Equienergetic Graphs

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Abstract

The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of G. Two graphs G_1 and G_2 are said to be equienergetic if $E(G_1) = E(G_2)$. In this paper we outline various classes of equienergetic graphs. These results enables construction of pairs of noncospectral equienergetic graphs of same order and of same size.

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1 Introduction

Let G be a graph of order n and size m. The eigenvalues of the adjacency matrix of G denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ are said to be the eigenvalues of G

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and they form the spectrum of G. Two nonisomorphic graphs of same order are said to be cospectral if they have same spectra [4]. The energy of a graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

This concept was introduced by I. Gutman [7] in 1978 and is intensively studied in chemistry, since it can be used to approximate the total π-electron energy of a molecule [11]. Recently this concept started to attract both mathematicians and chemists [6, 8 - 10, 13, 14].

Two graphs G_1 and G_2 are said to be equienergetic if $E(G_1) = E(G_2)$. Equienergetic graphs were first time considered in 2004 in [1, 17]. For obvious reasons, cospectral graphs are equienergetic. Therefore we are interested in noncospectral equienergetic graphs.

The simplest nontrivial example of equienergetic graphs is formed by the triangle G_a, quadrangle G_b and $2K_2$ (see Fig. 1) whose eigenvalues are $2, -1, -1$; $2, 0, 0, -2$; and $1, 1, -1, -1$ respectively. Therefore $E(G_a) = E(G_b) = E(2K_2) = 4$.

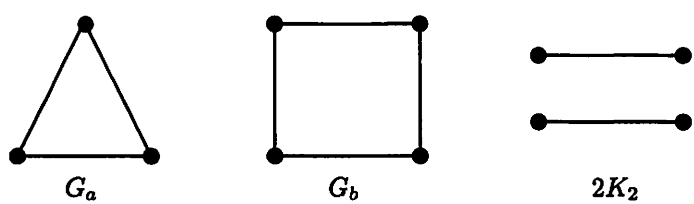

Fig. 1

The smallest pair of noncospecral connected equienergetic graphs with same number of vertices are the pentagon G_c and the tetragonal pyramid G_d (see Fig. 2) whose eigenvalues are 2, $(\sqrt{5}+1)/2$, $(\sqrt{5}-1)/2$, $-(\sqrt{5}+1)/2$, $-(\sqrt{5}-1)/2$ and $\sqrt{5}+1$, 0, 0, -2, $-\sqrt{5}+1$ respectively and for which $E(G_c) = E(G_d) = 2\sqrt{5}+2$.

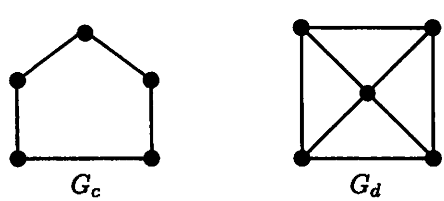

Fig. 2

We are now interested in graphs that are noncospectral, connected, having equal number of vertices, equal number of edges and equienergetic.

2 Equienergetic line graphs

The line graph of G will be denoted by $L(G)$. For $k = 1, 2, \ldots$ the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$ [12].

The line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = (n_0 r_0)/2$ and of degree $r_1 = 2r_0 - 2$. Consequently the order and degree of $L^k(G)$ are [2, 3]

$$n_k = \frac{r_{k-1} n_{k-1}}{2} \qquad \text{and} \qquad r_k = 2r_{k-1} - 2$$

where n_i and r_i stands for order and degree of $L^i(G)$, $i = 0, 1, \ldots$.
Therefore

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{2}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \tag{3}$$

Theorem 2.1. [18, 19] *If G is a regular graph of order n and of degree $r \geq 3$ then*

$$E(L^2(G)) = 2nr(r-2).$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a regular graph G of order n and of degree $r \geq 3$, then by Sachs Theorem [4, 20] the eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} \lambda_i + r - 2 & i = 1, 2, \ldots, n & \text{and} \\ -2 & n(r-2)/2 \ \ \text{times} & \end{array}\right\} \tag{4}$$

In view of the fact that $L(G)$ is also a regular graph of order $nr/2$ and of degree $2r - 2$, from Eqn.(4) the eigenvalues of $L^2(G)$ are easily calculated as

$$\left.\begin{array}{rll} \lambda_i + 3r - 6 & i = 1, 2, \ldots, n & \text{and} \\ 2r - 6 & n(r-2)/2 \ \ \text{times} & \text{and} \\ -2 & nr(r-2)/2 \ \ \text{times} & \end{array}\right\} \tag{5}$$

If d_{max} is the greatest vertex degree of a graph then all its eigenvalues belong to the interval $[-d_{max}, +d_{max}]$ [4]. In particular the eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$, $i = 1, 2, \ldots, n$.

If $r \geq 3$ then $\lambda_i + 3r - 6 \geq 0$ and $2r - 6 \geq 0$.
Thus

$$E(L^2(G)) = \sum_{i=1}^{n} |\lambda_i + 3r - 6| + |2r - 6|\frac{n(r-2)}{2} + |-2|\frac{nr(r-2)}{2}$$

$$= 2nr(r-2) \qquad\qquad \text{since} \quad \sum_{i=1}^{n} \lambda_i = 0.$$

□

Corollary 2.2. [19] *Let G be a regular graph of order n_0, of degree $r_0 \geq 3$ and let for $k \geq 1$ the k-th iterated line graph of G be of degree r_k and possess n_k vertices then*

$$E(L^{k+1}(G)) = 2n_k(r_k - 2).$$

Corollary 2.3. [19] *If G is a regular graph of order n_0 and of degree $r_0 \geq 3$ then in the notation specified in Corollary 2.2, for any $k \geq 1$*

$$E(L^{k+1}(G)) = 2n_0(r_0 - 2)\prod_{i=0}^{k-1}(2^i r_0 - 2^{i+1} + 2).$$

Corollary 2.4. [19] *If G is a regular graph of order n_0 and of degree $r_0 \geq 3$ then in the notation specified in Corollary 2.2, for any $k \geq 2$*

$$E(L^k(G)) = 4(n_k - n_{k-1}) = 4n_k\frac{r_k - 2}{r_k + 2}.$$

Lemma 2.5. [19] *Let G_1 and G_2 be two regular graphs of the same order and of the same degree. Then for any $k \geq 1$ the folowing holds:*

(a) $L^k(G_1)$ and $L^k(G_2)$ are of the same order and have the same nmber of edges.

(b) $L^k(G_1)$ and $L^k(G_2)$ are cospectral if and only if G_1 and G_2 are cospectral.

Proof. Statement (a) follows from the equations (2) and (3) and the fact that the number of edges of $L^k(G)$ is equal to the number of vertices of $L^{k+1}(G)$. Statement (b) follows from relation (4), applied a sufficient number of times. □

Combining Lemma 2.5 with the Corollary 2.3 we arrive at:

Theorem 2.6. [19] *Let G_1 and G_2 be two noncospectral regular graphs of the same order and of the same degree $r \geq 3$. Then for $k \geq 2$ the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ form a pair of noncospectral equienergetic graphs of equal order and of equal number of edges. If in addition, G_1 and G_2 are chosen to be connected then $L^k(G_1)$ and $L^k(G_2)$ are also connected.*

It is now easy to generate large families of equienergetic graphs satisfying the requirements given in Theorem 2.6. For instance there are 2, 5, 19 and 85 connected regular graphs of degree 3 of order 6, 8, 10 and 12 respectively. No two of these are cospectral [4]. Their second and higher iterated line graphs form families consisting of 2, 5, 19, 85, ... equienergetic graphs.

3 Equienergetic Complement Graphs

Let $\overline{G}$ be the complement of G. The results of this section can be proved in similar manner to that of Section 2.

Theorem 3.1. [15] *If G is a regular graph of order n and of degree $r \geq 3$ then*

$$E(\overline{L^2(G)}) = (nr-4)(2r-3)-2.$$

Corollary 3.2. [22] *Let G be a regular graph of order n_0 and of degree $r_0 \geq 3$. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 2$ then*

$$\begin{aligned} E(\overline{L^{k+1}(G)}) &= (n_{k-2}r_{k-2}-4)(2r_{k-2}-3)-2 \\ &= (2n_{k-1}-4)(r_{k-1}-1)-2. \end{aligned}$$

Corollary 3.3. [22] *If G is a regular graph of order n_0 and of degree $r_0 \geq 3$ then in the notation specified in Corollary 3.2, for any $k \geq 2$*

$$E(\overline{L^k(G)}) = \left[\frac{n_0}{2^{k-2}}\prod_{i=0}^{k-2}(2^i r_0 - 2^{i+1}+2)-4\right](2^{k-1}r_0-2^k+1)-2.$$

Corollary 3.4. [22] *If G is a regular graph of order n_0 and of degree $r_0 \geq 3$ then in the notation specified in Corollary 3.2, for any $k \geq 2$*

$$E(\overline{L^k(G)}) = \frac{4n_k r_k}{2+r_k} - 2(r_k+1).$$

Corollary 3.5. [15] *Let G_1 and G_2 be two noncospectral regular graphs on n vertices, of degree $r \geq 3$, then for $k \geq 2$ both $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are regular, noncospectral possessing same number of vertices, same number of edges and equienergetic.*

4 Equienergetic Graph Products

Definition 4.1. *The strong product of two graphs G and H is the graph $G \otimes H$ whose vertex set is $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent in $G \otimes H$ if u is adjacent to x in G and v is adjacent to y in H.*

Bearing in mind that the eigenvalues of $G_1 \otimes G_2$ are just the products of the eigenvalues of G_1 and G_2 [4], R. Balakrishnan [1] observed that for any two graphs G_1 and G_2,

$$E(G_1 \otimes G_2) = E(G_1)E(G_2).$$

Using this he has proved that for any graph G of order n, $E(K_2 \otimes K_2 \otimes G) = E(C_4 \otimes G) = 4E(G)$.

Thus he has constructed pairs of equienergetic graphs for $n \equiv 0(mod 4)$.

The same result was also reported independently by Stevanović [21] and he constructed equienergetic graphs for $n \equiv 0(mod 5)$.

Following result gives that construction of equienergetic graphs for all $n \geq 9$.

Definition 4.2. *The complete product of two graphs G_1 and G_2 denoted by $G_1 \nabla G_2$ is the graph obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2.*

If G_1 is r_1-regular graph on n_1 vertices and G_2 is r_2-regular graph on n_2 vertices then the characteristic polynomial of $G_1 \nabla G_2$ is [4, 5]

$$\phi(G_1 \nabla G_2 : \lambda) = \frac{(\lambda - r_1)(\lambda - r_2) - n_1 n_2}{(\lambda - r_1)(\lambda - r_2)} \phi(G_1 : \lambda)\phi(G_2 : \lambda).$$

With this characteristic polynomial we get the following result.

Theorem 4.3. [16] *If G_1 is r_1-regular graph on n_1 vertices and G_2 is r_2-regular graph on n_2 vertices then*

$$E(G_1 \nabla G_2) = E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2). \tag{6}$$

Corollary 4.4. [16] *If H_a and H_b are equienergetic regular graphs on n vertices and of same degree r then for any regular graph G, the graphs $G \nabla H_a$ and $G \nabla H_b$ are also equienergetic.*

Theorem 4.5. [17] *There exists infinitely many pairs of noncospectral equienergetic graphs on n vertices, $n \geq 9$.*

Proof. Consider the graphs G_1 and G_2 as shown in Fig. 3.

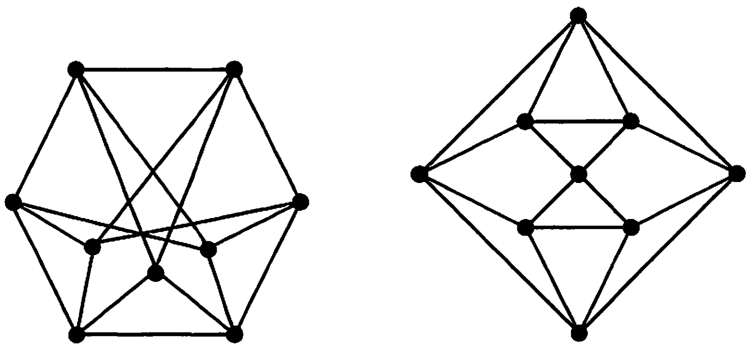

Fig. 3

$$\phi(G_1 : \lambda) = (\lambda - 4)(\lambda - 1)^4(\lambda + 2)^4 \tag{7}$$

$$\phi(G_2 : \lambda) = (\lambda - 4)(\lambda - 2)(\lambda - 1)^2(\lambda + 1)^2(\lambda + 2)^3 \tag{8}$$

Both G_1 and G_2 are regular, connected on 9 vertices and of degree 4. And $E(G_1) = 16 = E(G_2)$.

Let H be any regular graph on p vertices and of degree r, $0 \leq r \leq p-1$. Then by Theorem 4.3,

$$E(G_1 \nabla H) = E(G_2 \nabla H) = 12 + E(H) - r + \sqrt{36p + (r-4)^2}.$$

Thus $G_1 \nabla H$ and $G_2 \nabla H$ are equienergetic. By Eqns. (7) and (8), G_1 and G_2 are noncospectral, so $G_1 \nabla H$ and $G_2 \nabla H$ are noncospectral. Further $G_1 \nabla H$ and $G_2 \nabla H$ are connected and possesses equal number of vertices $n = 9 + p$, $p = 0, 1, 2, \ldots$

It is now easy to construct large families of pairs of equienergetic graphs on n vertices, $n \geq 9$. □

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