Improved Bounds for Some of the Radio k-chromatic Numbers of Paths

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Abstract

For a path P_n of order n and for any odd integer k, $1 \leq k \leq n-3$, Chartrand et al. have given an upper bound for the radio k-chromatic number of P_n as $\frac{k^2+2k+1}{2}$. Here we improve this bound for $\frac{n-4}{2} \leq k < \frac{2n-5}{3}$ and $\frac{2n-5}{3} \leq k \leq n-3$. They are $\frac{k^2+k+4}{2}$ and $\frac{k^2+k+2}{2}$ respectively. Also, we improve the lower bound of Kchikech et al. from $\frac{k^2+3}{2}$ to $\frac{k^2+5}{2}$ for odd integer k, $3 \leq k \leq n-3$.

Keywords. Radio k-coloring, span of a radio k-coloring, radio k-chromatic number.
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1 Introduction

Let G be a connected graph. For any two vertices u and v of G, the *distance* $d(u,v)$ between u and v is the length of the shortest $u-v$ path in G. The *eccentricity* of a vertex v in G is the distance between v and a vertex in G farthest from v. The diameter of G, denoted by *diamG*, is the maximum eccentricity of the vertices of G. For any positive integer k, $1 \leq k \leq \text{diam}G$, a *radio k-coloring* is an assignment f of positive integers to the vertices of G such that $|f(u) - f(v)| \geq 1 + k - d(u,v)$ for every pair of distinct vertices u, v of G. The maximum positive integer assigned to a vertex of G is called the *span* of f which is denoted by $rc_k(f)$. The minimum span of all radio k-colorings of G is called the *radio k-chromatic number*, denoted by $rc_k(G)$, of G. The radio 1-chromatic number is nothing but the chromatic number $\chi(G)$ of G. If $diamG = d$, then the radio d-coloring is called the *radio coloring* of G and the radio d-chromatic number is the *radio number* of G that was introduced in [1]. Radio k-coloring of a graph was defined by Chartrand et. al. in [2, 3]. The problem of finding radio k-chromatic number of a graph is highly nontrivial. In fact radio k-chromatic number of a path P_n of order n is not yet known for $1 < k \leq n-3$, only bounds are there which are stated below.

Theorem 1.1. [3] *For any integer k, $1 \leq k \leq n-3$,*

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+2k+2}{2}, & if\ k\ is\ even \\ \frac{k^2+2k+1}{2}, & if\ k\ is\ odd. \end{cases}$$

Theorem 1.2. [6] *For any integer k, $1 \leq k \leq n-3$,*

$$rc_k(P_n) \geq \begin{cases} \frac{k^2+6}{2}, & if\ k\ is\ even \\ \frac{k^2+3}{2}, & if\ k\ is\ odd. \end{cases}$$

In [4] and [5] the exact value of $rc_{n-1}(P_n)$ and $rc_{n-2}(P_n)$ are determined respectively and the value of $rc_k(P_n)$, $k \geq n$ are given in [6]. In this paper we improve the bounds of $rc_k(P_n)$, k odd, given in Theorem 1.1 and Theorem 1.2 for $\frac{n-4}{2} \leq k \leq n-3$ and $3 \leq k \leq n-3$ respectively.

2 Results

Before we give the main result we will see an easy but important observation which is also used in many papers, namely ([3], [6]).

Observation 2.1. *For any positive integers m and n, if $m \leq n$ then $rc_k(P_m) \leq rc_k(P_n)$.*

This observation is true because a radio k-coloring of P_n also gives a radio k-coloring of P_m.

The following theorem is the main result of this paper.

Theorem 2.2. *For a positive integer n and an odd positive integer k,*

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+k+2}{2}, & if\ \frac{2n-5}{3} \leq k \leq n-3 \\ \frac{k^2+k+4}{2}, & if\ \frac{n-4}{2} \leq k < \frac{2n-5}{3} \end{cases}$$

Proof. Let P_n be the path $a_1\ a_2\ a_3 \ldots\ a_n$. Note that $d(a_i, a_j) = j - i$ if $j > i$.

Case I: We first prove that for any odd positive integer k and $n = \frac{3k+5}{2}$, the coloring f given as

$$\begin{aligned} f(a_i) &= \frac{k+3}{2} + (i-1)k, for\ 1 \le i \le \frac{k+1}{2} \\ f(a_{\frac{k+3}{2}+j}) &= jk+1, \qquad for\ 0 \le j \le \frac{k+1}{2} \\ f(a_{k+3+l}) &= \frac{k+1}{2} + lk, \qquad for\ 0 \le l \le \frac{k-1}{2} \end{aligned}$$

is a radio k-coloring of P_n

(i) For $1 \le i_1 < i_2 \le \frac{k+1}{2}$, $|f(a_{i_1}) - f(a_{i_2})| = \left|\frac{k+3}{2} + (i_1 - 1)k - \left(\frac{k+3}{2} + (i_2 - 1)k\right)\right|$
$= |(i_1 - i_2)k| \ge k \ge 1 + k - (i_2 - i_1)$

(ii) For $1 \le i \le \frac{k+1}{2}$ and $1 \le j \le \frac{k+1}{2}$, $|f(a_i) - f(a_{\frac{k+3}{2}+j})| = \left|\frac{k+3}{2} + (i-1)k - (jk+1)\right| = \left|\frac{k-1}{2} - (i-j)k\right| = \left|\frac{k-1}{2} + (i-j) - (i-j)(k+1)\right| \ge \frac{k-1}{2} + (i-j)$, for $(i-j) \le 0$. For $i - j > 0$, one can easily check that $(i-j)(k+1) \ge 2\left(\frac{k-1}{2} + (i-j)\right)$.
Therefore for every $1 \le i, j \le \frac{k+1}{2}$, we get $\left|f(a_i) - f\left(a_{\frac{k+3}{2}+j}\right)\right| \ge \frac{k-1}{2} + (i-j) = k + 1 - \left(\frac{k+3}{2} + j - i\right) = k + 1 - d\left(a_i, a_{\frac{k+3}{2}+j}\right)$.

(iii) For $1 \le i \le \frac{k+1}{2}$ and $1 \le l \le \frac{k-1}{2}$, $|f(a_i) - f(a_{k+3+l})| = \left|\frac{k+3}{2} + (i-1)k - \left(\frac{k+1}{2} + lk\right)\right| = |(i-l) - 2 + (k-1)(i-l-1) + 2| \ge (i-l) - 2$, for every i and l. Now $(i-l) - 2 = k + 1 - (k+3+l-i) = k + 1 - d(a_i, a_{k+3+l})$.

(iv) For $0 \le j_1 < j_2 \le \frac{k+1}{2}$, $\left|f\left(a_{\frac{k+3}{2}+j_1}\right) - f\left(a_{\frac{k+3}{2}+j_2}\right)\right| = |(j_1 k + 1) - (j_2 k + 1)| = |(j_1 - j_2)k| \ge k \ge 1 + k - \left(\frac{k+3}{2} + j_2 - \frac{k+3}{2} - j_1\right)$.

(v) For $0 \le j \le \frac{k+1}{2}$ and $0 \le l \le \frac{k-1}{2}$, $\left|f\left(a_{\frac{k+3}{2}+j}\right) - f(a_{k+3+l})\right| = \left|(jk+1) - \left(\frac{k+1}{2} + lk\right)\right| = \left|\frac{k-1}{2} - (j-l)k\right| = \left|\frac{k-1}{2} + (j-l) - (j-l)(k+1)\right| \ge \frac{k-1}{2} + (j-l)$ by (ii). $\frac{k-1}{2} + (j-l) = 1 + k - \left(k+3+l - \frac{k+3}{2} - j\right) = k + 1 - d\left(a_{\frac{k+3}{2}+j}, a_{k+3+l}\right)$.

(vi) For $0 \le l_1 < l_2 \le \frac{k-1}{2}$, $|f(a_{k+3+l_1}) - f(a_{k+3+l_2})| = \left|\frac{k+1}{2} + l_1k - \frac{k+1}{2} - l_2k\right| = |(l_1 - l_2)k| \ge k \ge 1 + k - (k + 3 + l_2 - k - 3 - l_1)$ since $l_2 - l_1 \ge 1$.

Therefore f is a radio k-coloring of $P_{\frac{3k+5}{2}}$ and thus $rc_k(P_{\frac{3k+5}{2}}) \le \frac{k^2+k+2}{2}$. From Observation 2.1, we can say that for any odd k, $rc_k(P_n) \le \frac{k^2+k+2}{2}$, for $n \le \frac{3k+5}{2}$.

In other words $rc_k(P_n) \le \frac{k^2+k+2}{2}$, for $k \ge \frac{2n-5}{3}$. Thus $rc_k(P_n) \le \frac{k^2+k+2}{2}$, for $\frac{2n-5}{3} \le k \le n-3$.

Case II : In this case also we first prove that for any odd integer k and $n = 2k+4$, the coloring f defined as

$$\begin{aligned} f(a_{1+i}) &= 2 + ik, && for\ 0 \le i \le \frac{k+1}{2} \\ f(a_{\frac{k+5}{2}+j}) &= \frac{k+3}{2} + jk, && for\ 0 \le j \le \frac{k-1}{2} \\ f(a_{k+3+l}) &= lk + 1, && for\ 0 \le l \le \frac{k+1}{2} \\ f(a_{\frac{3k+9}{2}+m}) &= \frac{k+1}{2} + mk, && for\ 0 \le m \le \frac{k-1}{2} \end{aligned}$$

is a radio k-coloring of P_n.

(i) For $0 \le i_1 < i_2 \le \frac{k+1}{2}$, $|f(a_{1+i_1}) - f(a_{1+i_2})| = |2 + i_1k - 2 - i_2k| = |(i_1 - i_2)k| \ge k \ge 1 + k - (1 + i_2 - (1 + i_1)) = 1 + k - d(a_{1+i_1}, a_{1+i_2})$.

(ii) For $0 \le i \le \frac{k+1}{2}$ and $0 \le j \le \frac{k-1}{2}$, $\left|f(a_{1+i}) - f\left(a_{\frac{k+5}{2}+j}\right)\right| = \left|2 + ik - \left(\frac{k+3}{2} + jk\right)\right| = \left|(i-j)k - \frac{k-1}{2}\right| = \left|\frac{k-1}{2} + (i-j) - (i-j)(k+1)\right| \ge \frac{k-1}{2} + (i-j)$ by (ii) of case I. Now $\frac{k-1}{2} + (i-j) = 1 + k - \left(\frac{k+5}{2} + j - (1+i)\right) = 1 + k - d\left(a_{1+i}, a_{\frac{k+5}{2}+j}\right)$.

(iii) For $0 \le i \le \frac{k+1}{2}$ and $0 \le l \le \frac{k+1}{2}$, $|f(a_{1+i}) - f(a_{k+3+l})| = |2 + ik - (lk+1)| =$
$|1 + (i-l)k| = |(i-l) - 1 + (i-l)(k-1) + 2| \ge |(i-l) - 1| \ge (i-l) - 1 = 1 + k - (k + 3 + l - (1+l)) = 1 + k - d(a_{1+i}, a_{k+3+l})$.

(iv) For $0 \le i \le \frac{k+1}{2}$ and $0 \le m \le \frac{k-1}{2}$, $\left|f(a_{1+i}) - f\left(a_{\frac{3k+9}{2}+m}\right)\right| = \left|2 + ik - (\frac{k+1}{2} + mk)\right| = \left|(i-m) - \frac{k+5}{2} + (i-m)(k-1) + 4\right| \ge (i-m) - \frac{k+5}{2} = 1 + k - \left(\frac{3k+9}{2} + m - (1+i)\right)$.

(v) For $0 \le j_1 < j_2 \le \frac{k-1}{2}$, $\left|f\left(a_{\frac{k+5}{2}+j_1}\right) - f\left(a_{\frac{k+5}{2}+j_2}\right)\right| = \left|\frac{k+3}{2} + j_1 k - \left(\frac{k+3}{2} + j_2 k\right)\right| = |(j_1 - j_2)k| \ge k \ge 1 + k - \left(\frac{k+5}{2} + j_2 - \left(\frac{k+5}{2} + j_1\right)\right) = 1 + k - d\left(a_{\frac{k+5}{2}+j_1}, a_{\frac{k+5}{2}+j_2}\right)$.

(vi) For $0 \le j \le \frac{k-1}{2}$ and $0 \le l \le \frac{k+1}{2}$, $\left|f\left(a_{\frac{k+5}{2}+j}\right) - f(a_{k+3+l})\right| = \left|\frac{k+3}{2} + jk - (lk+1)\right| = \left|(j-l)k + \frac{k+1}{2}\right| = \left|\frac{k+1}{2} + (j-l) + (j-l)(k-1)\right| \ge (j-l) + \frac{k+1}{2} = 1 + k - \left(k + 3 + l - \left(\frac{k+5}{2} + j\right)\right) = 1 + k - d\left(a_{\frac{k+5}{2}+j}, a_{k+3+l}\right)$.

(vii) For $0 \le j \le \frac{k-1}{2}$ and $0 \le m \le \frac{k-1}{2}$, $\left|f\left(a_{\frac{k+5}{2}+j}\right) - f\left(a_{\frac{3k+9}{2}+m}\right)\right| = \left|\frac{k+3}{2} + jk - \left(\frac{k+1}{2} + mk\right)\right| = |(j-m)k + 1| = |(j-m) - 1 + (j-m)(k-1) + 2| \ge (j-m) - 1 = 1 + k - \left(\frac{3k+9}{2} + m - \left(\frac{k+5}{2} + j\right)\right) = 1 + k - d\left(a_{\frac{k+5}{2}+j}, a_{\frac{3k+9}{2}+m}\right)$.

(viii) For $0 \le l_1 < l_2 \le \frac{k+1}{2}$, $|f(a_{k+3+l_1}) - f(a_{k+3+l_2})| = |(l_1 k + 1) - (l_2 k + 1)| = |(l_1 - l_2)k| \ge k \ge 1 + k - (k + 3 + l_2 - (k + 3 + l_1))| = 1 + k - d(a_{k+3+l_1}, a_{k+3+l_2})$.

(ix) For $0 \le l \le \frac{k+1}{2}$ and $0 \le m \le \frac{k-1}{2}$, $\left|f(a_{k+3+l}) - f\left(a_{\frac{3k+9}{2}+m}\right)\right| = \left|(lk+1) - \left(\frac{k+1}{2} + mk\right)\right| = \left|(l-m)k - \frac{k-1}{2}\right| = \left|\frac{k-1}{2} - (l-m)k\right| = \left|\frac{k-1}{2} + (l-m) - (l-m)(k+1)\right| \ge \frac{k-1}{2} + (l-m) = 1 + k - \left(\frac{3k+9}{2} + m - (k+3+l)\right) = 1 + k - d\left(a_{k+3+l}, a_{\frac{3k+9}{2}+m}\right)$.

(x) For $0 \le m_1 < m_2 \le \frac{k-1}{2}$, $\left|f\left(a_{\frac{3k+9}{2}+m_1}\right) - f\left(a_{\frac{3k+9}{2}+m_2}\right)\right| = \left|\frac{k+1}{2} + m_1 k - \left(\frac{k+1}{2} + m_2 k\right)\right| = |(m_1 - m_2)k| \ge k \ge 1 + k - \left(\frac{3k+9}{2} + m_2 - \left(\frac{3k+9}{2} + m_1\right)\right)$.

Therefore f is a radio k-coloring of P_{2k+4} and thus $rc_k(P_{2k+4}) \le \frac{k^2+k+4}{2}$. By the same argument as in Case I, $rc_k(P_n) \le \frac{k^2+k+4}{2}$, for $k \ge \frac{n-4}{2}$. Hence Theorem 2.2 is proved. □

Example 2.3. *Here we illustrate Theorem 2.2 by giving example of radio k-colorings of some paths below.*
(i) *For $k = 5$ and $n = \frac{3k+5}{2} = 10$, the labeling below of Case I (Theorem 2.2) improves the upper bound from* 18 *(of Theorem 1.1) to* 16.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
4	*9*	*14*	*1*	*6*	*11*	*16*	*3*	*8*	*13*

(ii) *For* $k = 5$ *and* $n = 2k + 4 = 14$, *the labeling below of Case II (Theorem 2.2) improves the upper bound from* 18 *(of Theorem 1.1) to* 17.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
2	*7*	*12*	*17*	*4*	*9*	*14*	*1*	*6*	*11*	*16*	*3*	*8*	*13*

Next we use a result of Khennoufa and Togni [4], given below, and improve the lower bound of $rc_k(P_n)$, for k an odd integer and $3 \leq k \leq n-3$.

Theorem 2.4. [4] *For* $n = 2p + 1$, $p \geq 2$ *is an integer,* $rc_{n-2}(P_n) = 2p^2 - 2p + 3$.

Theorem 2.5. *For* $n \geq 5$ *and an odd positive integer* k *with* $3 \leq k \leq n - 3$, $rc_k(P_n) \geq \frac{k^2+5}{2}$.

Proof. Since $k \leq n - 3$, from Observation 2.1, we have $rc_k(P_{k+2}) \leq rc_k(P_n)$. From Theorem 2.4 it is easy to check that $rc_k(P_{k+2}) = \frac{k^2+5}{2}$, if k is odd. □

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