(G_m, H_m)-Multifactorization of λK_m

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Abstract

A (G, H)-multifactorization of λK_m is a partition of the edge set of λK_m into G-factors and H-factors with at least one G-factor and one H-factor. Atif Abueida and Theresa O'. Neil have conjectured that for any integer $n \geq 3$ and $m \geq n$, there is a (G_n, H_n)-multidecomposition of λK_m where $G_n = K_{1,n-1}$ and $H_n = C_n$. In this paper it is shown that the above conjecture is true for $m = n$ when

(i) $G_m = K_{1,m-1}$; $H_m = C_m$,

(ii) $G_m = H_{1,m-1}$; $H_m = P_m$ and

(iii) $G_m = P_m$; $H_m = C_m$.

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1 Introduction

Let G and H be subgraphs of K_m. A *G-factor* of K_m is a spanning subgraph of K_m such that each component of it is isomorphic to G. A path with end vertices u and v is denoted by $P(u, v)$. A *Hamilton cycle decomposition* of G is a partition of the edge set of G into Hamilton cycles. A (G, H)-*multifactorization* of λK_m is a partition of the edge set of λK_m into G-factors and H-factors with at least one G-factor and one H-factor. A (G, H)-*multidecomposition* of λK_m is a partition of the edge set of λK_m into copies of G and H with at least one copy of G and H. The study of (G, H)-multidecomposition was introduced by Abueida and Daven in [1] and [2]. Recently, Atif Abueida and Theresa O'. Neil [3] settled the problem of existence of (G_n, H_n)-multidecomposition of λK_m when $G_n = K_{1,n-1}$

and $H_n = C_n$ for $n = 3, 4, 5$ and $m \geq n$. They have also conjectured that for any integer $n \geq 3$ and $m \geq n$, there is a multidecomposition of λK_m into small cycles and claws. In this paper, necessary and sufficient conditions for the existence of (G_m, H_m)-multifactorization of λK_m are obtained when (G_m, H_m) belongs to one of the following.

(i) $(G_m, H_m) = (K_{1,m-1}, C_m)$,

(ii) $(G_m, H_m) = (K_{1,m-1}, P_m)$ and

(iii) $(G_m, H_m) = (P_m, C_m)$.

In fact our results prove the above conjecture affirmatively when $m = n$ not only for $(G_m, H_m) = (K_{1,m-1}, C_m)$, and also for $(G_m, H_m) = (K_{1,m-1}, P_m)$, and $(G_m, H_m) = (P_m, C_m)$.

To prove our results we require the following.

Result 1.1. (Walecki's Construction [4]) *The graph K_{2n+1} has a Hamilton cycle decomposition.*

Corollary 1.2. *The graph K_{2n} has a Hamilton path decomposition.*

Result 1.3. *For even $m \geq 2$, the graph $2K_{2m}$ has a C_{2m} decomposition.*

Proof. Let $V(K_{2m}) = \{v_0, v_1, v_2, \ldots, v_{2m-1}\}$ and $C = (v_0 v_1 v_2 v_{2m-1} v_3 v_{2m-2}, \ldots, v_{m-1} v_{m+2} v_m v_{m+1} v_0)$ be a Hamilton cycle in K_{2m}. Let the permutation σ be $\sigma = (0)(1, 2, 3, \ldots, 2m-1)$. Then $\{C, \sigma(C), \sigma^2(C), \sigma^3(C), \ldots, \sigma^{2m-2}(C)\}$ is a C_{2m} factorization of $2K_{2m}$. □

Corollary 1.4. *The graph $2K_{2m-1}$ has a Hamilton path decomposition.*

Result 1.5. *The graph $2K_m$ has a $K_{1,m-1}$ factorization.*

Proof. Let $V(K_m) = \{1, 2, \ldots, m\}$. We construct a star factorization of $2K_m$ as follows. Let $(i : i + 1, i + 2, \ldots, i + m - 1)$, $1 \leq i \leq m$, denote a star $K_{1,m-1}$ of K_m with center at i, where the additions are taken modulo m with residues $1, 2, \ldots, m$. When i varies we get a $K_{1,m-1}$ factorization of $2K_m$. □

2 Multifactorization of λK_m into Stars and Cycles

Let $(G_m, H_m) = (K_{1,m-1}, C_m)$.

Lemma 2.1. *There exists a (G_m, H_m)-multifactorization of λK_m for all $\lambda \geq 3$ and odd $m \geq 3$.*

Proof. Let $m = 2n+1$. We can write $\lambda K_{2n+1} = (\lambda-2)K_{2n+1} + 2K_{2n+1}$. By Result 1.1, K_{2n+1} has a C_{2n+1} factorization. Further, $2K_{2n+1}$ has a $K_{1,2n}$ factorization by Result 1.5. Thus λK_m has a (G_m, H_m)-multifactorization for all $\lambda \geq 3$ and odd $m \geq 3$. □

Lemma 2.2. *There exists a (G_{2n}, H_{2n})-multifactorization in $2\lambda K_{2n}$ for all $\lambda > 1$ and $n \geq 2$.*

Proof. We can write $2\lambda K_{2n} = (\lambda-1)2K_{2n} + 2K_{2n}$. Also, $2K_{2n}$ has a C_{2n} factorization by Result 1.3 and $(\lambda-1)2K_{2n}$ has a $K_{1,2n-1}$ factorization by Result 1.5. Thus there exists a (G_{2n}, H_{2n})-multifactorization in $2\lambda K_{2n}$ for all $\lambda > 1$ and $n \geq 2$. □

Theorem 2.3. *For $m \geq 3$, λK_m has a (G_m, H_m)-multifactorization if and only if*

(i) $|E(\lambda K_m)| = (m-1)r + ms$, *where* $r, s \geq 1$,

(ii) $\lambda \geq 3$ *and*

(iii) $r \equiv 0 (mod\ m)$.

Proof. Let λK_m has a (G_m, H_m)-multifactorization. Then there exist integers $r, s \geq 1$ such that $\lambda K_m = rK_{1,m-1} \oplus sC_m$. Hence $|E(\lambda K_m)| = (m-1)r + ms$. If $\lambda = 1, 2$ then $|E(\lambda K_m)| = (m-1)r + ms$ by (i). But no r and s satisfy the above equation. Hence $\lambda \geq 3$.

Now, to prove (iii), in (i) sm is the number of edges in s copies of C_m and it forms a spanning subgraph in λK_m. Therefore, r copies of $K_{1,m-1}$ should also form a regular subgraph in λK_m, which is possible only if each vertex is a center vertex for the same number of copies of $K_{1,m-1}$. Hence $r \equiv 0(mod\ m)$.

Conversely, suppose (i), (ii) and (iii) are satisfied.

Case (a) m is odd and $\lambda \geq 3$.

By Lemma 2.1, we have a (G_m, H_m)-multifactorization.

Case (b) m is even.

Suppose $\lambda \geq 3$ is odd. Then by (i), $|E(\lambda K_m)| = (m-1)r + ms$. i.e., $\lambda m(m-1)/2 = r(m-1) + sm$. By (iii) RHS of the above equation is congruent to $0(mod\ m)$ but not the LHS. Hence λ cannot be odd. Therefore, by Lemma 2.2, there exists a (G_m, H_m)-multifactorization of λK_m for all even $\lambda > 3$. □

3 Multifactorization of λK_m into Stars and Paths

Let $(G_m, H_m) = (K_{1,m-1}, P_m)$.

Lemma 3.1. *There exists a (G_{2n}, H_{2n})-multifactorization of λK_{2n} for $\lambda \geq 2$.*

Proof. **Case a.** $\lambda = 2$.

We have $|E(2K_{2n})| = 2n(2n-1) = 2(2n-1) + (2n-2)(2n-1)$. Now we factorize $2K_{2n}$ into 2 stars and $2n-2$ paths as follows. Let $V(K_{2n}) = \{\alpha, \beta, 1, 2, \ldots, 2n-2\}$. By removing α, β from $2K_{2n}$, we get $2K_{2n-2}$. By Corollary 1.2, $2K_{2n-2}$, has a Hamilton path decomposition $\{P(i, n-1+i), 1 \leq i \leq 2n-2\}$ where the additions are taken modulo $2n-2$. We obtain Hamilton paths of $2K_{2n}$ by joining the end vertices of the Hamilton paths of $2K_{2n-2}$ with vertices α, β as $\alpha P(i, n-1+i)\beta, i = 1, 2, \ldots, 2n-2$. The remaining edges of $2K_{2n}$ incident with α and β form two stars centered at α, β. Thus there exists a (G_{2n}, H_{2n})-factorization of λK_{2n} for $\lambda = 2$.

Case b. $\lambda > 2$.

We have $\lambda K_{2n} = 2K_{2n} + (\lambda - 2)K_{2n}$. The result follows from Case (a) and Corollary 1.2. □

Lemma 3.2. *There exists a (G_{2n+1}, H_{2n+1})-multifactorization of $2\lambda K_{2n+1}$ for $\lambda \geq 1$.*

Proof. When $\lambda = 1$, we know that $2K_{2n+1}$ has a Hamilton cycle decomposition. Label the vertices of the graph as $\alpha, 1, 2, \ldots, 2n$. When we remove the vertex α from $2K_{2n+1}$, by Corollary 1.2, the resulting graph $2K_{2n}$, has a Hamilton path decomposition $\{P(i, n+i), 1 \leq i \leq 2n,\}$ where the additions are taken modulo $2n$. We obtain Hamilton paths of $2K_{2n+1}$ by joining α to an end vertex of the path $P(i, n+i)$. The remaining edges of $2K_{2n+1}$ incident with α form a star $K_{1,2n}$. Thus we have a (G_{2n+1}, H_{2n+1})-multifactorization of $2K_{2n+1}$ and hence the result follows. □

Theorem 3.3. *λK_m has a (G_m, H_m)-multifactorization if and only if*

(i) $|E(\lambda K_m)| = (m-1)r + (m-1)s$, *where* $r, s \geq 1$,

(ii) $\lambda \geq 2$ *when* m *is even and*

(iii) λ *is even when* m *is odd.*

Proof. Let λK_m has a (G_m, H_m)- multifactorization. Then there exist integers $r, s \geq 1$ such that $\lambda K_m = rP_m \oplus sC_m$. Hence $|\, E(\lambda K_m) \,| = (m-1)r + (m-1)s$. To prove (ii) let $m = 2n$. If $\lambda < 2$, then $K_{2n} - K_{1,2n-1}$ has no P_{2n}, a contradiction to the hypothesis. Thus $\lambda \geq 2$. To prove (iii) let $m = 2n+1$. Then by (i), $|\, E(\lambda K_{2n+1}) \,| = r(2n) + s(2n)$. i.e., $\lambda(2n+1)(2n)/2 = 2n(r+s)$. The RHS of the above equation is congruent to $0 (mod\ 2n)$, but not the LHS when λ is odd. Thus λ is even.

The converse follows from Lemmas 3.1 and 3.2. □

4 Multifactorization of λK_m into Paths and Cycles

Let $(G_m, H_m) = (P_m, C_m)$.

Lemma 4.1. *If G is a regular incomplete graph of order m, then G does not admit a P_m-decomposition.*

Proof. Suppose G has a P_m decomposition. Then every path in the P_m decomposition of G exhaust one degree at its end vertices and two degree at its middle vertices. As G is regular, every vertex of G must be an end vertex for equal number of paths in the path decomposition of G. Hence, there exist at least $\frac{m}{2}$ paths in the path decomposition of G. As each path is of length $m-1$, G must have at least $\frac{m(m-1)}{2}$ edges, which is a contradiction to the hypothesis that $| E(G) | < \frac{m(m-1)}{2}$. □

Theorem 4.2. *For $m \geq 3$, λK_m has (G_m, H_m)-multifactorization if and only if*

(i) $| E(\lambda K_m) | = (m-1)r + ms$, *where* $r, s \geq 1$ *and*

(ii) $\lambda > 2$.

Proof. Assume that λK_m has a (G_m, H_m)-multifactorization. Then there exist integers $r, s \geq 1$ such that $\lambda K_m = rP_m \oplus sC_m$. Hence $| E(\lambda K_m) | = (m-1)r + ms$. Also if $\lambda = 1$, by hypothesis $K_m = rP_m \oplus sC_m$, $r, s \geq 1$. Thus $K_m - sC_m = rP_m$. But $K_m - sC_m$ is a regular incomplete graph and is not decomposable into P_m, by Lemma 4.1. This contradicts that $K_m - sC_m = rP_m$. Thus $\lambda \neq 1$. If $\lambda = 2$, $| E(2K_m) | = m(m-1)$. By (i) $m(m-1) = (m-1)r + ms$. But no $r, s \geq 1$ satisfy the above equation. Thus $\lambda > 2$.

Conversely, Suppose $\lambda > 2$. Then $\lambda K_m = (\lambda - 2)K_m + 2K_m$. If m is even, $(\lambda - 2)K_m$ has a P_m decomposition by Corollary 1.2 and $2K_m$ has a C_m decomposition by Result 1.3. If m is odd, $(\lambda - 2)K_m$ has a C_m decomposition by Result 1.1 and $2K_m$ has a P_m decomposition by Corollary 1.4. □

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References

[1] A. Abueida and M. Daven, Multidesigns for graph-pairs of order 4 and 5, *Graphs Combinatorics,* **19**(4)(2003), 433-447.

[2] A. Abueida and M. Daven, Multidecomposition of the complete graph, *Ars Combin.,* **72**(2004).

[3] A. Abueida, Theresa O'. Neil, Multidecomposition of λK_m into small cycles and claws, *Bulletin of ICA.,* **49**(2007), 32-40.

[4] B. Alspach and J.C. Bermond D. Sotteau, *Decomposition into cycles I: Hamilton Decompositions, in cycles and rays,* (Eds. G. Hahn et al.) Kluwer Academic publishers (1990), 9-18.