Fuzzy Finite State Automaton with Unique Membership Transition on an Input Symbol

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Abstract

In this paper fuzzy finite state automaton with unique membership transition on an input symbol (uffsa) is defined. It is proved and illustrated that for a given fuzzy finite state automaton (ffsa), there exists an equivalent uffsa. Some closure properties of fuzzy regular languages are studied.

Keywords. Fuzzy finite state automaton, fuzzy regular languages.

1 Introduction

For basic terminology in automata and fuzzysets we refer to the books[2,3] and [4] respectively. There are many ways in which the ordinary automata and languages have been fuzzyfied [1, 5–11]. There is no universally standardized terminology of defining fuzzy automata in actual practice. However, fuzzy automata have the common property that they have membership values, which are in [0, 1], associated with the state transitions as well as initial and final state distributions.

In a fuzzy finite state automaton(ffsa), there may be more than one fuzzy transitions from a state on an input symbol with a given membership value [6,9–11]. This development was followed by the postulation called deterministic fuzzy finite state automaton (dffsa) [6], 'that is, $\forall p \in Q$, $a \in \Sigma$, $\mu(p, a, q) > 0$ and $\mu(p, a, q') > 0$ for some q, $q' \in Q$, then $q = q'$. However, it only acts as a deterministic fuzzy recognizer, that is, for any fuzzy recognizer M there is a deterministic fuzzy recognizer M_1 with the same behaviour in the sense $L(M) = L(M_1)$. Therefore, for any string $x \in \Sigma^*$, $\deg(x)$ in M and that in M_1 need not be the same. In this paper, we introduce an ffsa by incorporating a condition that the membership function has a unique transition on an input symbol, that is, $\forall p \in Q$, $a \in \Sigma$, $\mu(p, a, q) = \mu(p, a, q')$ for some q, $q' \in Q$, then $q = q'$. We denote such a fuzzy finite state automaton by uffsa, where u refers to the unique transition. The usual ffsa can have more than one transition with a membership value on an input, so uffsa is much simpler than ffsa.

In section 2, we recall the basic definitions. In Section 3, we formally define the uffsa and prove that for any ffsa M there exists an equivalent uffsa M_1. Therefore for any $x \in \Sigma^*$, deg(x) in M and deg(x) in M_1 will be the same, which is illustrated with an example. Section 4 includes the results that if L_1 and L_2 are fuzzy regular languages accepted by uffsa's then so are $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 L_2$ and L_1^*.

2 Preliminaries

We recall some basic definitions on fuzzy languages and fuzzy automata, many of them can be found in [1, 6, 10].

Definition 2.1. *Let Σ be a finite alphabet and L the fuzzy subset of Σ^*, i.e., $L : \Sigma^* \to [0, 1]$ is called the fuzzy language over Σ. For $x \in \Sigma^*$, $L(x)$ is the membership value (degree) of x.*

Definition 2.2. *Let L_1 and L_2 be two fuzzy languages over Σ. Then the basic operations on L_1 and L_2 are defined below :*

(i) The union of L_1 and L_2 is a fuzzy language denoted by $L_1 \cup L_2$ and $L = L_1 \cup L_2$ is defined by $L(x) = \vee\{L_1(x), L_2(x)\} \forall\, x \in \Sigma^*$.

(ii) The intersection of L_1 and L_2 is a fuzzy language denoted by $L_1 \cap L_2$ and $L = L_1 \cap L_2$ is defined by $L(x) = \wedge\{L_1(x), L_2(x)\} \forall x \in \Sigma^*$.

(iii) The concatenation of L_1 and L_2 is a fuzzy language denoted by $L_1 L_2$ and $L = L_1 L_2$ is defined by $L(x) = \vee\{L_1(u) \wedge L_2(v) \mid x = uv, u, v \in \Sigma^*\}$

(iv) The Kleene's closure of L_1 is a fuzzy language denoted by L_1^* and $L = L_1^*$ is defined by

$$L(x) = \vee\{L_1(x_1) \wedge L_1(x_2) \wedge \ldots \wedge L_1(x_n) \mid x = x_1 x_2 \cdots x_n, \\ x_1, x_2, \ldots, x_n \in \Sigma^*, n \geq 0\} \forall x \in \Sigma^*$$

In other words, the fuzzy subset L_1^* of Σ^* defined by $L_1^*(x) = \vee\{L_1^n(x) \mid n = 0, 1, \ldots\} \forall x \in \Sigma^*$.

Since fuzzy languages are just a special class of fuzzy sets, the equivalence and inclusion relations between two fuzzy languages are the equivalence and inclusion relations between two fuzzy sets. Let L_1 and L_2 be two fuzzy languages over Σ. Then,

$$L_1 = L_2 \quad \text{if and only if} \quad L_1(x) = L_2(x) \forall x \in \Sigma^* \quad \text{and}$$
$$L_1 \subseteq L_2 \quad \text{if and only if} \quad L_1(x) \leq L_2(x) \forall x \in \Sigma^*$$

Definition 2.3. *A fuzzy finite state machine (ffsm) is a triple $M = (Q, \Sigma, \mu)$ where Q and Σ are finite non-empty sets and μ is a fuzzy subset of $Q \times \Sigma \times Q$ i.e., $\mu : Q \times \Sigma \times Q \to [0,1]$.*

This structure includes only the fuzzy transition function and does not have the fuzzy initial states and fuzzy final states.

Definition 2.4. *A fuzzy finite state automaton (ffsa) is a quintuple $M = (Q, \Sigma, \mu, i, f)$ where*

(i) Q is a finite non-empty set of states.

(ii) Σ is a finite non-empty set of input symbols.

(iii) the fuzzy subset $\mu : Q \times \Sigma \times Q \to [0,1]$ is a function, called the fuzzy transition function,

(iv) i is a fuzzy subset of Q, i.e., $i : Q \to [0,1]$ called the fuzzy subset of initial states, and

(v) f is a fuzzy subset of Q, i.e., $f : Q \to [0,1]$ called the fuzzy subset of final states.

Clearly, if $M = (Q, \Sigma, \mu, i, f)$ is an ffsa, then $N = (Q, \Sigma, \mu)$ is a fuzzy finite state machine (ffsm). We call N, the fuzzy finite state machine associated with the ffsa.

Definition 2.5. *Let $M = (Q, \Sigma, \mu, i, f)$ be an* ffsa, *the extended fuzzy transition function for M is the fuzzy subset $\mu^* : Q \times \Sigma^* \times Q \to [0,1]$ has been defined as follows:*
for all $p, q \in Q$, $a \in \Sigma$, $x \in \Sigma^*$,

$$\mu{*}p\lambda q = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q. \end{cases}$$

$$\mu{*}pxaq = \vee\{\mu{*}pxr \wedge \mu raq \mid r \in Q\}$$

Definition 2.6. *Let $M = (Q, \Sigma, \mu, i, f)$ be an ffsa. Let $x \in \Sigma^*$. Then x is said to be recognized by M if* $\deg_M(x) = \vee\Big\{\{i(p) \wedge \mu{*}pxq \wedge f(q) \mid q \in Q\} \mid p \in Q\Big\} > 0$.

Definition 2.7. *Let* $M = \big(Q, \Sigma, \mu, i, f\big)$ *be an ffsa. Let* $L(M) = \big\{x \in \Sigma^* \mid \deg_M(x) > 0\big\}$. $L(M)$ *is called the language recognized by the ffsa* M.

Definition 2.8. *The ffsa* M *defined in Definition 2.4 is also referred to fuzzy finite state recognizer and* $A \subset \Sigma^*$ *is said to be recognized if* $L(M) = A$.

Definition 2.9. *A deterministic fuzzy finite state automaton is an ordered five tuple* $M = \big(Q, \Sigma, \mu, i, f\big)$ *such that*

(i) there exists a unique $s_0 \in Q$ *such that* $i(s_0) > 0$*;* s_0 *is called the initial state,*

(ii) μ *is a fuzzy function of* $Q \times \Sigma$ *into* Q *i.e.,* $\mu : Q \times \Sigma \times Q \to [0,1]$ *and if* $\forall\ q \in Q,\ a \in \Sigma,\ \mu qap > 0$ *and* $\mu qap' > 0$ *for some* $p,\ p' \in Q$*, then* $p = p'$.

(iii) $\forall\ x \in \Sigma^*$*, there exists a unique* $q_x \in Q$ *such that* $\mu{*}s_0xq_x > 0$.

Let $F = \big\{q \in Q \mid f(q) > 0\big\}$. F *is called the set of final states of* M.

Definition 2.10. *The fuzzy language accepted by an ffsa* $M = \big(Q,\ \Sigma,\ \mu,\ i,\ f\big)$ *is a fuzzy subset of* Σ^* *and is denoted by* L_M, $L_M : \Sigma^* \to [0,1]$ *is defined by*

$$L_M(x) = \vee\Big\{\big\{i(p) \wedge \mu{*}pxq \wedge f(q) \mid q \in Q\big\} \mid p \in Q\Big\}$$

Definition 2.11. *A fuzzy subset* $L : \Sigma^* \to [0,1]$ *is called a fuzzy language over* Σ*. A fuzzy language* L *over* Σ *is called a fuzzy regular language if there exists an ffsa* M *such that the fuzzy language accepted by it is the same as* L*, i.e.,* $L_M = L$.

3 Definition and Example

Definition 3.1. *A fuzzy finite state automaton with unique membership transition on an input symbol is denoted by uffsa and is defined by* $M = (Q, \Sigma, \mu, i, f)$*, where*

(i) Q *is a finite non-empty set of states*

(ii) Σ *is a finite non-empty set of input symbols.*

(iii) the fuzzy subset $\mu : Q \times \Sigma \times Q \to [0,1]$ *is a fuzzy function of* $Q \times \Sigma \times [0,1]$ *into* Q *i.e., if* $\forall p \in Q, a \in \Sigma, m \in [0,1]$, $\mu(p,a,q) = \mu(p,a,q')$ *for some* $q, q' \in Q$ *then* $q = q'$.

(iv) i is a fuzzy subset of Q, i.e., $i : Q \to [0,1]$, called the fuzzy subset of initial states.

(v) f is a fuzzy subset of Q, i.e., $f : Q \to [0,1]$ called the fuzzy subset of final states.

Theorem 3.2. *Let $M = (Q, \Sigma, \mu, i, f)$ be an* uffsa. *Then $\mu{*}pxyq = \vee\{\mu{*}pxr \wedge \mu{*}ryq \mid r \in Q\}\ \forall\ p, q \in Q$ and $\forall x, y \in \Sigma^*$.*

The result can easily be proved by induction.

Theorem 3.3. *Let $M = (Q, \Sigma, \mu, i, f)$ be an ffsa and L_M is the fuzzy language accepted by M. Then there exists an uffsa $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ such that $L_{M_1} = L_M$.*

Proof. Let $M = (Q, \Sigma, \mu, i, f)$ be the ffsa and L_M be the fuzzy language accepted by M. Let $Q_1 = P(Q)$, the set of all subsets of Q, every state in Q_1 is of the form $\{p_1, p_2, \ldots, p_r\}$, $r \geq 1$, $p_i \in Q$, $i = 1, 2, \ldots, n$.

Define $\mu_1 : Q_1 \times \Sigma \times Q_1 \to [0,1]$ by

$$\mu_1(S, a, S') = \begin{cases} m, & \text{if} \quad S' = \{q \in Q \mid \mu paq = m, p \in S\} \\ 0, & \text{if} \quad S^1 = \phi \end{cases}$$

Define $i_1 : Q_1 \to [0,1]$ by

$$i_1(S) = \begin{cases} i(p), & \text{if} \quad S = \{p\}, p \in Q, i(p) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Define $f_1 : Q_1 \to [0,1]$ such that

$$f_1(S) = \begin{cases} \vee\{f(p) \mid p \in S\}, & \text{if } S \neq \phi \\ 0, & \text{if } S = \phi \end{cases}$$

Reduce Q_1 by removing all the states in Q_1 which are not reachable from any state of the form $\{p\}, p \in Q$. Construct $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$. Let $S, S_1, S_2 \in Q_1$, $\mu_1 SaS_1 = m$ and $\mu_1 SaS_2 = m$. By the definition of μ_1, $\mu_1 SaS_1 = m$, which implies that $S_1 = \{q \in Q \mid \mu paq = m, p \in S\}$.

Similarly, $\mu_1 SaS_2 = m$ and so $S_2 = \{q \in Q \mid \mu paq = m, p \in S\}$. Therefore $S_1 = S_2$. Hence μ_1 is a fuzzy function from $Q_1 \times \Sigma \times [0,1]$ into Q_1. Therefore M_1 is an uffsa. Now, we can easily prove that, for $p, p' \in Q$, $\mu{*}pxp' = \mu_1{*}\{p\}xS, p' \in S$.

Let L_{M_1} be the fuzzy language accepted by M_1. Now for $x \in \Sigma^*$

$$\begin{aligned} L_M(x) &= \vee\Big\{\{i(p) \wedge \mu{*}pxq \wedge f(q) \mid q \in Q\} \mid p \in Q\Big\} \\ &= \vee\Big\{i_1(\{p\}) \wedge \mu_1{*}\{p\}xS \wedge \big(\vee f(q) \mid q \in S\big) \mid p \in Q\Big\} \\ &= \vee\Big\{\{i_1(\{p\}) \wedge \mu_1{*}\{p\}xS \wedge f_1(S) \mid S \in Q_1\} \mid \{p\} \in Q_1\Big\} \\ &= L_{M_1}(x). \end{aligned}$$

Therefore $L_{M_1} = L_M$. Hence the theorem. □

Example 3.4. *Consider the ffsa* $M = (Q, \Sigma, \mu, i, f)$, *where* $Q = \{q_1, q_2, q_3, q_4, q_5\}$, $\Sigma = \{a, b\}, \mu : Q \times \Sigma \times Q \to [0, 1]$ *is shown in the following transition diagram:*
$i : Q \to [0, 1]$ *such that* $i(q_1) = 0.7$, $i(q_2) = 0.1$, $i(q_4) = 0.8$ *and*
$f : Q \to [0, 1]$ *such that* $f(q_3) = 0.2$, $f(q_5) = 0.4$.

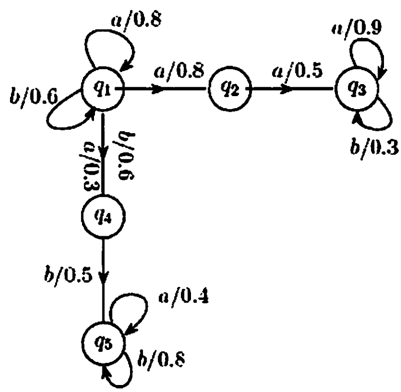

Figure 1

The language then accepted by M *is* $L_M : \Sigma^* \to [0, 1]$, *where*

$$L_M(x) = \begin{cases} 0.2, & \text{if } x \in \{a,b\}^*aa\{a,b\}^* \\ 0.1, & \text{if } x \in a\{a,b\}^* \\ 0.4, & \text{if } x \in \{a,b\}^*bb\{a,b\}^* \\ 0.3, & \text{if } x \in \{a,b\}^*ab\{a,b\}^* \\ 0.4, & \text{if } x \in b\{a,b\}^* \\ 0, & \text{otherwise} \end{cases}$$

By Theorem 3.3 we obtain the following uffsa $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ *where* $Q_1 = \{\{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}, \{q_1, q_2\}, \{q_1, q_4\}\}$ $\mu_1 : Q_1 \times \Sigma \times Q_1 \to$

$[0, 1]$ *is shown in the following fuzzy transition diagram:*
$i_1 : Q_1 \to [0, 1]$ *by* $i_1(\{q_1\}) = 0.7$, $i_1(\{q_2\}) = 0.1$, $i_1(\{q_4\}) = 0.8$ *and*
$f_1 : Q_1 \to [0, 1]$ *by* $f_1(\{q_3\}) = 0.2$, $f_1(\{q_5\}) = 0.4$.

The fuzzy language accepted by M_1 is the same as L_M.

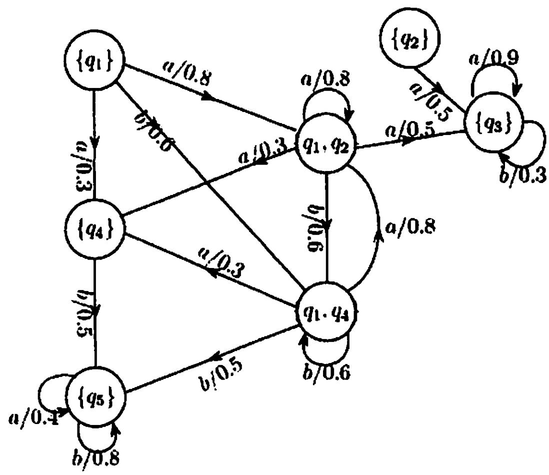

Figure 2

4 Closure Properties of Fuzzy Regular Languages

Now we consider some of the closure properties of fuzzy regular languages, such as, union, intersection, concatenation (product) and Kleene's closure. For union and intersection the system required can be defined in the classical way and the related theorems can be proved accordingly [6, 10]. We design new systems for concatenation and Kleene's closure and the theorems are proved in this section.

Definition 4.1. *Let* $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ *and* $M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2)$ *be two uffsa's,* $Q_1 \cap Q_2 = \phi$. *The union of* M_1 *and* M_2 *is the uffsa* $M_1 \cup M_2 = (Q, \Sigma, \mu, i, f)$, *where* $Q = Q_1 \cup Q_2$, $\mu : Q \times \Sigma \times Q \to [0, 1]$ *is defined as follows:*
$\forall p, q \in Q$,

$$\mu paq = \begin{cases} \mu_1 paq, & if \quad p, q \in Q_1 \\ \mu_2 paq, & if \quad p, q \in Q_2 \\ 0, & otherwise \end{cases}$$

$i : Q \to [0,1]$ *is defined by* $i(p) = \begin{cases} i_1(p), & \text{if } p \in Q_1 \\ i_2(p), & \text{if } p \in Q_2 \end{cases}$ *and*

$f : Q \to [0,1]$ *is defined by* $f(p) = \begin{cases} f_1(p), & \text{if } p \in Q_1 \\ f_2(p), & \text{if } p \in Q_2 \end{cases}$

Theorem 4.2. *Let $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2)$ be two uffsa's with the fuzzy regular languages L_1 and L_2 respectively. Then L is a fuzzy regular language accepted by $M = M_1 \cup M_2$, where $L = L_1 \cup L_2$.*

Definition 4.3. *Let $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2)$ be uffsa's, $Q_1 \cap Q_2 = \phi$. Then the ffsa $M_1 \cap M_2 = (Q_1 \times Q_2, \Sigma, \mu_1 \wedge \mu_2, i_1 \wedge i_2, f_1 \wedge f_2)$, where $\mu_1 \wedge \mu_2 : (Q_1 \times Q_2) \times \Sigma \times (Q_1 \times Q_2) \to [0,1]$ is defined by $(\mu_1 \wedge \mu_2)\big((p_1, p_2), a, (q_1, q_2)\big) = \mu_1 p_1 a q_1 \wedge \mu_2 p_2 a q_2$. $i_1 \wedge i_2 : Q_1 \times Q_2 \to [0,1]$ is defined by $(i_1 \wedge i_2)(p_1, p_2) = i_1(p_1) \wedge i_2(p_2)$. $f_1 \wedge f_2 : Q_1 \times Q_2 \to [0,1]$ is defined by $(f_1 \wedge f_2)(p_1, p_2) = f_1(p_1) \wedge f_2(p_2)$.*

Theorem 4.4. *Let $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2)$ be uffsa's with L_1 and L_2 as the fuzzy regular languages respectively. Then L is the fuzzy regular language accepted by an uffsa M such that $L = L_1 \cap L_2$.*

From the Definition 4.3, we need not have $M_1 \cap M_2$ be uffsa, but by using Theorem 3.3 we can obtain the uffsa M.

Theorem 4.5. *Let A and B be recognizable sets over Σ^* with fuzzy regular languages L_1 and L_2, which are accepted by uffsa's M_1 and M_2 respectively. Then the set AB is recognizable with fuzzy regular language L, accepted by an uffsa M such that $L = L_1 L_2$.*

Proof. Let $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2)$, $Q_1 \cap Q_2 = \phi$, with the fuzzy regular languages be L_1 and L_2 respectively.

$$\begin{aligned} i.e., \quad & L_1 : \Sigma^* \to [0,1] \text{ such that } && L_1(x) > 0 \, \forall x \in A. \\ & L_2 : \Sigma^* \to [0,1] \text{ such that } && L_2(x) > 0 \, \forall x \in B. \end{aligned}$$

Let $Q = Q_1 \cup Q_2$. Define ffsa $M_3 = (Q, \Sigma, \mu, i, f)$, where $\mu : Q \times \Sigma \times Q \to [0,1]$ is defined as follows:

(i) $\forall p, q \in Q_1$, $\mu paq = \mu_1 paq$

(ii) $\forall p, q \in Q_2$, $\mu paq = \mu_2 paq$

(iii) $\forall p \in Q_1$, $q \in Q_2$ and if $f_1(p) > 0$, $r \in Q_2$ with $i_2(r) > 0$ and $\mu_2 raq > 0$ then

$$\mu paq = \begin{cases} f_1(p) \wedge i_2(r) \wedge \mu_2 raq \\ 0, \quad \text{otherwise} \end{cases}$$

$i : Q \to [0,1]$ is defined by $i(p) = \begin{cases} i_1(p), & \text{if} \quad p \in Q_1 \\ 0, & \text{otherwise} \end{cases}$

$f : Q \to [0,1]$ is defined by $f(p) = \begin{cases} f_2(p), & \text{if} \quad p \in Q_2 \\ 0, & \text{otherwise} \end{cases}$

For $L_3(w) = 0$ the result is obvious.

Let $L_3(w) > 0$. $L_3(w) = \vee\{\{i(p) \wedge \mu{*}pwq \wedge f(q) \mid q \in Q\} \mid p \in Q\}$ Since Q is finite and w is of finite length, from definition of M, there exists $x = a_1a_2\cdots a_n \in A$, $y = b_1b_2\cdots b_m \in B$, $w = xy$ and $p_0, p_1, p_2, \ldots, p_n, q_1, \ldots, q_m \in Q$ such that $L_3(w) = i(p_0) \wedge \mu p_0 a_1 p_1 \wedge \mu p_1 a_2 p_2 \wedge \ldots \wedge \mu p_{n-1} a_n p_n \wedge \mu p_n b_1 q_1 \wedge \mu q_1 b_2 q_2 \wedge \ldots \wedge \mu p_{m-1} b_m q_m \wedge f(q_m)$ Each term is non-negative, therefore there exists $q_0 \in Q_2$ such that $\mu p_n b_1 q_1 = f_1(p_n) \wedge i_2(q_0) \wedge \mu_2 q_0 b_1 q_1$. Therefore,

$$\begin{aligned} L_3(w) &= i(p_0) \wedge \mu p_0 a_1 p_1 \wedge \mu p_1 a_2 p_2 \wedge \cdots \wedge \mu p_{n-1} a_n p_n \\ &\quad \wedge f_1(p_n) \wedge i_2(q_0) \wedge \mu_2 q_0 b_1 q_1 \wedge \mu q_1 b_2 q_2 \wedge \cdots \\ &\qquad \wedge \mu q_{m-1} b_m q_m \wedge f(q_m) \\ &= \big(i_1(p_0) \wedge \mu_1 p_0 a_1 p_1 \wedge \mu_1 p_1 a_2 p_2 \wedge \cdots \\ &\qquad \wedge \mu_1 p_{n-1} a_n p_n \wedge f_1(p_n)\big) \\ &\quad \wedge \big(i_2(q_0) \wedge \mu_2 q_0 b_1 q_1 \wedge \mu_2 q_1 b_2 q_2 \wedge \cdots \\ &\qquad \wedge \mu_2 q_{m-1} b_m q_m \wedge f_2(q_m)\big) \end{aligned}$$

$$\begin{aligned} &\leq \Big(\vee \{i_1(p_0) \wedge \mu_1 p_0 a_1 p_1 \wedge \mu_1 p_1 a_2 p_2 \wedge \cdots \\ &\qquad \wedge \mu_1 p_{n-1} a_n p_n \wedge f_1(p_n) \mid p_0, p_1, \ldots p_n \in Q_1\} \\ &\quad \wedge \Big(\vee \{i_2(q_0) \wedge \mu_2 q_0 b_1 q_1 \wedge \mu_2 q_1 b_2 q_2 \wedge \cdots \\ &\qquad \wedge \mu_2 q_{m-1} b_m q_m \wedge f_2(q_m) \mid q_0, q_1, \ldots, q_m \in Q_2\}\Big) \\ &= L_1(x) \wedge L_2(y) \end{aligned}$$

Therefore, $L_3(w) \leq L_1(x) \wedge L_2(y)$. This is true for any w such that $w = xy$. Thus

$$L_3(w) \leq \vee\{L_1(x) \wedge L_2(y) \mid w = xy,\ x \in A,\ y \in B\} \tag{1}$$

Let $x \in A$, and $y \in B$. Therefore $L_1(x) > 0$ and $L_2(y) > 0$.

$$L_1(x) = \vee\{i_1(p_0) \wedge \mu_1 p_0 a_1 p_1 \wedge \mu_1 p_1 a_2 p_2 \wedge \ldots \wedge \mu_1 p_{n-1} a_n p_n \wedge f_1(p_n) \mid p_0, p_1, \ldots, p_n \in Q_1\}$$

Since Q_1 is finite, there exists $p_0, p_1, \ldots, p_n \in Q_1$ such that

$$L_1(x) = i_1(p_0) \wedge \mu_1 p_0 a_1 p_1 \wedge \mu_1 p_1 a_2 p_2 \wedge \ldots \wedge \mu_1 p_{n-1} a_n p_n \wedge f_1(p_n), \text{ where } i_1(p_0) > 0, f_1(p_n) > 0, x = a_1 a_2 \cdots a_n$$

Similarly

$$L_2(y) = \vee\{i_2(q_0) \wedge \mu_2 q_0 b_1 q_1 \wedge \mu_2 q_1 b_2 q_2 \wedge \ldots \wedge \mu_2 q_{m-1} b_m q_m \wedge f_2(q_m) \mid q_0, q_1, \ldots, q_m \in Q_2\}$$

Since Q_2 is finite, there exists $q_0, q_1, \ldots, q_m \in Q_2$ such that

$$L_2(y) = i_2(q_0) \wedge \mu_2 q_0 b_1 q_1 \wedge \mu_2 q_1 b_2 q_2 \wedge \ldots \wedge \mu_2 q_{m-1} b_m q_m \wedge f_2(q_m), \text{ where } i_2(q_0) > 0, f_2(q_m) > 0, \; y = b_1 b_2 \cdots b_m.$$

$$L_1(x) \wedge L_2(y) = i_1(p_0) \wedge \mu_1 p_0 a_1 p_1 \wedge \mu_1 p_1 a_2 p_2 \wedge \cdots \mu_1 p_{n-1} a_n p_n \wedge f_1(p_n) \wedge i_2(q_0) \wedge \mu_2 q_0 b_1 q_1 \wedge \mu_2 q_1 b_2 q_2 \wedge \cdots \wedge \mu_2 q_{m-1} b_m q_m \wedge f_2(q_m)$$

Since $f_1(p_n) > 0$, $i_2(q_0) > 0$ and $\mu_2 q_0 b_1 q_1 > 0$, we have $\mu p_n b_1 q_1 = f_1(p_n) \wedge i_2(q_0) \wedge \mu_2 q_0 b_1 q_1$. Therefore,

$$\begin{aligned}
&L_1(x) \wedge L_2(y)\\
&= i(p_0) \wedge \mu p_0 a_1 p_1 \wedge \mu p_1 a_2 p_2 \wedge \ldots \wedge \mu p_{n-1} a_n p_n \wedge \\
&\quad \mu p_n b_1 q_1 \wedge \mu q_1 b_2 q_2 \wedge \ldots \wedge \mu q_{m-1} b_m q_m \wedge f(q_m)\\
&\le \vee\{i(p_0) \wedge \mu p_0 a_1 p_1 \wedge \mu p_1 a_2 p_2 \wedge \ldots \wedge \mu p_{n-1} a_n p_n \wedge \\
&\quad \mu p_n b_1 q_1 \wedge \mu q_1 b_2 q_2 \wedge \ldots \wedge \mu q_{m-1} b_m q_m \wedge f(q_m) \mid \\
&\qquad p_0, p_1, \ldots p_n, \; q_1, \ldots q_m \in Q\\
&= L_3(a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m)\\
&= L_3(xy)
\end{aligned}$$

Therefore, $L_1(x) \wedge L_2(y) \le L_3(w)$. This is true for any $x \in A$, $y \in B$, such that $w = xy$. Thus

$$\vee\{L_1(x) \wedge L_2(y) \mid w = xy, \; x \in A, y \in B\} \le L_3(w) \qquad (2)$$

From (1) and (2), $L_3(w) = \vee\{\{L_1(x) \wedge L_2(y)\} \mid w = xy, x \in A, y \in B\}$. Hence AB is a recognizable set over Σ^* with fuzzy regular language L_3 which is accepted by M_3. By Theorem 3.3 there exists an equivalent uffsa M with fuzzy regular language L such that $L = L_3$. Therefore $L(w) = \vee\{L_1(x) \wedge L_2(y) \mid w = xy, \; x \in A, y \in B\}$.

Hence $L = L_1 L_2$. □

Theorem 4.6. *Let $A \subseteq \Sigma^*$ be a recognizable set with fuzzy regular language L_1, which is accepted by an uffsa M_1. Then A^* is recognizable with fuzzy regular language L, accepted by an uffsa M such that $L = L_1^*$.*

Proof. Let $M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)$ be an uffsa with L_1 being the language accepted by it. We have $L_1(x) > 0\ \forall x \in A$. Define ffsa $M_2 = (Q, \Sigma, \mu, i, f)$ with fuzzy language L_2, where $Q = Q_1$, $\mu : Q \times \Sigma \times Q \to [0,1]$ is defined as follows:

(i) $\mu paq = \mu_1 paq \forall p,\ q \in Q_1,\ a \in \Sigma$

(ii) For $p,\ q \in Q$, if $r \in Q_1$, $i_1(q) > 0$, and $f_1(r) > 0$, $\mu_1 par > 0$ then include $\mu paq = \mu_1 par \wedge f_1(r) \wedge i_1(q)$.

$i : Q \to [0,1]$ is defined by $i(q) = i_1(q)\,\forall p \in Q_1$,
$f : Q \to [0,1]$ is defined by $f(q) = f_1(q)\,\forall q \in Q_1$.

For $L_2(w) = 0$, the result is obvious. Let $w \in A^*$, $L_2(w) > 0$. $L_2(w) = \vee\Big\{\{i(p) \wedge \mu{*}pwq \wedge f(q) \mid q \in Q\} \mid p \in Q\Big\}$. Since Q is finite and w is of finite length, we have

$$\begin{aligned}
L_2(w) = i(p_1) &\wedge \mu p_1 a_{11} p_{11} \wedge \mu p_{11} a_{12} p_{12} \wedge \cdots \wedge \mu p_{1n_1-1} a_{1n_1} p_2 \wedge \\
&\mu p_2 a_{21} p_{21} \wedge \mu p_{21} a_{22} p_{22} \wedge \cdots \wedge \mu p_{2n_2-1} a_{2n_2} p_3 \wedge \\
&\vdots \\
&\mu p_m a_{m1} p_{m1} \wedge \mu p_{m1} a_{m2} p_{m2} \wedge \ldots \wedge \mu p_{mn_m-1} a_{mn_m} p_{m+1} \\
&\qquad\qquad\qquad\qquad \wedge f(p_{m+1}).
\end{aligned}$$

where $p_i,\ p_{ij}, p_{m+1} \in Q$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, (n_i - 1)$ and $x_i = a_{i_1} a_{i_2} \cdots a_{in_i} \in A$ (n_i is the least integer from 1 satisfying this), and $w = x_1 x_2 \cdots x_m$. From definition of M, there exists $p_{in_i} \in Q_1$, $f_1(p_{in_i}) > 0$, $i_1(p_{i+1}) > 0$ such that $\mu p_{in_i-1} a_{in_i} p_{i+1} = \mu_1 p_{in_i-1} a_{in_i} p_{in_i} \wedge f_1(p_{in_i}) \wedge i_1(p_{i+1})$, $i = 1, 2, \ldots, m-1$.

Therefore

$$\begin{aligned}
L_2(w) = i_1(p_1) &\wedge \mu_1 p_1 a_{11} p_{11} \wedge \ldots \wedge \mu_1 p_{1n_1-1} a_{1n_1} p_{1n_1} \wedge f_1(p_{1n_1}) \wedge \\
&i_1(p_2) \wedge \mu_1 p_2 a_{21} p_{21} \wedge \ldots \wedge \mu_1 p_{2n_2-1} a_{2n_2} p_{2n_2} \wedge f_1(p_{2n_2}) \wedge \\
&\vdots \\
&i_1(p_m) \wedge \mu_1 p_m a_{m_1} p_{m_1} \wedge \cdots \wedge \mu_1 p_{mn_m-1} a_{mn_m} p_{m+1} \wedge f_1(p_{m+1}) \\
\le \Big(&\vee \{ i_1(p_1) \wedge \mu_1 p_1 a_{11} p_{11} \wedge \cdots \wedge \mu_1 p_{1n_1-1} a_{1n_1} p_{1n_1} \wedge \\
&f_1(p_{1n_1}) \mid p_1, p_{1j} \in Q_1,\ j = 1, 2, \ldots, n_1 \} \Big)
\end{aligned}$$

$$\wedge \Big(\vee \{ i_1(p_2) \wedge \mu_1 p_2 a_{21} p_{21} \wedge \cdots \wedge \mu_1 p_{2n_2-1} a_{2n_2} p_{2n_2} \wedge f_1(p_{2n_2})$$
$$\mid p_2, p_{2j} \in Q_1,\ j = 1,2,\ldots,n_2 \} \Big)$$
$$\vdots$$
$$\wedge \Big(\vee \{ i_1(p_m) \wedge \mu_1 p_m a_{m1} p_{m1} \wedge \cdots \wedge \mu_1 p_{mn_m-1} a_{mn_m} p_{m+1} \wedge$$
$$f_1(p_{m+1}) \mid p_m, p_{m+1}, p_{mj} \in Q_1,\ j = 1,2,\ldots,n_m - 1 \} \Big)$$
$$= L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m).$$

Therefore $L_2(w) \leq L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m)$, $x_i \in A$, $i = 1,2,\ldots,m$, which is true for any $x_1, x_2, \ldots, x_m$ such that $w = x_1 x_2 \cdots x_m$. Therefore

$$L_2(w) \leq \vee \{ L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m) \mid w = x_1 x_2 \cdots x_m,$$
$$x_i \in A,\ i = 1,2,\ldots,m \} \tag{3}$$

Let $x_i \in A$, therefore $L_1(x_i) > 0$, $i = 1,2,\ldots,m$. Therefore $L_1(x_i) = \vee \Big\{ \{ i_1(p) \wedge \mu_1 {*} p x_i q \wedge f_1(q) \mid q \in Q_1 \} \mid p \in Q_1 \Big\}$. Since Q_1 is finite, we have $L_1(x_i) = i_1(p_i) \wedge \mu_1 p_i a_{i1} p_{i1} \wedge \mu_1 p_{i1} a_{i2} p_{i2} \wedge \cdots \wedge \mu_1 p_{in_i-1} a_{in_i} p_{in_i} \wedge f_1(p_{in_i})$, where p_i, $p_{ij} \in Q_1$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n_i$ and each factor is non-negative. Also $x_i = a_{i1} a_{i2} \cdots a_{in_i}$

$$\text{Now} \quad L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m)$$
$$= \Big(i_1(p_1) \wedge \mu_1 p_1 a_{11} p_{11} \wedge \mu_1 p_{11} a_{12} p_{12} \wedge \cdots$$
$$\wedge \mu_1 p_{1n_1-1} a_{1n_1} p_{1n_1} \wedge f_1(p_{1_{n_1}}) \Big)$$
$$\wedge \Big(i_1(p_2) \wedge \mu_1 p_2 a_{21} p_{21} \wedge \mu_1 p_{21} a_{22} p_{22} \wedge \cdots$$
$$\wedge \mu_1 p_{2n_2-1} a_{2n_2} p_{2n_2} \wedge f_1(p_{2n_2}) \Big)$$
$$\vdots$$
$$\wedge \Big(i_1(p_m) \wedge \mu_1 p_m a_{m1} p_{m1} \wedge \mu_1 p_{m1} a_{m2} p_{m2} \wedge \cdots$$
$$\wedge \mu_1 p_{mn_m-1} a_{mn_m} p_{mn_m} \wedge f_1(p_{mn_m}) \Big)$$

From the definition of M, $\mu_1 p_{in_i-1}\ a_{in_i} p_{in_i} \wedge f_1(p_{in_i}) \wedge i_1(p_{i+1}) = \mu p_{in_i-1}$ $a_{in_i} p_{i+1}$, $i = 1,2,\ldots m-1$. Therefore

$$
\begin{aligned}
&L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m)\\
&= i_1(p_1) \wedge \mu p_1 a_{11} p_{11} \wedge \mu p_{11} a_{12} p_{12} \wedge \cdots \wedge \mu p_{1n_1-1} a_{1n_1} p_2\\
&\wedge \mu p_2 a_{21} p_{21} \wedge \mu p_{21} a_{22} p_{22} \wedge \cdots \wedge \mu p_{2n_2-1} a_{2n_2} p_3\\
&\vdots\\
&\wedge \mu p_m a_{m1} p_{m1} \wedge \mu p_{m1} a_{m2} p_{m2} \wedge \cdots \wedge p_{mn_m-1} a_{mn_m} p_{mn_m}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge f(p_{mn_m})\\
&\le \vee\Big\{ i(p_1) \wedge \mu p_1 a_{11} p_{11} \wedge \mu p_{11} a_{12} p_{12} \wedge \cdots \wedge \mu p_{1n_1-1} a_{1n_1} p_2 \wedge\\
&\quad \mu p_2 a_{21} p_{21} \wedge \mu p_{21} a_{22} p_{22} \wedge \cdots \wedge \mu p_{2n_2-1} a_{2n_2} p_3 \wedge\\
&\quad \vdots\\
&\quad \mu p_m a_{m1} p_{m1} \wedge \mu p_{m1} a_{m2} p_{m2} \wedge \cdots \wedge \mu p_{mn_m-1} a_{mn_m} p_{mn_m} \wedge\\
&\quad f(p_{mn_m}) \mid p_i, p_{ij}, i = 1,2,\ldots,m,\ j = 1,2,\ldots,n_i \in Q \Big\}\\
&= L_2\big(a_{11} a_{12} \cdots a_{1n_1}\, a_{21} a_{22} \cdots a_{2n_2}\, \cdots a_{m1} a_{m2} \cdots a_{mn_m}\big)\\
&= L_2(x_1 x_2 \cdots x_m)\\
&= L_2(w), \quad \text{where } w = x_1 x_2 \cdots x_m
\end{aligned}
$$

Therefore, $L_1(x_1) \wedge L_2(x_2) \wedge \cdots \wedge L_1(x_m) \le L_2(w)$. Varying $x_1, x_2, \ldots, x_m$ such that $w = x_1 x_2 \cdots x_m$, we get

$$
\begin{aligned}
&\vee \Big\{ L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m) \mid w = x_1 x_2 \cdots x_m,\\
&\qquad\qquad\qquad\qquad\qquad x_i \in A, i = 1,2,\ldots,m \Big\}\\
&\le L_2(w) \qquad (4)
\end{aligned}
$$

From (3) and (4),

$$
\begin{aligned}
L_2(w) = \vee\Big\{ &L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m) \mid w = x_1 x_2 \cdots x_m,\\
&\qquad\qquad\qquad x_i \in A, i = 1,2,\ldots,m \Big\}
\end{aligned}
$$

Hence A^* is a recognizable set over Σ^* with fuzzy regular language L_2 which is accepted by M_2. By Theorem 3.3, there exists an equivalent uffsa M with fuzzy regular language L such that $L = L_2$.

$$
\begin{aligned}
L(w) = \vee\Big\{ &L_1(x_1) \wedge L_1(x_2) \wedge \cdots \wedge L_1(x_m) \mid w = x_1 x_2 \cdots x_m,\\
&\qquad\qquad\qquad x_i \in A,\ i = 1,2,\ldots,m \Big\}
\end{aligned}
$$

Hence $L = L_1^*$. □

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