Mathematical Programs for Drawing Nonplanar Graphs in the Plane

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Abstract

Many approaches to drawing graphs in the plane can be formulated and solved as mathematical programming problems. Here, we consider only drawings of a graph where each edge is drawn as a straight-line segment, and we wish to minimize the number of edge crossings over all such drawings. Some formulations of this problem are presented that lead very naturally to other unsolved problems, some solutions, and some new open problems associated with drawing nonplanar graphs in the plane.

Keywords. Drawing a graph, good drawing, grid drawing, rectilinear drawing, crossing number, rectilinear crossing number.
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1 Introduction

At least since 1954 with the publication of the paper by Zarankiewicz that attempted to solve Turan's Brick Factory Problem [45], the study of crossing numbers of graphs has continued to progress both in theory and for applications. The original problem remains unsolved; however, engineers and entrepreneurs have developed an entire scientific and computational culture centered on the applications of crossing numbers to the design of printed circuit boards and VLSI circuits. Regarding theoretical approaches the results are scattered. The theory of graphs that can be drawn with

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no crossings (i.e., planar graphs) is well understood, and there are numerous efficient algorithms for determining that fact and exploiting it to construct drawings such as VLSI layouts, information diagrams, flowcharts, etc. However, minimizing the number of crossings is known to be NP-hard, and exact solutions are exceedingly difficult to obtain (for example, Turan's Brick Factory Problem).

This paper concerns *drawings* of a simple graph $G = (V, E)$ with vertex set V and edge set E in the plane where the vertices are mapped to distinct points. We are mainly concerned with *good drawings*; these are drawings where each edge is mapped to a homeomorphic image of the closed interval $[0, 1]$ with its ends as endpoints. The interior (called an *arc*) contains no endpoint, arcs incident with a common endpoint do not intersect, and no two arcs intersect in more than one point. Any drawing where the edges are straight-line segments is called a *rectilinear drawing.* The goal of (rectilinear) crossing minimization is to find a (rectilinear) drawing of G with as few edge crossings as possible. This minimum value is called the *(rectilinear) crossing number* $cr(G)$ (respectively, $\overline{cr}(G)$) of G. Both of the corresponding decision problems are known to be NP-hard. Moreover, the crossing number problem is known to be NP-complete [20], but so far no one has determined whether the rectilinear crossing number problem is in NP. This is somewhat surprising, since it is much easier to determine if two straight-line segments intersect than it is to determine if two curved lines do.

This paper presents some new results on rectilinear crossing minimization. In Section 2 we construct a mathematical formulation of the problem in terms of a linear objective function with simple quadratic constraints. In Section 3 we give a comparative sizing of the problem for various graph families in an attempt to provide some evidence for ranking problems according to how difficult they might be to solve computationally. In an earlier paper [15] we predicted that as more advanced techniques were developed and applied to the formulation given there, some of these problems would be solved. Indeed they have, and so we return to this problem to clean up some details, explain the connection to current ongoing research by others, and to identify additional research topics. One of these topics is the screen size of a graph which will be introduced in Section 4. Finally, in Section 5 we conclude with some related open problems.

2 Quadratic Constraints Formulation

In this section we construct a mathematical programming formulation of the rectilinear crossing number problem where, for a given graph G on n vertices, most of the variables are binary, they appear only in linear form, and the $2n$ continuous variables appear in quadratic form. Of course, many mathematical problems have useful mathematical programming formulations (for example, see [34] and [44] for related results), but until now no such formulation has appeared for the rectilinear crossing number problem.

For any rectilinear drawing of a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ we define $\text{Area2}(i, j, k) = x_i y_j - y_i x_j - x_i y_k + y_i x_k + x_j y_k - y_j x_k$ where (x_i, y_i), (x_j, y_j), (x_k, y_k) are the coordinates for the vertices i, j, k, respectively. It's an old exercise in Analytic Geometry to prove that $\text{Area2}(i, j, k)$ is twice the signed area determined by the triangle ijk where $\text{Area2}(i, j, k)$ is positive if vertex k lies to the left of line ij, negative if it lies to the right, and zero if k lies on the line ij. When we speak of a line ij (not a line segment) we mean the two-way infinite directed line determined by the ordered pair of vertices i, j. We must traverse the line i, j so that i is encountered before j. Thus, the order of the triple i, j, k is important.

For any graph G, let $I(G)$ denote the set of independent (i.e., disjoint) edge sets of size two. We wish to solve the following problem which we call the *Quadratic Constraints Formulation* QCF2(G) for G, or simply QCF2 when there is no particular graph G under consideration.

$$\text{Minimize} \sum_{\substack{1 \le i < j, i < k < l, \\ \{ij, kl\} \in I(G)}} c_{ijkl}$$

subject to

$$
\begin{aligned}
\text{Area2}(i,j,k) &\le M(1 - c_{ijkl}) + M t_{ijkl} - 1 && (1)\\
-\text{Area2}(i,j,l) &\le M(1 - c_{ijkl}) + M t_{ijkl} - 1 && (2)\\
-\text{Area2}(i,j,k) &\le M(1 - c_{ijkl}) + M(1 - t_{ijkl}) - 1 && (3)\\
\text{Area2}(i,j,l) &\le M(1 - c_{ijkl}) + M(1 - t_{ijkl}) - 1 && (4)\\
\text{Area2}(k,l,i) &\le M(1 - c_{ijkl}) + M p_{ijkl} - 1 && (5)\\
-\text{Area2}(k,l,j) &\le M(1 - c_{ijkl}) + M p_{ijkl} - 1 && (6)\\
-\text{Area2}(k,l,i) &\le M(1 - c_{ijkl}) + M(1 - p_{ijkl}) - 1 && (7)\\
\text{Area2}(k,l,j) &\le M(1 - c_{ijkl}) + M(1 - p_{ijkl}) - 1 && (8)\\
\text{Area2}(i,j,k) &\le M c_{ijkl} + M t_{ijkl} + M p_{ijkl} - 1 && (9)\\
\text{Area2}(i,j,l) &\le M c_{ijkl} + M t_{ijkl} + M p_{ijkl} - 1 && (10)\\
-\text{Area2}(i,j,k) &\le M c_{ijkl} + M(1 - t_{ijkl}) + M p_{ijkl} - 1 && (11)\\
-\text{Area2}(i,j,l) &\le M c_{ijkl} + M(1 - t_{ijkl}) + M p_{ijkl} - 1 && (12)\\
\text{Area2}(k,l,i) &\le M c_{ijkl} + M t_{ijkl} + M(1 - p_{ijkl}) - 1 && (13)\\
\text{Area2}(k,l,j) &\le M c_{ijkl} + M t_{ijkl} + M(1 - p_{ijkl}) - 1 && (14)\\
-\text{Area2}(k,l,i) &\le M c_{ijkl} + M(1 - t_{ijkl}) + M(1 - p_{ijkl}) - 1 && (15)\\
-\text{Area2}(k,l,j) &\le M c_{ijkl} + M(1 - t_{ijkl}) + M(1 - p_{ijkl}) - 1 && (16)
\end{aligned}
$$

where the variables i, j, k, l satisfy $\{ij, k\ell\} \in I(G)$, $1 \leq i < j, i < k < l$, the functions c, t, p are 0-1 binary variables, and M is a sufficiently large integer constant.

Now we explain why QCF2 determines the rectilinear crossing number and produces an optimal rectilinear drawing of any graph. Notice first that the -1 in the functional constraint forces the area determined by any three points forming a 3-cycle to be nonzero, actually at least $1/2$ in absolute value. Rescaling can be used to make all nonzero areas satisfy this condition, and so the -1 really only ensures that we never have an edge passing through a vertex, and for complete graphs this forces the points to be in general position (not considered in [15]). This is known as a *good drawing* (see [18], [23], [30]). So, the function c_{ijkl} counts the number of times edges ij and kl cross (i.e., 0 or 1), and the objective function counts the total number of crossings for the drawing.

Suppose $c_{ijkl} = 1$. Then constraints (9)-(16) become irrelevant, and constraints (1)-(4) ensure that points k and l lie on opposite sides of the line ij. Similarly, constraints (5)-(8) ensure that points i and j lie on opposite sides of the line kl. Therefore, the line segments ij and kl must cross.

Now suppose $c_{ijkl} = 0$. Then constraints (1)-(8) are irrelevant, and either $p_{ijkl} = 0$ which ensures that points k and l lie on the same side of the line ij (see constraints (9)-(12)) or $p_{ijkl} = 1$ which ensures that points i and j lie on the same side of the line kl (see constraints (13)-(16)). Thus, ij and kl do not cross.

3 Comparative Sizing for Graph Families

Research in integer and nonlinear programming provides some hope that some useful instances of QCF2 can be solved. One breakthrough was made by Crowder, Johnson and Padberg [14] whose paper won the Lanchester Prize for solving binary integer programming (BIP) problems with up to 2,756 variables and no special structure. Roy and Wolsey [40] succeeded in solving mixed BIPs with nearly 1,000 binary variables and an even larger number of real variables. Their paper won the Orchard-Hays Prize. In this section we provide some data, such as the number of binary and real variables and the number of functional constraints in QCF2, on certain families of graphs, to help us decide which of these instances can be solved now or in the foreseeable future. Also, we hope that implementation experts in the field can somehow take advantage of the special structure and ranking of problems revealed here so that the newest algorithmic developments can be applied to solve these problems. In fact since our earlier paper [15] on these families of graphs, several of the problems thought then to be intractable have now been solved using methods (order types) which are essentially equivalent to QCF2 (see [4] and [1]).

Let $r(G), b(G), f(G)$ denote the number of real variables, the number of binary variables, and the number of functional constraints in QCF2(G), respectively. Also, let $i(G)$ denote the number of matchings of size two in G (i.e., $i(G) = |I(G)|$). Then $r(G) = 2n$, $b(G) = 3i(G)$ and $f(G) = 16i(G)$. Letting e denote the number of edges we have

$$i(G) = \frac{1}{2}(e^2 + e - \sum_{x \in V} d^2(x)) \tag{17}$$

from which it follows that

$$i(K_n) = n(n-1)(n-2)(n-3)/8 = 3\binom{n}{4}.$$

Drawings of these graphs have been investigated extensively in the literature because of various theoretical implications and applications to VLSI and computer network design.

Some believe that $\overline{cr}(K_n) > cr(K_n)$ for $n \geq 10$; however, no proof has been published. Of course for rectilinear drawings, putting the vertices of K_n on the circumference of a circle creates the maximum number $\binom{n}{4}$ of crossings. The best upper bounds have been established by producing a drawing, as Singer did in an unpublished manuscript (see [19]) to show $\overline{cr}(K_{10}) \leq 62$. In the same paper he showed $\overline{cr}(K_{10}) \geq 61$. In 2001 the result $\overline{cr}(K_{10}) = 62$ was proved by two independent groups of researchers, one [13] by a purely combinatorial argument and the other [2] using a computational approach requiring the enumeration of all inequivalent point set configurations (i.e., order types, see Section 5) of size 10 . The value of $\overline{cr}(K_n)$ for $n \geq 11$ is unknown. It is known that the parity of the number of crossings in all drawings of K_n is the same whenever n is odd (see [7], [28], and [31]), and so $\overline{cr}(K_{11}) \neq 101$. Guy[25] proved that $cr(K_m) \geq \binom{m}{n} cr(K_n) \Big/ \binom{m-4}{n-4}$ for $m \geq n \geq 5$, and this result provides lower bounds for K_{11} and K_{12} of 100 and 150, respectively. However, the exact values were found later by enumerating all inequivalent point sets of size 11 [3]. The exact value of $cr(K_n)$ is known for all $n \leq 17$ [4]. The bounds list above the line in the following chart are actually known, exact values. Those listed below the line are upper bounds obtained from Aichholzer's web site [1].

n	$r(K_n)$	$b(K_n)$	$f(K_n)$	Best Bound for $\overline{cr}(K_n)$
4	8	9	48	0
5	10	45	240	1
6	12	135	720	3 [23]
7	14	315	1680	9 [23]
8	16	630	3360	19 [23]
9	18	1134	6048	36 [23]
10	20	1890	10080	62 [13], [2]
11	22	2970	15840	102 [3]
12	24	4455	23760	153 [3]
13	26	6435	34320	229 [4]
14	28	9009	48048	324 [4]
15	30	12285	65520	447 [4]
16	32	16380	87360	603 [4]
17	34	21420	114240	798 [4]
18	36	27540	146880	1029 [1]
19	38	34884	186048	1318 [1]
20	40	43605	232560	1657 [1]
21	42	53865	287280	2055 [1]
22	44	65835	351120	2528 [1]

The following equation known as Zarankiewicz's Conjecture has been solved only for $\min\{p,q\} \leq 6$ by Kleitman [30] and for $K_{7,7}$ and $K_{7,9}$ by Woodall [46].

$$\overline{cr}(K_{p,q}) = cr(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$$

The question "$\overline{cr}(K_{p,q}) = cr(K_{p,q})$?" could be treated as a separate problem. Because of a counting argument this leaves $K_{7,11}$ and $K_{9,9}$ as the two smallest unsolved cases. From Equation 17 we see that

$$i(K_{p,q}) = pq(p-1)(q-1)/2 = 2\binom{p}{4}\binom{q}{4}$$

which is used to construct the following table.

p	q	$r(K_{p,q})$	$b(K_{p,q})$	$f(K_{p,q})$	Best Bounds on $\overline{cr}(K_{p,q})$
3	3	12	54	288	1
4	4	16	216	1152	4
5	5	20	600	3200	16 [30]
6	6	24	1350	7200	36 [30]
7	7	28	2646	14112	81 [46]
8	8	32	4704	25088	144 [46]
7	11	36	6930	36960	≤ 175
9	9	36	7776	41472	≤ 256

The main conjecture on the crossing number of the product $C_p \times C_q$ is due to Harary, Kainen and Schwenk [27] and states that $cr(C_p \times C_q) = (p-2)q$ for $p \leq q$. For a survey on this problem including a discussion of the main proof techniques, see Myers [36]. The conjecture has been verified for $p \leq 7$. The drawings used to establish the upper bound for the conjecture are easy to transform into a rectilinear drawing, and so $\overline{cr}(C_p \times C_q) \leq (p-2)q$ for $p \leq q$. Recently, both Bruce Richter and Gelasio Salazar commented (personal communication) that the case $C_8 \times C_8$ can be settled using existing techniques, but the proof would require at least 100 pages! Our sizing of this problem is based on the formula

$$i(C_p \times C_q) = pq(2pq - 7)$$

from Equation 17 and appears in the following table.

p	$r(C_p \times C_p)$	$b(C_p \times C_p)$	$f(C_p \times C_p)$	Best Bounds on $\overline{cr}(C_p \times C_p)$
3	18	297	1584	3 [27]
4	32	1200	6400	8 [17]
5	50	3225	17200	15 [37]
6	72	7020	37440	24 [5]
7	98	13377	71344	35 [6]
8	128	23232	123904	≤ 48 [27]
9	162	37665	200880	≤ 63 [27]

Let Q_p denote the p-dimensional cube; this is the graph whose vertex set consists of all possible p-tuples of 0's and 1's where two vertices are adjacent if and only if they differ in exactly one coordinate. Eggleton and Guy [18] announced the inequality

$$cr(Q_p) \leq \frac{5}{32}4^p - \left\lfloor \frac{p^2+1}{2} \right\rfloor 2^{p-2},$$

but an error was found in the construction [24]. So this upper bound is now only a conjecture. In fact, they conjecture that equality holds, and (if true) this would imply $cr(Q_5) = 56$ and $cr(Q_p) = 352$. So far, Madej [33] has proved the best upper bound $cr(Q_p) < 4^p/6$, and Sýkora and Vrto [43] have proved the best lower bound $cr(Q_p) > 4^p/20 - (p^2+1)2^{p-1}$. These bounds are useless for small p. Equation 17 gives

$$i(Q_p) = 2^{p-2}p(2^{p-1}p - 2p + 1)$$

from which we construct the following table.

p	$r(Q_p)$	$b(Q_p)$	$f(Q_p)$	Best Bounds on $\overline{cr}(Q_p)$
3	16	126	672	0
4	32	1200	6400	8 [17],[18]
5	64	8520	45440	
6	128	52128	278016	

Let $\overline{C}_n$ denote the graph obtained from K_n by deleting a hamiltonian cycle; that is, $\overline{C}_n$ is the complement of C_n. Guy and Hill [26] determined the exact values of $cr(\overline{C}_n)$ for $n \leq 10$ and $\overline{cr}(\overline{C}_n)$ for $n \leq 9$. In general, they showed that

$$cr(\overline{C}_n) \leq \frac{1}{64}(n-3)^2(n-5)^2 \text{ for } n \text{ odd}$$

and

$$cr(\overline{C}_n) \leq \frac{1}{64}n(n-4)(n-6)^2 \text{ for } n \text{ even.}$$

Equality is conjectured in both cases. Application of Equation 17 gives the formula

$$i(\overline{C}_n) = \frac{1}{8}n(n-3)(n^2-7n+14)$$

which is used to construct the following table.

n	$r(\overline{C}_n)$	$b(\overline{C}_n)$	$f(\overline{C}_n)$	Best Bound for $\overline{cr}(\overline{C}_n)$
5	10	15	80	0
6	12	54	288	0
7	14	147	784	1 [26]
8	16	330	1760	2 [26]
9	18	648	3456	9 [26]
10	20	1155	6160	≥ 15 [26]
11	22	1914	10208	≥ 36 [26]

Jendrol' and M. Ščerbová [29] proved that $cr(K_{1,3} \times P_n) = n$ for $n \geq 2$. Both Beineke and Ringeisen (see [10]) and Jendrol' and Scerbov [29] proved that $cr(K_{1,3} \times C_n) = 1$ for $n = 3$, 2 for $n = 4$, 4 for $n = 5$, and n for $n \geq 6$. The crossing numbers of the products of cycles with all other graphs of orders 3 and 4 were determined earlier by Beineke and Ringeisen (see [9]).

Of the numerous unsolved instances of crossing number problems we have presented in the previous five tables only two instances from five different families of graphs. It seems reasonable that the formulations with fewer binary variables should be easier to solve. Based on this criterion we obtain the following ordering of these ten graphs from easiest to hardest, where two solved instances K_{11}, K_{12} are included just for comparison.

$$\overline{C}_{10}, \overline{C}_{11}, K_{11}, K_{12}, K_{7,11}, K_{9,9}, Q_5, C_8 \times C_8, C_9 \times C_9, Q_6$$

4 Grid Drawings

Besides having just a few crossings, other aesthetic criteria for a nice drawing of a graph include small area and a large angular resolution (i.e., the minimum angle between incident edges) (see [38]). All of them are believed to be computationally difficult to solve; for example, as mentioned earlier, it is NP-hard to decide whether a graph has a rectilinear drawing with at most a prescribed number of crossings [20]. Further, minimizing the number of crossings seems to conflict with the goal of maximizing the angular resolution [21]. Let the size of a grid drawing of a graph G be the number of nodes on the side of the smallest square grid that encloses the drawing, and let the screen size $s(G)$ of G be the smallest such number over all crossing minimal drawings of G. For example, $s(K_{1,8}) = 3$ and $s(K_5) = 4 = s(K_{3,3})$. If G is planar, then $s(G) \leq n$ [41]. However, the problem of deciding whether a graph has a crossing minimal drawing of screen size bounded above by a prescribed number seems to be very difficult, even for planar graphs (see [32]).

5 Conclusions and Open Problems

It may seem that the crossing number and the rectilinear crossing number are very closely related; however in this section we summarize by pointing out some differences between these two parameters and some open problems on the rectilinear crossing number. Along the way we consider other approaches to rectilinear crossing minimization. First, consider the simulated annealing approach taken in [16] to minimize the number of crossings in a rectilinear drawing of a graph. Starting with some initial drawing, nodes were repeatedly moved to random locations, and the drawing was saved if the number of crossings was reduced. This approach produced the best results known at that time. A modification of that approach may still be competitive. For example, if we use an integer lattice so that all coordinates are integers, we can search for an optimal location to place a given vertex k keeping all other vertices fixed. The corresponding formulation of QCP2 reduces to an integer program (IP) with only two nonbinary variables, x_k and y_k.Without the integer lattice,we simply have a mixed BIP where x_k and y_k are the only nonbinary variables.

There are several examples of graphs G with $\overline{cr}(G) > cr(G)$. Is the difference $\overline{cr}(G) - cr(G)$ unbounded? Yes. In [12] the authors showed that, for every integer $k \geq 4$, there is an infinite family of graphs of crossing number k, but unbounded rectilinear crossing number. On the other hand, they prove that any graph G for which $cr(G) \leq 3$ must also satisfy $cr(G) = \overline{cr}(G)$, an extension of the classic theorem of Steinitz and Wagner (also called Fary's theorem) that *every planar graph has a rectilinear embedding in the plane.*

Fixing the binary variables in QCF2(G) except the c_{ijkl}'s reduces to the problem of deciding whether, for the given graph G , a set of coordinates exists for a rectilinear drawing of G that is consistent with these constraints. Call this problem the *QCF2 Realization Problem.* For a configuration of points $p_1, p_2, \ldots, p_n$ in general position, the order type of the configuration is a mapping that assigns to each ordered triple (p_i, p_j, p_k) a sign $+$ or $-$ depending on whether p_k lies to the right or left of the oriented line $p_i p_j$. This is precisely what is accomplished by our use of the signed area function Area2(i, j, k) Hence, the QCF2 realization problem is the same as the problem of realizing a prescribed order type which is discussed by other authors (for example, see [4]). This problem turns out to be NP-hard; however even more is true. A pseudoline is a homeomorph in $\Re^2$ of the closed unit interval. In an *arrangement* of pseudolines every two lines meet at exactly one interior point where they must cross. A *rectilinear realization* of an arrangement is an arrangement where each pseudoline is a straight-line segment, and an arrangement is said to be stretchable if such a realization exists. Shor [42] proved that the problem of determining whether a pseudoline arrangement is stretchable is NP-hard. This also follows from a paper by Mnev [35] which implies the stronger result that deciding stretchability is equivalent to the existential theory of the reals. Bienstock [11] proved that any given arrangement of p pseudolines can be forced to occur in every crossing minimal drawing of an appropriate graph, with $O(p^3)$ edges and $5p(p-1)$ crossings by using some results of Goodman, Pollack, and Sturmfels [22] on arrangements whose straight-line realizations require vertex coordinates with exponentially many bits. Therefore, assuming that *the coordinates must be stored and that drawing a graph consumes time proportional to the size of the drawing,* this yields that there is no polynomial-time algorithm for producing a rectilinear drawing of a graph which achieves the rectilinear crossing number.

For the following questions we do not make the assumptions mentioned above.

Open Problem 1. *Is the QCF2 Realization Problem NP-hard, even when restricted to complete graphs?*

Open Problem 2. *Is $cr(G) < \overline{cr}(G)$ for n sufficiently large?*

Open Problem 3. *Is testing whether $cr(G) = \overline{cr}(G)$ NP-hard?*

Open Problem 4. *Determine all obstructions to stretchability of graph layouts.* (See [25], [39] and [37].)

Open Problem 5. *Is it NP-hard to decide whether a given graph has a crossing minimal drawing of screen size bounded above by a prescribed number?*

Open Problem 6. *Is it NP-hard to decide whether a given planar graph has a crossing minimal drawing of screen size bounded above by a prescribed number?*

Our final two questions are related to approaches for simplifying QCF2 to make it more practical.

Open Problem 7. *Does some crossing minimal drawing of K_{n+1} contain a crossing minimal drawing of K_n?*

Currently, all known optimal drawings of K_n have the following property.

Open Problem 8. (Problem 1 in [8]) *Does every rectilinear drawing of a graph K_n ($n \geq 3$) with exactly $\overline{cr}(K_n)$ crossings have a triangle as its convex hull?*

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