$.*

Corollary 4.2. *A signed graph S satisfies $L_\times(S) \sim J(S)$ if and only if S^u is isomorphic to any of the graphs K_2, P_5, C_5, $P_3 o K_1$, $K_{3,3} - e$, $K_{3,3}$.*

5 $\times$-line signed graphs and $\{-1, 0, 1\}$-matrices

Let $A = (a_{ij})$ be any $m \times n$ matrix in which each entry belongs to the set $\{-1, 0, 1\}$; we shall call such a matrix a $(0, \pm 1)$-*matrix*. Given any such matrix A, one can construct a *balanced matrix signed graph* **Sg**(A) of A as follows: The vertex set of **Sg**(A) consists of the nonzero entries in A and the edge set consists of distinct pairs of vertices corresponding to the nonzero entries of A that lie in the same row of A or in the same column of A; the sign $\sigma(uv)$ of an edge uv in **Sg**(A) is defined as the product of the signs of the entries in A that correspond to u and v. By a well known result of Sampathkumar [24], it follows that **Sg**(A) is a balanced signed graph. When no entry in A is negative, **Sg**$(A) =: G(A)$ is the graph of the $(0, 1)$-matrix A originally defined by Hedetniemi [18]. Next, Mishra [21] extended Cook's [13] notion of the *term graph* $T(B)$ of a $(0, 1)$-matrix B to that of the *term signed graph* $T(A)$ of a $(0, \pm 1)$-matrix A as follows: The vertex set of $T(A)$ consists of the m row labels $r_1, r_2, \ldots, r_m$ and the n column labels $c_1, c_2, \ldots, c_n$ of A, the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$ and the sign of an edge $r_i c_j$ being the sign of the nonzero entry a_{ij}. In the case of $(0, 1)$-matrices B, Hedetniemi [18] has shown that $G(B) \cong L(T(B))$ and Mishra [21] has generalized this relationship to demonstrate that $\mathbf{S}(A) \cong L(T(A))$ where $\mathbf{S}(A)$ is the *matrix signed graph* of A which differs in structure from **Sg**(A) just by the rule to assign signs to its edges; in fact, an edge uv in $\mathbf{S}(A)$ is signed negative if and only if both the nonzero entries of A corresponding to the vertices u and v happen to be negative. The following is a new observation, whose proof follows from the fundamental facts just mentioned and the definition of the $\times$-line signed graphs.

Theorem 5.1. *For any $(0, \pm 1)$-matrix A, Sg$(A) \cong L_\times(T(A))$.*

Mishra [21] defined the 'Kronecker product' (popularly known as *tensor product*) of two signed graphs S_1 and S_2, denoted $S_1 \bigotimes S_2$ as follows: $V(S_1 \bigotimes S_2) := V(S_1) \times V(S_2)$, $E(S_1 \bigotimes S_2) := \{(u_1, v_1)(u_2, v_2) : u_1 u_2 \in E(S_1) \;\; v_1 v_2 \in E(S_2)\}$ and the sign of the edge $(u_1, v_1)(u_2, v_2)$ is the product of the sign of $u_1 u_2$ in S_1 and the sign of $v_1 v_2$ in S_2. In the following result, $A(S)$ will denote the usual *adjacency matrix* of the given signed graph S (*see* [17]) and $A \bigotimes B$ denotes the standard tensor product of the given matrices A and B.

Theorem 5.2. [21] *For any two signed graphs S_1 and S_2, $A(S_1 \bigotimes S_2) = A(S_1) \bigotimes A(S_2)$.*

Theorem 5.3. [21] *For any signed graph S, $T(A(S)) \cong S \bigotimes K_2^+$, where K_2^+ denotes the complete graph K_2 with its only edge treated as being positive.*

Next, Mishra [21] goes on to define the *adjacency signed graph* $\mathfrak{d}(S)$ of a given signed graph S as the matrix signed graph $\mathbf{S}(A(S))$ of the adjacency matrix $A(S)$ of S. The following result may be easily derived by applying the fact that $S(A) \cong L(T(A))$ and Theorems 5.1 and 5.3.

Theorem 5.4. [21] *For any signed graph S, $\mathfrak{d}(S) \cong L(S \bigotimes K_2^+)$.*

We define the *balanced adjacency signed graph* $\mathfrak{d}_\times(S)$ of a given signed graph S as the balanced matrix signed graph $\mathbf{Sg}(A(S))$ of the adjacency matrix $A(S)$ of S. Then, analogous to Theorem 5.4, one may easily deduce the following result.

Theorem 5.5. *For any signed graph S, $\mathfrak{d}_\times(S) \cong L_\times(S \bigotimes K_2^+)$.*

References

[1] R.P. Abelson and M.J. Rosenberg, Symbolic psychologic: A model of attitudinal cognition, *Behav. Sci.*, **3** (1958), 1-13.

[2] B.D. Acharya and S. Joshi, On the complement of an ambisidigraph, *Electronic Notes on Discrete Mathematics,* **15**, 2003.

[3] B.D. Acharya, Shalini Joshi and S.B. Rao, A Ramsey theorem for strongly connected ambisidigraphs, In: *Proc. II International Conference On Social Network Analysis* held in The Indian Statistical Institute, Kolkata, ISI Platinum Jubilee Monograph Series, World Scientific Press, Boston, USA, 2008 (To appear).

[4] B.D. Acharya, A characterization of consistent marked graphs, *Nat. Acad. (India) Sci.-Letters,* **6** (12)(1983), 431–440.

[5] B.D. Acharya, Some further properties of consistent marked graphs, *Indian J. pure appl. math.,* **15** (8)(1984), 837–842.

[6] M. Acharya and D. Sinha, A characterization of signed graphs that are switching equivalent to their jump signed graphs, *Graph Theory Notes of New York,* **XLIII:1** (2002), 7–8.

[7] M. Acharya and D. Sinha, A characterization of sigraphs whose line sigraphs and jump sigraphs are switching equivalent, *Graph Theory Notes of New York,* **XLIV:6** (2003), 30–34.

[8] M. Behzad and G.T. Chartrand, Line coloring of signed graphs, *Element der Mathematik, Band* **24**(3) (1969), 49–52.

[9] L.W. Beineke and F. Harary, Consistency in marked graphs, *J. Mathl. Psychol.,* **18** (3)(1978), 260–269.

[10] L.W. Beineke and F. Harary, Consistent graphs with signed points, *Riv. Math. per Sci. Econom. Sociol.*, **1** (1978), 81–83.

[11] G.T. Chartrand, *Graphs as Mathematical Models*, Prindle, Weber and Schmidt, Inc., Boston, Massachusetts, 1977.

[12] G.T. Chartrand, H. Hevia, E.B. Jarett and M. Schultz, Subgraph distance in graphs defined by edge-transfers, *Discrete Math.*, **170**(1997), 63–79.

[13] C.R. Cook, *Graphs associated with* $(0,1)$*-arrays*, Themis Project Tech.Rep. No.28, University of Iowa, 1970.

[14] M.K. Gill, *Contributions to some topics in graph theory and its applications*, The Indian Institute of Technology, Bombay, 1983.

[15] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass., (1972).

[16] F. Harary, A characterization of balanced signed graphs, *Mich. Math. J.*, **2** (1953), 143–146.

[17] F. Harary, R.Z. Norman and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed graphs*, Wiley, New York, 1965.

[18] S.T. Hedetniemi, *Graphs of* $(0,1)$*-matrices*, In: Recent trends in graph theory, Lecture Notes in Mathematics, No.186, Springer-Verlag, New York, 1970.

[19] C. Hoede, A characterization of consistent marked graphs, *J. Graph Theory,* **16** (1)(1992), 17–23.

[20] Mukti Acharya and Deepa Sinha, Common-edge sigraphs, *AKCE J. Graphs. Combin.*, **3**(2)(2006), 115-130.

[21] V. Mishra, *Graphs associated with* $[0,1]$ *and* $[0,+1,-1]$ *matrices*, Ph.D. Thesis, The Indian Institute of Technology, Bombay, 1975.

[22] S.B. Rao, Characterization of harmonious marked graphs and consistent nets, *J. Combin., Infom. Syst. Sci.,* **9** (1984), 97–112.

[23] F.S. Roberts and Shaoji Xu, Characterizations of consistent marked graphs, *Discrete Appl. Math.,* **127** (2003), 357-371.

[24] E. Sampathkumar, *Point-signed and line-signed graphs*, Karnatak University Graph theory Research Report No.1, 1973. [Also, see Abstract No.1 in Graph theory Newsletter, **2** (2)(1972); National Academy Science-Letters, **7** (1984), 91–93.]

[25] D. Sinha, *New frontiers in the theory of signed graphs*, Ph.D. Thesis, University of Delhi, 2005.

[26] T. Sozansky, Enumeration of weak isomorphism classes of signed graphs, *J. Graph Theory,* 4 (2)(1980), 127–144.

[27] B. Wu and X. Guo, Diameters of jump graphs and self- complementary jump graphs, *Graph Theory Notes of New York,* **XL: 6**(2001), 31–34.

[28] T. Zaslavsky, Signed graphs, *Discrete Appl. Math.,* **4** (1)(1982), 47–74.