Edge Roman Domination in Graphs

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Abstract

An *Edge Roman dominating function* of a graph $G = (V, E)$ is a function $f' : E \to \{0, 1, 2\}$ satisfying the condition that every edge x for which $f'(x) = 0$ is adjacent to at least one edge y for which $f'(y) = 2$. The *weight* of an Edge Roman dominating function is the value $f'(E) = \sum_{x \in E} f'(x)$. The minimum weight of an Edge Roman dominating function on a graph G is called the *Edge Roman domination number of G*. In this paper we initiate a study of this parameter.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [5].

For any vertex $v \in V$, the open neighbourhood $N(v)$ and the closed neighbourhood $N[v]$ are defined by $N(v) = \{v \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$ respectively. Similarly for an edge $x \in E$, we define $N(x) = \{y \in E : y \text{ is adjacent to } x\}$ and $N[x] = N(x) \cup \{x\}$. Also if $S \subseteq V$, we define

$N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The degree of an edge $x = uv$ is defined by $d(x) = |N(x)| = d(u) + d(v) - 2$.

In a connected graph, the distance between two vertices u and v is the length of a shortest path joining u and v and is denoted by $d(u, v)$. If $v \in V$ and $S \subset V$, then $d(u, S)$ denotes the minimum distance between u and any vertex of S. The *radius* and *diameter* of G are defined by $rad(G) = \min_{v \in V} \max_{w \in V} d(v, w)$ and $diam(G) = \max_{v,w \in V} d(v, w)$. A *caterpillar* is a tree T in which the removal of all end vertices leaves a path which is called the *spine* of the caterpillar. A *lobster* is a tree in which the removal of all end vertices leaves a caterpillar. Let $v \in S \subseteq V$. A vertex u is called a *private neighbor* of v, with respect to S (denoted by u is an S - pn of v) if $u \in N[v] - N[S - \{v\}]$ An S -pn of v is external if it is a vertex of $V - S$. The set $pn(v, S) = N[v] - N[S - \{v\}]$ of all S-pns of V is called the *private neighborhood* of v with respect to S. Let $x \in F \subset E$. An edge x is called a *private neighbor* of y with respect to F (denoted by, x is an F-pn of y) if x in $N[y] - N[F - \{y\}]$. An F-pn of y is *external* if it is an edge of $E - F$. The set $pn(y, F) = N[y] - N[F - \{y\}]$ of all F-pns of y is called the *private neighborhood set* of y with respect to F.

A set S is a dominating set if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G and a dominating set S of minimum cardinality is called a γ - *set* of G. A set S of vertices is called *independent* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ is the minimum cardinality of a set S of vertices which is both independent and dominating . A set S of vertices is called a *2-packing* if for every pair of vertices $u, v \in S, N[u] \cap N[v] = \phi$. A set S of vertices is called a *vertex cover* if for every edge $uv \in E$, either $u \in S$ or $v \in S$. The recent book *Fundamentals of Domination in Graphs* [5] lists, in an appendix, many varieties of dominating sets.

The concept of edge domination was introduced by Mitchell and Hedetnimi [9]. Arumugam and Velammal [1] have obtained further results on edge domination. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ is the minimum cardinality taken over all edge dominating sets of G. A set X of edges is called *independent* if no two edges in X are adjacent. The *independent edge domination number* $i'(G)$ is the minimum cardinality of a set X of edges which is both independent and dominating. A set F of edges is called a *2-edge packing* if it is independent and for every pair of edges $x, y \in F, N[x] \cap N[y] = \phi$. An *edge cover* of G is a subset L of E such that each vertex of G is an end of some edge in L.

A variant of the domination number was suggested by an article in Scientific American by Ian Stewart [16]. Independently ReVelle [11]-[13]

has suggested the concept of Roman domination a few years earlier. Since then, several papers have been published on Roman domination number [3],[7]-[10], [12]-[16]. A Roman dominating function of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Let (V_0, V_1, V_2) be the ordered partition induced by f where $V_i = \{v \in V : f(v) = i\}$. The weight of a Roman dominating function of G is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function of G is called the Roman domination number of G and is denoted by $\gamma_R(G)$ [4]. The definition of a Roman domination function is given implicitly in [2] and [16].

In this paper we introduce the concept of an *Edge Roman dominating function* and *Edge Roman domination number* and initiate a study of this parameter.

2 Main Results

We assume throughout that $G = (V, E)$ is a graph without isolated vertices.

Definition 2.1. *Let $G = (V, E)$ be a graph. A function f' from $E \rightarrow \{0, 1, 2\}$ satisfying the condition that every edge x for which $f'(x) = 0$ is adjacent to at least one edge y for which $f'(y) = 2$ is called an Edge Roman dominating function (EDRF) of the graph. The weight of f' is defined by $f'(E) = \sum_{e \in E} f(e)$. The minimum weight of an ERDF of G is called the Roman domination number of G and is denoted by $\gamma'_R(G)$. An ERDF f' with $f'(E) = \gamma'_R(G)$ is called a γ'_R-function of G.*

Observation 2.2. *For a graph $G = (V, E)$, let $f' : E \rightarrow \{0, 1, 2\}$ and let (E_0, E_1, E_2) be the ordered partition of E induced by f' where $E_i = \{e \in E : f'(e) = i\}$ and $|E_i| = q_i$, for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the functions $f' : E \rightarrow \{0, 1, 2\}$ and the ordered partitions (E_0, E_1, E_2) of E. Thus we will write $f' = (E_0, E_1, E_2)$.*

Clearly $f' = (E_0, E_1, E_2)$ is an Edge Roman dominating function (ERDF) *if and only if $E_2 \succ E_0$, where $\succ$ signifies that the set E_2 dominates the set E_0. Also the weight of f' is $f'(E) = \sum_{e \in E} f(e) = 2q_2 + q_1$.*

The proofs of the following results are straightforward.

Proposition 2.3. *For any graph G, $\gamma'(G) \leq \gamma'_R(G) \leq 2\gamma'(G)$.*

Proposition 2.4. *Let $f' = (E_0, E_1, E_2)$ be any γ'_R - function. Then*

(a) In $G[E_1]$ the maximum degree of an edge is less than or equal to one.

(b) *No vertex of G is incident with* E_1 *and* E_2.

(c) *Each edge of* E_0 *is adjacent with at most two edges of* E_1.

(d) E_2 *is a* γ' *set of* $H = G[E_0 \cup E_2]$

(e) *Each* $e \in E_2$ *has at least two* $E_2 - pns$ *in* H.

(f) *If e is isolated in* $G[E_2]$ *and has precisely one external* $E_2 - pn(inH)$ *say* $w \in E_0$, *then* $N(w) \cap E_1 = \phi$.

Proposition 2.5. *Let* $f' = (E_0, E_1, E_2)$ *be a* $\gamma'_R(G)$ *- function of G, such that* q_1 *is a minimum. Then*

(a) E_1 *is independent and* $E_0 \cup E_2$ *is an edge cover.*

(b) $E_0 \succ E_1$

(c) *Each edge of* E_0 *is adjacent to at most one edge of* E_1. *i.e.* E_1 *is a 2 - edge packing.*

(d) *Let* $e \in G[V_2]$ *have exactly two external* $E_2 - pns$ w_1 *and* w_2 *in* E_0. *Then there do not exist edges* $y_1, y_2 \in E_1$, *such that* (y_1, w_1, e, w_2, y_2) *is the edge sequence of a path* P_6.

(e) $q_0 \geq \frac{3q}{7}$.

Proposition 2.6. *Let* P_n *and* C_n *denote respectively the path and cycle on n vertices. Then*

(a) $\gamma'_R(P_{3k}) = 2k$

(b) $\gamma'_R(P_{3k+1}) = 2k$

(c) $\gamma'_R(P_{3k+2}) = 2k + 1$

(d) $\gamma'_R(C_{3k}) = 2k$

(e) $\gamma'_R(C_{3k+1}) = 2k + 1$

(f) $\gamma'_R(C_{3k+2}) = 2(k + 1)$, *where* $k \geq 0$.

Proposition 2.7. *For any graph G,* $\gamma'(G) = \gamma'_R(G)$ *if and only if each component of G is a* K_2.

Proof. Suppose $\gamma'(G) = \gamma'_R(G)$. Let $f' = (E_0, E_1, E_2)$ be a γ' - function. Then $|E_1| + |E_2| = |E_1| + 2|E_2|$, so that $E_2 = \emptyset$. Hence $E_0 = \emptyset$ and $\gamma'_R(G) = |E_1| = |E| = q$. Thus $\gamma'(G) = q$, so that each component of G is K_2. The converse is obvious. □

Proposition 2.8. *Let G be a connected graph of size q and $p > 2$. Then $\gamma'(G) = 1$ and $\gamma'_R(G) = 2$ if and only if there exists an edge of degree $q-1$.*

Proof. If G has an edge e of degree $q-1$, then clearly $\gamma'(G) = 1$ and $\gamma'_R(G) = 2$. Conversely let $\gamma'(G) = 1$ and $\gamma'_R(G) = 2$ and let $f' = (E_0, E_1, E_2)$ be an $ERDF$ with weight 2. Then either $|E_2| = 1$ or $|E_2| = 0$. If $|E_2| = 1$, *then* $|E_1| = 0$ and since $E_2 \succ E_0$ it follows that the unique edge $e \in E_2$ has degree $q-1$.

If $|E_2| = 0$, then $|E_0| = 0$ and $|E_1| = 2$. In this case G is the path P_3 and hence the result follows. □

Proposition 2.9. *For any graph G, $\gamma'_R(G) = q$ if and only if each component of G is either a P_2 or P_3.*

Proof. It is sufficient to prove the result for connected graphs. If $G = P_2$ or P_3, then trivially $\gamma'_R(G) = q$. Conversely suppose $\gamma'_R(G) = q$. Then $2|E_2| + |E_1| = q$ and $|E_2| + |E_1| + |E_0| = q$.

Case i. $|E_2| = 0$.

Then $|E_0| = 0$ hence $|E_1| = q$. By Proposition 2.4(a), any edge in $G[E_1]$ has maximum degree less than or equal to one and hence $G = P_2$ or P_3.

Case ii. $|E_2| \neq 0$.

Then $|E_2| = |E_0|$ and each member of E_2 is incident to exactly one member of E_0. First we claim that $\Delta(G) = 2$. Suppose $\Delta(G) \geq 3$. Let w be a vertex in G such that $d(w) = \Delta$. Then clearly any γ'_R - function f' will label one of the edges incident at w as 2 and the remaining $\Delta - 1$ edges as 0, so that $|E_2| < |E_0|$ which is a contradiction. Then $\Delta(G) = 2$. Hence G is a path or a cycle. Since each edge of E_2 is incident with exactly one member of E_0, G cannot be a cycle and hence $G \cong P_3$. □

It follows from Propositions 2.3 and 2.7 that $\gamma'(G) \leq \gamma'_R(G) \leq 2\gamma'(G)$ and the lower bound is achieved only when each component of G is a K_2. Thus if G is a connected graph of order $p \geq 3$, then $\gamma'_R(G) \geq \gamma'(G) + 1$. We now proceed to characterize connected graphs with $\gamma'_R(G) = \gamma'(G) + 1$ and $\gamma'_R(G) = \gamma'(G) + 2$.

Theorem 2.10. *If G is a connected graph of order $p \geq 3$ then $\gamma'_R(G) = \gamma'(G) + 1$ if and only if there exists an edge e in $E(G)$ such that $d(e) = q - \gamma'(G)$.*

Proof. Suppose there exists an edge e in $E(G)$ such that $d(e) = q - \gamma'(G)$. Let $E_2 = \{e\}, E_1 = E - N[e]$ and $E_0 = E - (E_1 \cup E_2)$. Then $E_1 \cup E_2$ is a γ'-set of G and $f' = (E_0, E_1, E_2)$ is an ERDF with $f'(E) = \gamma'(G) + 1$. Since $\gamma'_R(G) \geq \gamma'(G) + 1$ for connected graphs of order $p \geq 3$ we have $\gamma'_R(G) = \gamma'(G) + 1$. Conversely, let G be a connected graph with $\gamma'_R(G) =$

$\gamma'(G)+1$. Let $f'=(E_0,E_1,E_2)$ be an ERDF with $\gamma'(G)+1$. Then either (*i*) $|E_1|=\gamma'(G)+1$ and $|E_2|=0$ or (*ii*) $|E_1|=\gamma'(G)-1$ and $|E_2|=1$.

In case (i) since $|E_2|=0, E_1=E$. Therefore $|E_1|=|E|$, so that $\gamma'(G)=q$. Hence it follows from Proposition 2.9, that G is P_3. Hence there exists an edge e in G satisfying the given condition. Now, suppose $|E_1|=\gamma'_R(G)-1$ and $|E_2|=1$. Let $e\in E_2$. Since no edge of E_1 is incident with e and $\{e\}\succ E_0$, $d(e)=|E_0|=q-|E_1|-|E_2|=q-\gamma'(G)$. □

Corollary 2.11. *If G is a connected graph, then $\gamma'_R(G)=\gamma'(G)+1$ if and only if G has a $\gamma'(G)$ - set E' which contains an edge e such that $\{e\}\succ E-E'$ and the set $E'-\{e\}$ is a 2-edge packing.*

Corollary 2.12. *If G is a connected graph and $\gamma'_R(G)=\gamma'(G)+1$ then $1\le rad(G)\le 2$ and $1\le diam(G)\le 4$. In particular if $\gamma'_R(G)\ge 3$, then $rad(G)=2$ and $diam(G)=4$.*

Corollary 2.13. *Let T be a tree of order $p>2$ and size q. Then $\gamma'_R(T)=\gamma'(T)+1$ if and only if one of the following holds.*

(a) T is a star $K_{1,p-1}$.

(b) T is a caterpillar whose spine is of length one.

(c) T is a lobster whose diameter is 4, spine is of length one and each vertex not on the spine is of degree at most 2.

Proof. Suppose $\gamma'_R(T)=\gamma'(T)+1$. Then $rad(T)=1$ or 2. If $rad(T)=1$, then $T=K_{1,p-1}$. If $rad(T)=2$, then $diam(T)=3$ or 4. If $rad(T)=2$ and $diam(T)=3$, then T is a caterpillar given in (b). If $rad(T)=2$ and $diam(T)=4$, then T is a lobster given in (c). The converse is obvious. □

Proposition 2.14. *Let G be a connected (p,q) graph. Then $\gamma'_R(G)=\gamma'(G)+2$ if and only if the following conditions are satisfied.*

(a) G does not have an edge e such that $d(e)=q-\gamma'(G)$.

(b) Either G has an edge e such that $d(e)=q-\gamma'(G)-1$ or there exist two edges x and y such that $|N[x]\cup N[y]|=q-\gamma'(G)+2$.

Proof. Suppose (a) and (b) are satisfied. It follows from Theorem 2.10 that $\gamma'_R(G)>\gamma'(G)+1$. If G has an edge $e=uv$ such that $d(u)+d(v)=q-\gamma'(G)+1$ then $f'=(E_0,E_1,E_2)$, where $E_0=N(e), E_1=E-N[e]$ and $E_2=\{e\}$ is an ERDF with $f'(E)=\gamma'(G)+2$.

If there exist edges x and y such that $|N[x]\cup N[y]=q-\gamma'(G)+2$, then $f'=(E_0,E_1,E_2)$ where $E_0=N[x]\cup N[y]-\{x,y\}$, $E_1=E-(N[x]\cup N[y])$ and $E_2=\{x,y\}$ is an ERDF with $f'(E)=\gamma'(G)+2$. Thus $\gamma'_R(G)=$

$\gamma'(G)+2$. Conversely, let G be a graph with $\gamma'_R(G)=\gamma'(G)+2$. Then (a) follows from Theorem 2.10. Now, let $f=(E_0,E_1,E_2)$ be an ERDF of G with weight $\gamma'(G)+2$. Then we have the following three cases.

Case i. $|E_1|=\gamma'(G)+2$ and $|E_2|=0$.

In this case $|E_2|=0$ so that $E_1=E$ and $\gamma'_R(G)=q$. Hence it follows from Proposition 2.9 that each component of G is isomorphic to P_2 or P_3. Now if m denote the number of components of G which are isomorphic to P_3 then $\gamma'_R(G)=\gamma'(G)+m$ and hence $m=2$. Let G_1 and G_2 be the two components of G, each isomorphic to P_3 and let $x\in E(G_4)$ and $y\in E(G_2)$. Clearly $|N(x)\cup N(y)|=q-\gamma'(G)+2$.

Case ii. $|E_1|=\gamma'(G)$ and $|E_2|=1$.

Let $E_2=\{e\}$. Clearly $d(e)=q-\gamma'(G)-1$.

Case iii. $|E_1|=\gamma'(G)-2$ and $|E_2|=2$.

Let $E_2=\{x,y\}$. Then $|N[x]\cup N[y]|=q-\gamma'(G)+2$. □

Corollary 2.15. *If G is a connected graph and $\gamma'_R(G)=\gamma'(G)+2$, then $2\le rad(G)\le 4$ and $3\le diam(G)\le 8$.*

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