On (k,d)-Multiplicatively Indexable Graphs

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Abstract

A (p,q)-graph G is said to be (k,d)-multiplicatively indexable if there exists an injection $f : V(G) \to \mathbb{N}$ such that $f^{\times}(E(G)) = \{k, k+d, \ldots, k+(q-1)d\}$, where $f^{\times} : E(G) \to \mathbb{N}$ is defined by $f^{\times}(uv) = f(u)f(v)$ for every $uv \in E(G)$. If further $f(V(G)) = \{1, 2, \ldots, p\}$, then G is said to be a (k,d)-strongly multiplicatively indexable graph. In this paper we initiate a study of graphs which admit such labellings.

Keywords. (k,d)-multiplicatively indexable graph, (k,d)-strongly multiplicatively indexable graph

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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. Terms not specifically defined in this paper may be found in Harary [6].

Over the past 30 years or so, more than 650 papers on various graph labelling methods have been published, identifying several classes of graphs admitting a given type of labelling (*see* Gallian [5]).

Acharya and Hegde [1], defined a (p, q)-graph $G = (V, E)$ to be multiplicative if there exists an injection $f : V(G) \rightarrow \mathbb{N}$ such that the induced function $f^{\times} : E(G) \rightarrow \mathbb{N}$ defined by $f^{\times}(uv) = f(u)f(v)$ for every $uv \in E(G)$, is also injective.

Beineke and Hegde [4], defined a (p, q)-graph $G = (V, E)$ to be strongly multiplicative if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the function $f^{\times}$ is injective.

Acharya [3] called such a function f a *multiplicative indexer* of G, keeping in analogy with its additive counterpart in the standard literature (*e.g.*, *see*, Acharya and Hegde [2]). Further, in the same spirit, Acharya et al. [3] used the term *multiplicatively indexable* instead of the term 'strongly multiplicative' in the above definition and a multiplicative indexer f of G, if it exists, is called a *strong multiplicative indexer* (abbreviated as 'SMI') of G if $f^{\times}(E(G)) := \{f^{\times}(uv) : uv \in E\} = \{2, 3, \ldots, q + 1\}$, since this set represents the first q positive integers each of which is a product of two other distinct positive integers; a graph that admits such a function is called a *strong multiplicatively indexable* (abbreviated as SMI-)*graph.*

In this paper we introduce the concept of (k, d)-multiplicatively indexable graphs and (k, d)-strongly multiplicatively indexable graphs, as a natural generalization of multiplicative graphs and multiplicatively indexable graphs. In fact strong multiplicatively indexable graphs (SMI- graphs) are nothing but $(2, 1)$-strongly multiplicatively indexable graphs.

2 Main results

Definition 2.1. *A (p,q)-graph G is said to be (k,d)-multiplicatively indexable if there exists an injection $f : V(G) \rightarrow N$ such that $f^{\times}(E(G)) = \{k, k+d, \ldots, k+(q-1)d\}$. Also f is called a (k, d)-multiplicative indexer of G. If further $f(V(G)) = \{1, 2, \ldots, p\}$, then G is said to be a (k,d)-strongly multiplicatively indexable graph and such a function f is called a (k, d)-strongly multiplicative indexer of G.*

We observe that if f is a (k, d)-multiplicative indexer of G and m is any positive integer, then mf is a (m^2k, m^2d)-multiplicative indexer. Hence if G is (k,d)-multiplicative, then G is (m^2k, m^2d)-multiplicative for any positive integer m.

Theorem 2.2. *Let $G = (V,E)$ be a (p,q)-graph which admits a (k,d)-multiplicative indexer f, where k and d are not simultaneously even. Let $X = \{u \in V(G) : f(u) \text{ is odd}\}$. Then the number of edges with both ends in X is*

(i) $\lfloor \frac{q+1}{2} \rfloor$ *if k and d are both odd,*

(ii) $\lfloor \frac{q}{2} \rfloor$ *if k is even and d is odd, and*

(iii) q *if k is odd and d is even.*

Proof. Since k and d are not simultaneously even, the number of odd numbers in $f^{\times}(E(G))$ is precisely given by (i), (ii) and (iii) in the respective cases. Further an edge e of G has both ends in X if and only if $f^{\times}(e)$ is odd and hence the result follows. □

Remark 2.3. *Let G be a connected (k,k)-multiplicatively indexable graph. Then for any (k,k)-multiplicative indexer f of G, if k divides $f(u)$ for every $u \in V(G) - \{v\}$, then $f(v) = 1$. However if $f(v) = 1$ for some $v \in V$, then it is not necessary that k divides $f(u)$ for all $u \in V(G) - \{v\}$. The graph given in Figure 1 is a $(3,3)$-multiplicatively indexable graph. Here $f(V(G)) = \{1,2,3,4,5,6,9\}$ and $f^{\times}(E(G)) = \{3,6,9,12,15,18\}$. Note that $1 \in f(V(G))$ and there exists a vertex u such that 3 does not divide $f(u)$.*

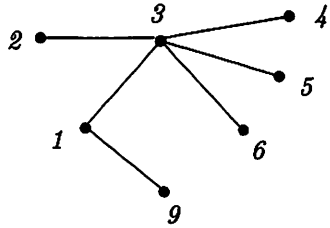

Figure 1

Remark 2.4. *If G is a (k,d)-multiplicatively indexable graph where k is a prime or square of a prime, or k and d are relatively prime, then $f(v) = 1$ for some $v \in V$.*

Theorem 2.5. *The cycle C_3 is (k,d)-multiplicatively indexable if and only if there exist three positive integers a, b and c such that $a < b < c$, a is the harmonic mean of b and c, $k = ab$ and $d = b(c-a)$.*

Proof. Suppose $C_3 = (u\ v\ w\ u)$ is (k,d)-multiplicatively indexable. Let $f : V(C_3) \to N$ defined by $f(u) = a$, $f(v) = b$ and $f(w) = c$ be a multiplicative indexer of G so that $ab = k$, $bc = k + d$ and $ca = k + 2d$. Clearly $a = \frac{2bc}{b+c}$ and $d = b(c-a)$. Conversely if there exist positive integers a, b, c such that $a < b < c$, a is the harmonic mean of b and c, $k = ab$ and $d = b(c-a)$, then $f : V(C_3) \to N$ defined by $f(u) = a$, $f(v) = b$ and $f(w) = c$ is a (k,d)- multiplicative indexer of G. □

Corollary 2.6. *If C_3 is (k,d)-multiplicatively indexable, then* $2d^2 \equiv 0(\text{mod}k)$.

Proof. Since $(k+d)(k+2d) = (bc)(ca) = kc^2$, the result follows. □

Corollary 2.7. *If f is a (k,d)-multiplicative indexer of the cycle $C_3 = (u\ v\ w\ u)$, then $1 \notin f(V(C_3))$.*

Proof. Suppose $f(u) = 1$. Let $f(v) = b, f(w) = c$ and $b < c$, so that $b = k, c = k+d$ and $bc = k+2d$. Hence $c = \frac{b}{2-b}$, where $b > 1$, so that $c < 0$, which is a contradiction. □

Theorem 2.8. *The cycle $C_4 = (u_1u_2u_3u_4u_1)$ is not (k,d)-multiplicatively indexable for any k, $d > 0$.*

Proof. Suppose C_4 admits a (k,d)-multiplicative indexer f and let $f(u_1) = a, f(u_2) = x_1, f(u_3) = b, f(u_4) = x_2$. Then $f^{\times}(E(C_4)) = \{ax_1, ax_2, bx_1, bx_2\} = \{k, k+d, k+2d, k+3d\}$. If $ax_1 = k$, $ax_2 = k+d$, $bx_1 = k+2d$, $bx_2 = k+3d$, then $a(x_2 - x_1) = b(x_2 - x_1) = d$ so that $a = b$, which is a contradiction. A similar contradiction arises for any other permutation of the values ax_1, ax_2, bx_1, bx_2. Hence C_4 is not (k,d)-multiplicatively indexable. □

Theorem 2.9. *The star $K_{1,b}$ is (k,d)-multiplicatively indexable.*

Proof. Let $V(K_{1,b}) = \{u_0, u_1, \ldots, u_b\}$ with $deg\ u_0 = b$. Then $f : V(K_{1,b}) \to N$ defined by $f(u_0) = 1$ and $f(u_i) = k + (i-1)d$ for $1 \le i \le b$, is a (k,d)-multiplicative indexer of $K_{1,b}$. □

Theorem 2.10. *Let k and d be positive integers such that d divides k. Then any (p,q)-graph G can be embedded into a (k,d)-multiplicatively indexable graph H.*

Proof. Let G be a graph with p vertices and let $V(G) = \{u_1, u_2, \ldots, u_p\}$. Define $f : V(G) \to N$ given by $f(u_1) = k$, $f(u_2) = k+d$ and $f(u_i) = f(u_{i-1})f(u_{i-2})$, $3 \le i \le p$. We observe that k divide $f(u_i)$ if $i \ne 2$. Consider the edge induced function $f^{\times}$ given by $f^{\times}(u_iu_j) = f(u_i)f(u_j)$. If there exist edges e_1 and e_2 such that $e_1 \ne e_2$ and $f^{\times}(e_1) = f^{\times}(e_2)$, then we choose an end vertex u of e_1 with $u \neq u_2$ and replace $f(u)$ by $k^\alpha f(u)$, where α is a positive integer chosen in such a way that $f^\times(e_1) \neq f^\times(e_2)$. This we may assume that $f^\times$ is injective. Clearly $f^\times(E(G))$ consists of terms of the sequence $\{k + rd,\ r \in Z^+\}$.

Now add a new vertex v, define $f(v) = 1$ and join v to the vertices u_1 and u_2 so that k, $k + d$ are labels of the edges vu_1 and vu_2. Also for each integer x with $k < x < \theta$, x is of the form $k + rd$, $r \in Z^+$ and x is not an edge label, we add a new vertex v_x, define $f(v_x) = x$ and join v_x to v. The resulting graph H is (k, d)-multiplicatively indexable and G is an induced subgraph of H. □

We now proceed to investigate properties of graphs which are (k, d)-strongly multiplicatively indexable.

Theorem 2.11. *The complete graph K_n is not (k, d)-strongly multiplicatively indexable for all $n \geq 3$.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let $f(v_i) = i$. Since $2, 3, \ldots, n \in f^\times(E)$, maximum edge label is $n(n-1)$ and $|E(K_n)| = \frac{n(n-1)}{2}$, it follows that K_n is not (k, d)-strongly multiplicatively indexable.

□

Theorem 2.12. *There does not exist a (p, q)-graph which is (k, d)-strongly multiplicatively indexable in the following cases:*

(i) *k is odd and d is even.*

(ii) *d is odd, $p \geq d$ and d does not divide k.*

Proof. (i) Suppose there exists a (p, q)-graph G which is (k, d)-strongly multiplicative indexable. Let f be a (k, d)-strongly multiplicative indexer of G, so that $f(V(G)) = \{1, 2, \ldots, p\}$. If k is odd and d is even, then any element of $f^\times(E(G))$ is odd. However $f^\times(e)$ is even for any edge incident at the vertex with $f(v) = 2$, which is a contradiction.

(ii) Suppose d is odd, $p \geq d$ and d does not divide k. Since $f^\times(E(G)) = \{k, k+d, \ldots, k+(q-1)d\}$, $f^\times(e) \equiv k (\bmod\ d)$ for every $e \in E(G)$. However, since $p \geq d$, there exists $v \in V(G)$ such that $f(v) = d$ and hence $f^\times(e) \equiv 0 (\bmod\ d)$ for any edge e incident at v. Therefore d divides k, which is a contradiction. □

Theorem 2.13. *The star $K_{1,n}$ is (k, k)-strongly multiplicatively indexable if and only if $n \leq k - 1$.*

Proof. Let $V(K_{1,n}) = \{u, v_1, v_2, \ldots, v_n\}$ with *deg* $u = n$. Suppose $K_{1,n}$ is (k, k)-strongly multiplicatively indexable with indexer f. Since $f(K_{1,n})) = \{1, 2, \ldots, (n+1)\}$ and $f^\times(E(K_{1,n})) = \{k, 2k, \ldots, nk\}$, it follows that $f(u) =$

k and $f(\{v_1, v_2, \ldots, v_n\}) = \{1, 2, \ldots, n\}$. Hence $n \leq k - 1$. The converse is obvious. □

Theorem 2.14. *Let k be a positive integer and let $n = k^2 - 2$. Then the graph G obtained from the star $K_{1,n}$ by subdividing exactly one edge is (k, k)-strongly multiplicatively indexable.*

Proof. Let $V(K_{1,n}) = \{u, u_1, \ldots, u_n\}$ where $n = k^2 - 2$, *deg* $u = n$ and let v be the vertex subdividing the edge uu_1. Then $f : V(G) \to \{1, 2, ..., n\}$ defined by $f(u) = k$, $f(v) = 1$ and

$$f(u_j) = \begin{cases} k^2 & \text{if } j = 1 \\ j & \text{if } 2 \leq j \leq k-1 \\ j+1 & \text{if } k \leq j \leq k^2 - 2 \end{cases}$$

is a (k, k)-strongly multiplicative indexer G. □

Example 2.15. *Figure 2 illustrates the theorem for $k = 3$.*

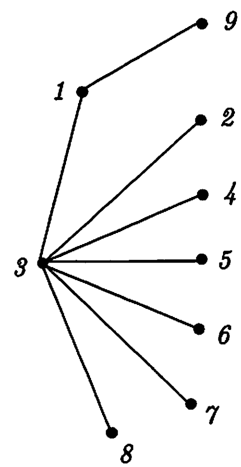

Figure 2

Theorem 2.16.

(i) *A graph G is $(2, 4)$-strongly multiplicatively indexable if and only if $G \cong P_3$.*

(ii) *A graph G is $(2, 3)$-strongly multiplicatively indexable if and only if $G \cong K_2$.*

(iii) *A graph G is $(2, 10)$-strongly multiplicatively indexable if and only if $G \cong 2K_2$.*

(iv) *A graph G is* $(3,5)$*-strongly multiplicatively indexable if and only if* $G \cong 2K_2$.

Proof. Let G be $(2,4)$-strongly multiplicative indexable graph with indexer f. Then $f^{\times}(e) \equiv 2(mod\ 4)$ for every $e \in E(G)$ so that $4 \notin f(V(G))$. Thus $|V(G)| = 3$ and hence $G \cong P_3$. Conversely if $G = P_3 = (v_1, v_2, v_3)$ then f defined by $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$ is a $(2,4)$-strongly multiplicative indexer of G.

The proofs for $(ii), (iii)$ and (iv) are similar. □

Problem 2.17. *Is it possible to embed a given connected graph G into a connected* (k,d)*-strongly multiplicatively indexable graph?.*

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