Skolem Graceful Signed Stars

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Abstract

In this paper, we obtain a necessary condition for the Skolem gracefulness of disjoint union of k signed stars K_{1,r_i}, $1 \leq i \leq k$, which we call a k-signed star $St(r_1, r_2, \ldots, r_k)$. We also present results on the Skolem gracefulness of the 2-signed star $St(r_1, r_2)$.

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1 Introduction

For standard terminology and notation in graph theory, we follow West [20] and for signed graphs we follow Chartrand [8] and Zaslavsky [21, 22]. For a dynamic survey on graph labelings we refer to Gallian [11].

Lee and Wui [16] introduced the concept of Skolem graceful graphs. A *Skolem graceful labeling* of a graph $G = (V, E)$ is a bijection $f : V \to \{1, 2, \ldots, |V|\}$ such that the induced edge labeling $g_f : E \to \{1, 2, \ldots, |E|\}$ defined by $g_f(uv) = |f(u) - f(v)|$ $\forall uv \in E$ is also bijective; if such a labeling exists for a given graph G, then it is called a *Skolem graceful graph.* If a graph G, with p vertices and q edges, is graceful then by definition one may see that $q \geq p - 1$, while if it is Skolem graceful, then $q \leq p - 1$. Thus, Skolem graceful labelings nearly complement graceful labelings and *a graph with $q = p - 1$ is graceful if and only if it is Skolem graceful.* Subsequent to this observation, subtle links of this notion with that of graceful graphs have been found (*e.g., see* [1]). A k-star $St(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a disconnected graph with k components $K_{1,\alpha_1}, K_{1,\alpha_2}, \ldots, K_{1,\alpha_k}$ where $K_{1,r}$ denotes the *star* with $r+1$ vertices and r edges.

A (p, m, n)-sigraph S is an ordered pair (G, s), where $G = (V, E)$ is a (p, q)-graph and s is a function which assigns to each edge of G a positive or negative sign and $|E^+(S)| = m, |E^-(S)| = n$ where $E^+(S)$ and $E^-(S)$ denote respectively the set of positive and negative edges of S. If a bijection $f : V(S) \to \{1, 2, \ldots, |V(S)|\}$ such that the induced edge function $g_f(uv) = s(uv)|f(u) - f(v)|, \forall uv \in E(S)$ assigns the numbers $1, 2, \ldots, m$ to the positive edges and $-1, -2, \ldots, -n$ to the negative edges of S, then f is called a *Skolem graceful* labeling of S and the signed graph which admits such a labeling is called *Skolem graceful sigraph.* Note that if $n = 0$, then above definition of Skolem graceful sigraphs coincides with the Skolem graceful graphs (e.g. see [2] - [6],[18], [19]).

By a *k-signed star* (or *k-signed star,* in short), we mean a sigraph on $St(\alpha_1, \alpha_2, \ldots, \alpha_k)$. In this paper, we investigate the Skolem gracefulness of a k-signed star. Some well known results on Skolem graceful graphs listed below will serve as background and are likely to be used in our investigation.

Theorem 1.1. [15] *Let $G = (V, E)$ be a (p, q)-graph. If G is Skolem graceful then*

(i) $p \geq q + 1$ *and*

(ii) it is possible to partition $V(G)$ into two subsets V_1 and V_2 such that the number of edges connecting vertices in V_1 with vertices in V_2 is exactly $\lfloor \frac{q+1}{2} \rfloor$.

Since a tree is Skolem graceful if and only if it is graceful, the problem of determining disconnected $(p, p-1)$–graphs that are Skolem graceful is open. The following result is known in literature towards this end.

Theorem 1.2. [15] *The graph nK_2 is Skolem graceful if and only if* $n \equiv 0$ *or* $1 (\mathrm{mod}\ 4)$.

The following results are known in literature for any disconnected (p, q)-graph for which $q \leq p - 1$.

Theorem 1.3. [16] *A 2-star $St(\alpha_1, \alpha_2)$ is Skolem graceful if and only if $\alpha_1 \alpha_2$ is even.*

Theorem 1.4. [16] *A 3-star $St(\alpha_1, \alpha_2, \alpha_3)$ is Skolem graceful if and only if $\alpha_1 \alpha_2 \alpha_3$ is even.*

Theorem 1.5. [13] *If a k-star $St(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is Skolem graceful then either some α_i is even or $k \equiv 0$ or $1 \pmod 4$.*

Theorem 1.6. [9, 17] *All 4-stars $St(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are Skolem graceful.*

Theorem 1.7. [13] *All 5-stars are Skolem graceful.*

Lee, Quach and Wang [14] showed that the disjoint union of the path P_n and the star of size m is Skolem graceful if and only if $n = 2$ and m is even or $n \geq 3$ and $m \geq 1$. Harary and Hsu [12] studied Skolem graceful graphs under the name of *node-graceful graphs*. Frucht [10] has shown that $P_m \cup P_n$ is Skolem graceful when $m+n \geq 5$. Bhat-Nayak and Deshmukh [7] have shown that $P_{n_1} \cup P_{n_2}$ is Skolem graceful for all $n_1, n_2 \geq 3$. In general, they have proved that $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$, $i \geq 4$, is Skolem graceful.

Theorem 1.8. [6, 18] *Let $S = (G, s)$ be any (p, m, n)-sigraph with $G = (V, E)$ as its underlying graph. If S is Skolem graceful then*

(i) $p \geq m + n + 1$ *and*

(ii) it is possible to partition $V(S)$ into two subsets V_o and V_e such that the numbers $m^+(V_o, V_e)$ and $m^-(V_o, V_e)$ of positive and negative edges of S respectively, each of which joins a vertex of V_o with one of V_e, are given by
$m^+(V_o, V_e) = \lfloor \frac{m+1}{2} \rfloor$ *and* $m^-(V_o, V_e) = \lfloor \frac{n+1}{2} \rfloor$.

Theorem 1.9. [6, 18] *The sigraph on k copies of K_2 is Skolem graceful if and only if one of the following statements holds:*

(1) $k \equiv 0 \pmod 4$ *and the number of negative edges is even*

(2) $k \equiv 1 \pmod 4$

(3) $k \equiv 2 \pmod 4$ *and the number of negative edges is odd.*

In this paper we discuss about the Skolem gracefulness of a sigraph on k-star $St(r_1, r_2, \ldots, r_k)$. Let the central vertices of $St(r_1, r_2, \ldots, r_k)$ be labeled as c_i, $1 \leq i \leq k$. Whenever c_i receives an odd label it is called *odd* c_i and *even* c_i is defined similarly.

2 Main Results

In the following theorem we obtain a necessary condition for a sigraph on $St(r_1, r_2, \ldots, r_k)$ to be Skolem graceful.

Theorem 2.1. *If a sigraph on $St(r_1, r_2, \ldots, r_k)$ is Skolem graceful then one of the following conditions holds:*

(1) *If every* r_i *is odd then*

(i) $k \equiv 0 (\text{mod } 4)$ *and* n *is even or*

(ii) $k \equiv 1 (\text{mod } 4)$ *or*

(iii) $k \equiv 2 (\text{mod } 4)$ *and* n *is odd.*

(2) *If every* r_i *is even then*

(i) $k \equiv 0 (\text{mod } 4)$ *and* n *is even (odd) if the number of odd* c_i*'s is even (odd) or*

(ii) $k \equiv 1 (\text{mod } 4)$ *and* n *is even (odd) if number of odd* c_i*'s is odd (even) or*

(iii) $k \equiv 2 (\text{mod } 4)$ *and* n *is even (odd) if number of odd* c_i*'s is odd (even) or*

(iv) $k \equiv 3 (\text{mod } 4)$ *and* n *is even (odd) if the number of odd* c_i*'s is even (odd).*

(3) *If the number of odd* r_i*'s is odd then*

(i) $k \equiv 0, 1 (\text{mod } 4)$ *and the number of odd* c_i*'s is even or*

(ii) $k \equiv 2$ *or* $3 (\text{mod } 4)$ *and the number of odd* c_i*'s is odd.*

Proof. Let us assume that the sigraph S on $St(r_1, r_2, \ldots, r_k)$ is Skolem graceful through a Skolem graceful labeling ψ. Let the central vertices of S be labeled by c_i, where $1 \leq i \leq k$. Let $a_{i,1}, a_{i,2}, \ldots, a_{i,\alpha_i}$ be the labels of the vertices adjacent to c_i such that $a_{i,j} > c_i$ for $1 \leq j \leq \alpha_i$, and let $b_{i,1}, b_{i,2}, \ldots, b_{i,\beta_i}$ be the labels of the vertices adjacent to c_i such that $b_{i,j} \leq c_i$ for $1 \leq j \leq \beta_i$ where $\alpha_i + \beta_i = r_i$.

The order and size of S are given by $p = r_1 + r_2 + \cdots + r_k + k = q + k$, and $q = |E| = r_1 + r_2 + \cdots + r_k$. By definition, the vertex labels are $1, 2, \ldots, q + k$, the positive edges receive the labels $1, 2, \ldots, m$ and the negative edges receive the labels $1, 2, \ldots, n$. Then

$$\sum_{v \in V(S)} \psi(v) = 1 + 2 + \cdots + q + k = \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} \sum_{j=1}^{\alpha_i} a_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{\beta_i} b_{ij}$$

Hence

$$\frac{(q+k)(q+k+1)}{2} = \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} \sum_{j=1}^{\alpha_i} a_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{\beta_i} b_{ij} \qquad (1)$$

$$\text{Also} \sum_{e\in E(S)} g_\psi(e) = \sum_{e\in E^+(S)} g_\psi(e) + \sum_{e\in E^-(S)} g_\psi(e)$$

$$= \sum_{i=1}^{k}(\sum_{j=1}^{\alpha_i}(a_{ij} - c_i)) + \sum_{i=1}^{k}(\sum_{j=1}^{\beta_i}(c_i - b_{ij})).$$

$$\text{Hence } \frac{m(m+1)}{2} + \frac{n(n+1)}{2} = \sum_{i=1}^{k}(\sum_{j=1}^{\alpha_i}(a_{ij} - c_i)) + \sum_{i=1}^{k}(\sum_{j=1}^{\beta_i}(c_i - b_{ij}))$$

Since $q = m + n$, the above equation can be written as

$$\frac{(q-n)(q-n+1)}{2} + \frac{n(n+1)}{2} = \sum_{i=1}^{k}(\sum_{j=1}^{\alpha_i}(a_{ij} - c_i) + \sum_{i=1}^{k}(\sum_{j=1}^{\beta_i}(c_i - b_{ij}) \qquad (2)$$

By summing (1) and (2) we get,

$$\frac{(q+k)(q+k+1)}{2} + \frac{(q-n)(q-n+1)}{2} + \frac{n(n+1)}{2}$$

$$= \sum_{i=1}^{k} c_i + \sum_{i=1}^{k}\sum_{j=1}^{\alpha_i} a_{ij} + \sum_{i=1}^{k}\sum_{j=1}^{\beta_i} b_{ij} + \sum_{i=1}^{k}(\sum_{j=1}^{\alpha_i}(a_{ij} - c_i)) + \sum_{i=1}^{k}(\sum_{j=1}^{\beta_i}(c_i - b_{ij}))$$

On simplification, we get

$$q(q+k+1-n) + n^2 + \frac{k(k+1)}{2} = \sum_{i=1}^{k}(1+r_i)c_i + 2\sum_{i=1}^{k}(\sum_{j=1}^{\alpha_i} a_{ij} - \alpha_i c_i) \qquad (3)$$

Case 1. All r_i's are odd.

Then $q(q+k+1-n)+n^2+\frac{k(k+1)}{2} \equiv 0(\bmod\ 2)$. (4)

If $k \equiv 3(\mathrm{mod}\ 4)$, then q is odd whence by substituting in (4) we get a contradiction.

If $k \equiv 0(\mathrm{mod}\ 4)$, then since all r_i's are odd, q must be even and hence from (4) n must be even.

If $k \equiv 1(\mathrm{mod}\ 4)$, then, q is odd, so that n is either even or odd.

Also if $k \equiv 2(\mathrm{mod}\ 4)$, then, q is even and it follows that n is odd.

Case 2. All r_i's are even.

Then the sum $\sum_{i=1}^{k}(1 + r_i)c_i$ in the right hand side of (3) is even or odd according as the number of odd c_i's is even or odd. Hence the results given in statement (2) of the theorem follow.

The proof is similar if the number of odd r_i's is odd. □

Conjecture 2.2. *Every sigraph on $St(r_1, r_2, \ldots, r_k)$ satisfying the necessary conditions stated in Theorem 2.1 is Skolem graceful.*

It follows from Theorem 2.1 that *any sigraph on 1- star is Skolem graceful.* We now proceed to investigate the Skolem gracefulness of any sigraph on a 2-star.

Theorem 2.3. *If neither r_1 and r_2 are both odd nor n is such that $1 < n < \frac{r_1+r_2}{2}$, then any sigraph on $St(r_1, r_2)$ is Skolem graceful.*

Proof. To prove the above theorem it is sufficient to provide a Skolem graceful labeling to any sigraph on $St(r_1, r_2)$ in each of the cases for the values of r_1 and r_2 excepting the values as specified in the statement of the theorem.

Towards this end, let ψ be a vertex labeling from the set of positive integers $\{1, 2, \ldots, |V|\}$. Then, there arise the following cases:

If a sigraph S on $St(r_1, r_2)$ is homogeneous then the proof follows from Theorem 1.3 [16].

Hence, we assume that S is a heterogeneous sigraph on $St(r_1, r_2)$. Let the central vertices of $St(r_1, r_2)$ be c_1 and c_2, let $a_1, a_2, \ldots, a_{r_1}$ and $b_1, b_2, \ldots, b_{r_2}$ be the pendant vertices adjacent to c_1 and c_2 respectively. We first consider the case where one component of S is all-negative and the other is all-positive. Then the numbering ψ defined by

$\psi(c_1) = 1$;
$\psi(c_2) = r_1 + 2$;
$\psi(a_i) = i + 1, \quad 1 \leq i \leq r_1$; and
$\psi(b_j) = r_1 + 2 + j, \quad 1 \leq j \leq r_2$.

is a Skolem graceful labeling of the sigraph (*e.g.*, *see* Figure 1).

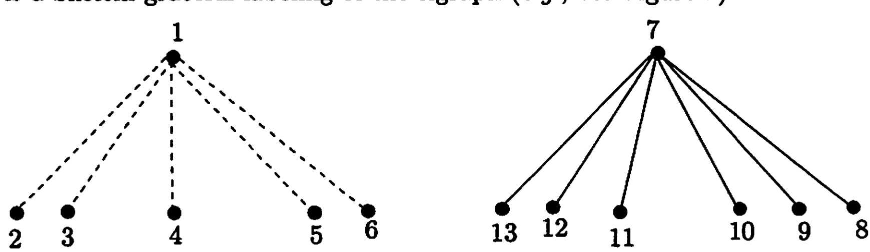


Figure 1. Example of Skolem graceful sigraph on $St(r_1, r_2)$.

Now suppose each component of S is heterogeneous. We have the following cases.

Case 1. r_1 and r_2 are both even.

Let the number of negative edges in K_{1,r_1} and K_{1,r_2} be $n_1 = \frac{r_1}{2} - k_1$ and $n_2 = \frac{r_2}{2} - k_2$ for some positive integers k_1 and k_2. Let $r_2 = r_1 + 2x$, where x is some positive integer. We define a numbering as follows:

$\psi(c_1) = 1$;
$\psi(c_2) = r_1 + k_1 + k_2 + x + 2$;
$\psi(a_i) = i + 1, \quad 1 \leq i \leq \frac{r_1}{2}$;
$\psi(a_i) = k_1 + k_2 + x + 1 + i, \quad \frac{r_1}{2} + 1 \leq i \leq r_1$;
$\psi(b_j) = \frac{r_1}{2} + 1 + j, \quad 1 \leq j \leq k_1 + k_2 + x$ and
$\psi(b_j) = r_1 + 2 + j, \quad k_1 + k_2 + x + 1 \leq j \leq r_2$.

It is easy to verify that the numbering so defined is a Skolem graceful labeling of the sigraph (*e.g., see* Figure 2).

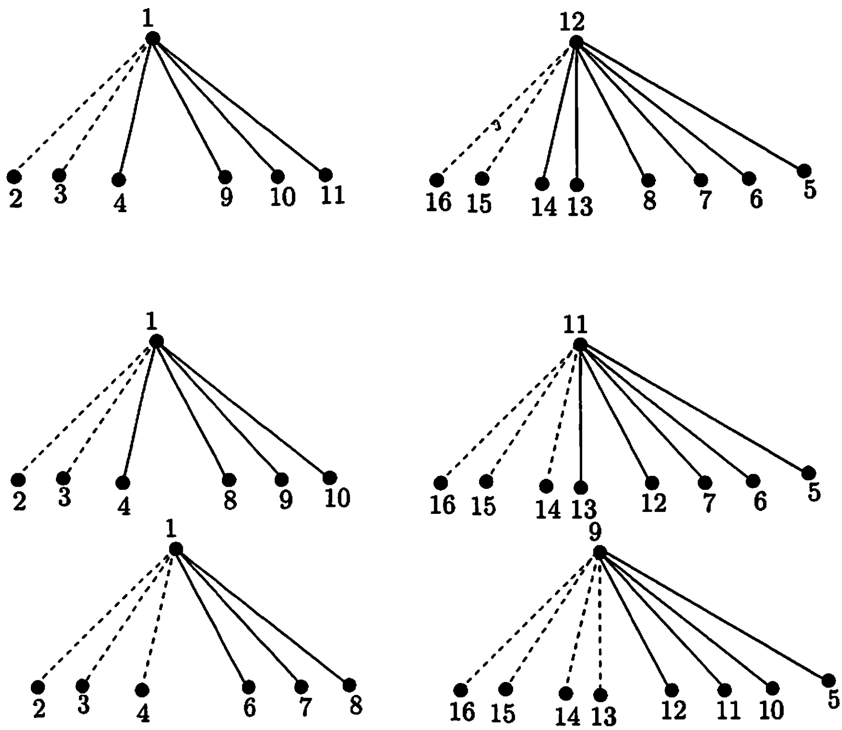


Figure 2. Three examples of Skolem graceful sigraphs on $St(r_1, r_2)$.

If $r_1 = r_2$ and $n = \frac{r_1}{2}$ in each copy of the signed star, then, we define a numbering as follows:
$\psi(c_1) = 1$;
$\psi(c_2) = r_1 + 2$;
$\psi(a_i) = i + 1, \quad 1 \leq i \leq r_1$; and
$\psi(b_j) = r_1 + 2 + j, \quad 1 \leq j \leq r_2$.

It is easy to verify that the numbering so defined is a Skolem graceful labeling of the sigraph (e.g., see Figure 3).

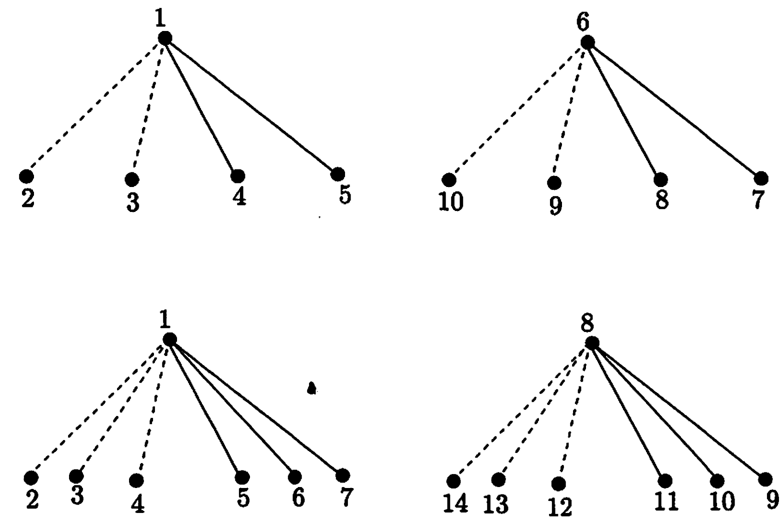


Figure 3. Two examples of Skolem graceful sigraphs on $St(r_1, r_2)$.

Case 2. r_1 is even and r_2 is odd.

Let the number of negative edges in K_{1,r_1} and K_{1,r_2} be $n_1 = \frac{r_1}{2} - k_1$ and $n_2 = \lfloor \frac{r_2}{2} \rfloor - k_2$ for some positive integers k_1 and k_2. Let $r_2 = r_1 + 2x - 1$ where x is some positive integer. Then, we define a numbering of S as follows:

$\psi(c_1) = 1;$

$\psi(c_2) = r_1 + k_1 + k_2 + x + 2;$

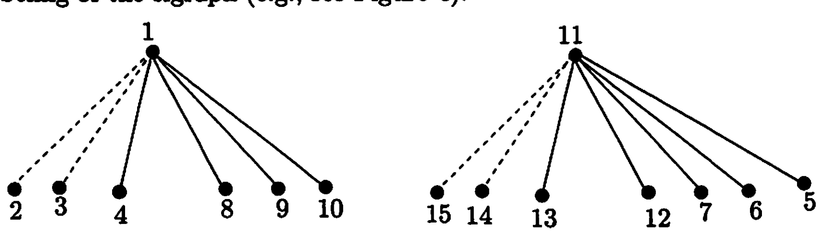
$\psi(a_i) = i + 1, \quad 1 \le i \le \frac{r_1}{2};$

$\psi(a_i) = k_1 + k_2 + x + 1 + i, \quad \frac{r_1}{2} + 1 \le i \le r_1;$

$\psi(b_j) = \frac{r_1}{2} + 1 + j, \quad 1 \le j \le k_1 + k_2 + x$ and

$\psi(b_j) = r_1 + 2 + j, \quad k_1 + k_2 + x + 1 \le j \le r_2.$

It is easy to verify that the numbering so defined is a Skolem graceful labeling of the sigraph (e.g., see Figure 4).



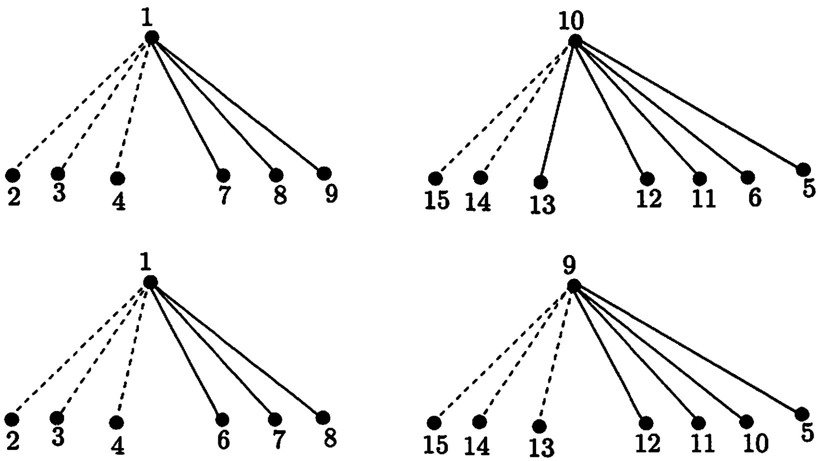


Figure 4. Three examples of Skolem graceful sigraphs on $St(r_1, r_2)$.

Case 3. Both r_1 and r_2 are odd.

If $n = 1$, we define a numbering as follows:

$\psi(c_1) = 1$;

$\psi(a_i) = i + 1, \quad 1 \leq i \leq \lceil(\frac{r_1}{2})\rceil$;

$\psi(a_i) = r_2 + i, \quad \lceil(\frac{r_1}{2})\rceil + 1 \leq i \leq r_1$;

$\psi(c_2) = r_1 + r_2 + 2$;

$\psi(b_j) = \lceil(\frac{r_1}{2})\rceil + 1 + j, \quad 1 \leq j \leq r_2 - 1$ and

$\psi(b_j) = r_1 + r_2 + 1$.

It is easy to verify that the numbering so defined is a Skolem graceful labeling of the sigraph S (e.g., see Figure 5).

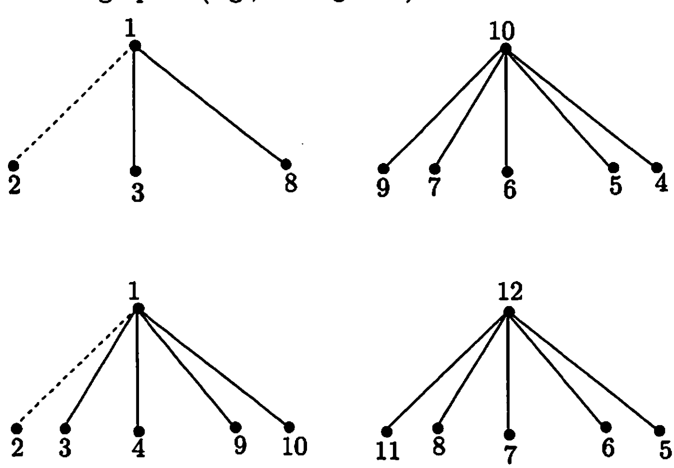


Figure 5. Two examples of Skolem graceful sigraphs on $St(r_1, r_2)$.

If $r_1 = r_2 = r$ and $n_1 = \lfloor \frac{r}{2} \rfloor$, $n_2 = \lceil \frac{r}{2} \rceil$, we define a numbering as follows:

$\psi(c_1) = 1$;

$\psi(a_i) = i + 1, \quad 1 \leq i \leq r_1$;

$\psi(c_2) = r_1 + 2$; and $\psi(b_j) = r_1 + 2 + j, \quad 1 \leq i \leq r_2$.

It is easy to verify that the numbering so defined is a Skolem graceful labeling of the sigraph S (e.g., see Figure 6).

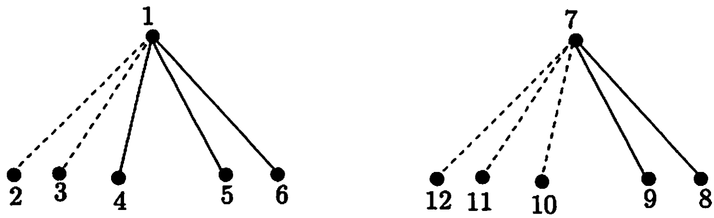


Figure 6. Skolem graceful sigraph on $St(r_1, r_2)$.

Thus the proof is seen to be complete. □

Note that when r_1 and r_2 are both odd and number of negative edges in each copy is more than one but less than $(\frac{q}{2})$, then the Skolem gracefulness of 2-signed star is not yet known but seems to be true.

We have started investigating the Skolem gracefulness of disjoint union of k signed caterpillars. We have obtained some preliminary results, which will be reported in a subsequent paper.

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