Cyclic Orthogonal Double Covers of Complete Graphs by Some Complete Multipartite Graphs

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Abstract

An *orthogonal double cover* (ODC) of the complete graph K_n is a collection $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of n subgraphs of K_n such that every edge of K_n belongs to exactly two of the G_i's and every pair of G_i's intersect in exactly one edge. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, n\}$, then $\mathcal{G}$ is an ODC of K_n by G. An ODC of K_n is *cyclic* (CODC) if the cyclic group of order n is a subgroup of its automorphism group. In this paper, we find CODCs of complete graphs by the complete multipartite graphs $K_{2,r,s}$, $K_{1,1,r,s}$ and $K_{1,1,1,1,r}$.

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1 Introduction

Let K_n be the complete graph on an n-element vertex set V. A collection $\mathcal{G} = \{G_i : i \in V\}$ of n subgraphs of K_n is an *orthogonal double cover* (briefly ODC) of K_n if it has the following properties:

1. *Double cover property*
   Every edge of K_n belongs to exactly two of the subgraphs.
2. *Orthogonality property*
   Any two distinct subgraphs intersect in exactly one edge.

The subgraphs G_i, $i \in V$, are called *pages* of $\mathcal{G}$, and if all the pages are isomorphic to some graph G, then $\mathcal{G}$ is called an *ODC of K_n by G*. An

ODC $\mathcal{G}$ of K_n by G is called *cyclic* (CODC) if the cyclic group of order n is a subgroup of the automorphism group of $\mathcal{G}$.

Let $V(K_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, the set of integers modulo n. The *length* of an edge xy, x, $y \in \mathbb{Z}_n$, is defined as $\ell(xy) = min\{| x - y |, n - | x - y |\}$.

Consider two edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$, $x_1, y_1, x_2, y_2 \in \mathbb{Z}_n$, with $\ell(e_1) = \ell(e_2)$. Their *rotationdistance* $r(e_1, e_2)$ is defined to be the shorter one of the two rotation mappings e_1 onto e_2, i.e., $r(e_1, e_2) = min\{r_1, r_2 : \{x_1 + r_1, y_1 + r_1\} = \{x_2, y_2\}, \{x_2 + r_2, y_2 + r_2\} = \{x_1, y_1\}\}$, where $+$ is addition modulo n. If $\ell(e_1) = \ell(e_2) = k$, then we denote $r(e_1, e_2)$ by $r(k)$.

A 1-1 mapping $\phi : V \to \mathbb{Z}_n$ of a graph $G = (V, E)$ with $| E | = n - 1$ is called an *orthogonal labelling* (OL) of G if the following conditions are satisfied:

1. For every $k \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, G contains exactly two edges of length k.
2. The set of all rotation-distances form a permutation of $\{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.

The following theorem by Gronau, Mullin and Rosa relates CODCs and OLs.

Theorem 1.1. [1] *A CODC of K_n by a graph G exists if and only if there exists an OL of G.*

It is natural to ask for OLs of complete graphs. In general, one can ask *ODCs* of complete graphs by complete graphs. But this question turns out to be hard. For every $n = \binom{k}{2} + 1$, an *ODC* of K_n by K_k corresponds to a biplane with block size k, that is, a symmetric $(n, k, 2)$ block design. So far, biplanes are only known for $k \in \{1, 2, 3, 4, 5, 6, 9, 11, 13\}$. In particular, there is no biplane with block size 7.

The complete k-partite graph in which partite sets are of sizes $n_1, n_2, \ldots, n_k$ is denoted by $K_{n_1, n_2, \ldots, n_k}$, and let us denote the vertices of the i-th partite set of $K_{n_1, n_2, \ldots, n_k}$ by $v_1^i, v_2^i, \ldots, v_{n_i}^i$, $i \in \{1, 2, \ldots, k\}$.

More generally, one can ask for OLs of complete k-partite graphs and *ODCs* of complete graphs by complete k-partite graphs. In [2], Sampathkumar and Simaringa obtained OLs for the complete bipartite graph $K_{r,s}$ and for the complete tripartite graph $K_{1,r,s}$. In this paper, we find OLs for the complete multipartite graphs $K_{2,r,s}$, $K_{1,1,r,s}$ and $K_{1,1,1,1,r}$.

The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

The complement of the simple graph G is denoted by G^c.

2 Results

Theorem 2.1. *Let G be a simple graph with n vertices, m edges and $n \leq m+1$. If there exists a CODC of K_{m+1} by G, then for any positive integer t, there exists a CODC of $K_{(m+1)(t+1)}$ by $(G \cup (m+1-n)K_1) \vee K_t^c$.*

Proof. Let $H = (G \cup (m+1-n)K_1) \vee K_t^c$. Given an OL $\phi : V(G) \to \mathbb{Z}_{m+1}$ of G, we define an injection $\Psi : V(H) \to \mathbb{Z}_{(m+1)(t+1)}$ as follows:

$\Psi(v) = (t+1)\phi(v)$, if $v \in V(G)$;

$\Psi(V((m+1-n)K_1)) = \{0, t+1, 2(t+1), \ldots, m(t+1)\} \backslash \{\Psi(v) : v \in V(G)\}$;

$\Psi(V(K_t^c)) = \{1, 2, \ldots, t-1, t\}$.

Case 1. m is even.

Let $A = \{(t+1), 2(t+1), \ldots, \frac{m}{2}(t+1)\}$. For every $k \in A$, H contains exactly two edges, of G, of length k. Also $\{r(k) : k \in A\} = A$.

Case 2. m is odd.

Let $A = \{(t+1), 2(t+1), \ldots, (\frac{m-1}{2})(t+1)\}$. For every $k \in A$, H contains exactly two edges, of G, of length k and it contains exactly one edge, of G, of length $(\frac{m+1}{2})(t+1)$. Further $\{r(k) : k \in A\} = A$.

Table 1. Verification of Ψ to be an OL of H.

New edges	Length	Rotation-distance
For $i \in \{1, 2, \ldots, t\}$, $\{0, i\}, \{t+1-i, t+1\}$	i	$t+1-i$
For $i \in \{1, 2, \ldots, t\}$ and $j \in \{1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor\}$, $\{i, (j+1)(t+1)\}$, $\{t+1-i, (m+1-j)(t+1)\}$	$(j+1)(t+1)$ $-i$	$j(t+1)+i$
For $j \in \{\lceil \frac{m+1}{2} \rceil, \lceil \frac{m+3}{2} \rceil, \ldots, m-1\}$ and $i \in \{1, 2, \ldots, t\}$, $\{i, (j+1)(t+1)\}$, $\{t+1-i, (m+1-j)(t+1)\}$	$(m-j)(t+1)$ $+i$	$(m-j+1)$ $(t+1)-i$

New edges	Length	Rotation-distance
For m even, $j = \frac{m}{2}$ and $i \in \{1, 2, \ldots, \lfloor \frac{t}{2} \rfloor\}$, $\{i, (j+1)(t+1)\}$, $\{t+1-i, (m+1-j)(t+1)\}$	$(m-j)(t+1)$ $+i$	$j(t+1)+i$
For m even, $j = \frac{m}{2}$, and for t odd, $i = \frac{t+1}{2}$, $\{i, (j+1)(t+1)\}$	$\frac{mt+m+t+1}{2}$	

The lengths and rotation distances for the new edges are given in Table 1. □

Corollary 2.1. *If there exist a CODC of K_{m+1} by an $(m+1)$-vertex tree T, then, for any positive integer t, there exist a CODC of $K_{(m+1)(t+1)}$ by $T \vee K_t^c$.*

Corollary 2.2. *If there exists a CODC of K_{m+1} by P_{m+1}, then there exists a CODC of K_{2m+2} by the fan $P_{m+1} \vee K_1$.*

Corollary 2.3. [2] *There exists a CODC of $K_{rs+r+s+1}$ by the complete tripartite graph $K_{1,r,s}$.*

Theorem 2.2. *The complete tripartite graph $K_{2,r,s}$ has an OL.*

Proof. Without loss of generality, assume that $r \le s$. Define $\phi : V(K_{2,r,s}) \to \mathbb{Z}_{rs+2r+2s+1}$ by $\phi(v_1^1) = 0$, $\phi(v_2^1) = s+1$, $\phi(v_i^2) = (i+1)(s+1)+i-1$, $i \in \{1, \ldots, r\}$ and $\phi(v_j^3) = j$, $j \in \{1, \ldots, s\}$. See Table 2 for verification.

Table 2. Verification of ϕ to be an OL of $K_{2,r,s}$.

Edges	Length	Rotation-distance
For $j \in \{1, ..., s\}$, $\{v_1^1, v_j^3\}, \{v_2^1, v_{s+1-j}^3\}$	j	$s+1-j$
For $i \in \{1, \ldots, \lfloor \frac{r}{2} \rfloor\}$, $\{v_1^1, v_i^2\}, \{v_2^1, v_{r+1-i}^2\}$	$i(s+2)+s$	$i(s+2)-1$
For $i \in \{\lceil \frac{r+2}{2} \rceil, \ldots, r\}$, $\{v_1^1, v_i^2\}, \{v_2^1, v_{r+1-i}^2\}$	$rs+2r+s+1$ $-i(s+2)$	$rs+2r+2s+2$ $-i(s+2)$
For r odd and $i = \frac{r+1}{2}$, $\{v_1^1, v_i^2\}, \{v_2^1, v_{r+1-i}^2\}$	$rs+2r+s+1$ $-i(s+2)$	$i(s+2)-1$

Edges	Length	Rotation-distance
For $i \in \{1,\ldots,\lfloor\frac{r}{2}\rfloor\}$ and $j \in \{1,\ldots,s\}$, $\{v_i^2, v_j^3\}, \{v_{r+1-i}^2, v_{s+1-j}^3\}$	$i(s+2)+s-j$	$i(s+2)+j-1$
For $i \in \{\lceil\frac{r+2}{2}\rceil,\ldots,r\}$ and $j \in \{1,\ldots,s\}$, $\{v_i^2, v_j^3\}, \{v_{r+1-i}^2, v_{s+1-j}^3\}$	$rs+2r+s+1$ $-i(s+2)+j$	$rs+2r+2s$ $-i(s+2)-j+2$
For r odd, $i = \frac{r+1}{2}$ and $1 \le j \le \lfloor\frac{s}{2}\rfloor$, $\{v_i^2, v_j^3\}, \{v_{r+1-i}^2, v_{s+1-j}^3\}$	$i(s+2)+s-j$	$i(s+2)+j-1$
If both r and s are odd, $\{v_{\frac{r+1}{2}}^2, v_{\frac{s+1}{2}}^3\}$	$\frac{rs+1}{2}+r+s$	

□

Theorem 2.3. *The complete 4-partite graph $K_{1,1,r,s}$ has an OL.*

Proof. Without loss of generality, assume that $r \le s$. Define
$\phi : V(K_{1,1,r,s}) \to \mathbb{Z}_{rs+2r+2s+2}$ by
$\phi(v_1^1) = 0$,
$\phi(v_1^2) = s+1$,
$\phi(v_i^3) = (i+1)(s+1)+i$, $i \in \{1,\ldots,r\}$ and
$\phi(v_j^4) = j$, $j \in \{1,\ldots,s\}$.
See Table 3 for verification.

Table 3. Verification of ϕ to be an OL of $K_{1,1,r,s}$.

Edges	Length	Rotation-distance
For $j \in \{1,\ldots,s\}$, $\{v_1^1, v_j^4\}, \{v_1^2, v_{s+1-j}^4\}$	j	$s+1-j$
$\{v_1^1, v_1^2\}, \{v_1^1, v_r^3\}$	$s+1$	$s+1$
For $i \in \{1,\ldots,\lfloor\frac{r-1}{2}\rfloor\}$, $\{v_1^1, v_i^3\}, \{v_1^1, v_{r-i}^3\}$	$i(s+2)+s+1$	$i(s+2)+s+1$
For $i \in \{1,\ldots,\lfloor\frac{r}{2}\rfloor\}$, $\{v_1^4, v_i^3\}, \{v_1^2, v_{r+1-i}^3\}$	$i(s+2)+s$	$i(s+2)$
For $i = \lfloor\frac{r+2}{2}\rfloor$, $\{v_1^4, v_i^3\}, \{v_1^2, v_{r+1-i}^3\}$	$rs+2r+s+2$ $-i(s+2)$	$i(s+2)$

Edges	Length	Rotation-distance
For r odd, $i=\frac{r+1}{2}$ and $s\geq 4$, $\{v_1^4,v_i^3\},\{v_1^2,v_{r+1-i}^3\}$	$i(s+2)+s$	$i(s+2)$
For r odd, $i=\frac{r+1}{2}$ and $s\leq 3$, $\{v_1^4,v_i^3\},\{v_1^2,v_{r+1-i}^3\}$	$rs+2r+s+2$ $-i(s+2)$	$i(s+2)$
For $i\in\{\lceil\frac{r+2}{2}\rceil,\ldots,r\}$, $\{v_1^4,v_i^3\},\{v_1^2,v_{r+1-i}^3\}$	$rs+2r+s+2$ $-i(s+2)$	$rs+2r+2s+2$ $-i(s+2)$
For $i\in\{1,\ldots,\lfloor\frac{r}{2}\rfloor\}$ and $j\in\{2,\ldots,s\}$, $\{v_i^3,v_j^4\},\{v_{r+1-i}^3,v_{s+2-j}^4\}$	$i(s+2)-j+s+1$	$i(s+2)+j-1$
For $i\in\{\lceil\frac{r+2}{2}\rceil,\ldots,r\}$ and $j\in\{2,\ldots,s\}$, $\{v_i^3,v_j^4\},\{v_{r+1-i}^3,v_{s+2-j}^4\}$	$rs+2r+s+1$ $-i(s+2)+j$	$rs+2r+2s+3$ $-i(s+2)-j$
For r odd, $i=\frac{r+1}{2}$ and $2\leq j\leq\lfloor\frac{s+1}{2}\rfloor$, $\{v_i^3,v_j^4\},\{v_{r+1-i}^3,v_{s+2-j}^4\}$	$i(s+2)-j+s+1$	$i(s+2)+j-1$
If r is odd and s is even, $\{v_{\frac{r+1}{2}}^3,v_{\frac{s+2}{2}}^4\}$	$\frac{rs}{2}+r+s+1$	
If r is even, $\{v_1^1,v_{\frac{r}{2}}^3\}$	$\frac{rs}{2}+r+s+1$	

□

Theorem 2.4. *The complete 5-partite graph $K_{1,1,1,1,r}$ has an OL.*

Proof. Define $\phi : V(K_{1,1,1,1,r}) \to \mathbb{Z}_{4r+7}$ by $\phi(v_1^1) = 0$, $\phi(v_1^2) = r+1$, $\phi(v_1^3) = 2r+2$, $\phi(v_1^4) = 3r+4$ and $\phi(v_i^5) = i$, $i \in \{1,\ldots,r\}$. See Tables 4(a) and 4(b) for verification.

Table 4(a). Verification of ϕ to be an OL of $K_{1,1,1,1,1}$

Edges	Length	Rotation-distance
$\{v_1^1,v_1^5\},\{v_1^2,v_1^5\}$	1	1
$\{v_1^1,v_1^2\},\{v_1^2,v_1^3\}$	2	2
$\{v_1^3,v_1^4\},\{v_1^3,v_1^5\}$	3	3
$\{v_1^1,v_1^3\},\{v_1^1,v_1^4\}$	4	4
$\{v_1^2,v_1^4\},\{v_1^4,v_1^5\}$	5	5

Table 4(b). Verification of ϕ to be an OL of $K_{1,1,1,1,r}$, $r \geq 2$.

Edges	Length	Rotation-distance
For $i \in \{1,...,r\}$, $\{v_1^1, v_i^5\}, \{v_1^2, v_{r+1-i}^5\}$	i	$r+1-i$
$\{v_1^1, v_1^2\}, \{v_1^2, v_1^3\}$	$r+1$	$r+1$
$\{v_1^3, v_1^4\}, \{v_1^3, v_r^5\}$	$r+2$	$r+2$
$\{v_1^1, v_1^4\}, \{v_1^3, v_{r-1}^5\}$	$r+3$	$2r+2$
For $i \in \{1,...,r-2\}$ and $r \geq 3$, $\{v_1^4, v_i^5\}, \{v_1^3, v_{r-1-i}^5\}$	$r+3+i$	$2r+2-i$
$\{v_1^1, v_1^3\}, \{v_1^4, v_{r-1}^5\}$	$2r+2$	$r+3$
$\{v_1^2, v_1^4\}, \{v_1^4, v_r^5\}$	$2r+3$	$2r+3$

□

Corollary 2.4.

1. *There exists a CODC of* $K_{2r+2s+rs+1}$ *by* $K_{2,r,s}$.
2. *There exists a CODC of* $K_{2r+2s+rs+2}$ *by* $K_{1,1,r,s}$.
3. *There exists a CODC of* K_{4r+7} *by* $K_{1,1,1,1,r}$.

In conclusion, we propose the following problem.

Problem 2.1. *Which complete k-partite graphs* $K_{n_1,n_2,...,n_k}$ *admit an OL?*

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