Edge Detour Graphs

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Abstract

For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. A set $S \subseteq V$ is called an *edge detour set* if every edge in G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an *edge detour basis* of G. A connected graph G is called an *edge detour graph* if it has an edge detour set. Certain general properties of these concepts are studied. The edge detour numbers of certain classes of graphs are determined. We show that for each pair of integers k and p with $2 \leq k < p$, there is an edge detour graph G of order p with $dn_1(G) = k$. An edge detour set S, no proper subset of which is an edge detour set, is a *minimal edge detour set*. The *upper edge detour number* $dn_1^+(G)$ of a graph G is the maximum cardinality of a minimal edge detour set of G. We determine the upper edge detour numbers of certain classes of graphs. We also show that for every pair a, b of integers with $2 \leq a \leq b$, there is an edge detour graph G with $dn_1(G) = a$ and $dn_1^+(G) = b$.

Keywords. Detour, detour set, detour number, edge detour set, edge detour basis, edge detour number.
2000 Mathematics Subject Classification: 05C12

1 Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1, 4].

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. It is known that the detour distance is a metric on the vertex set V. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand et.al. [2].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied by Chartrand et.al. [3].

For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v to $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. Thus every end-block is a branch of G at the cut-vertex v of G. The following theorems are used in the sequel.

Theorem 1.1. [3] *Every end-vertex of a non-trivial connected graph G belongs to every detour set of G. Also if the set S of all end-vertices of G is a detour set, then S is the unique detour basis for G.*

Theorem 1.2. [3] *If T is a tree with k end-vertices, then $dn(T) = k$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2 Edge Detour Number of a Graph

In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 2.1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivates us to introduce the concept of *edge detour graphs*.

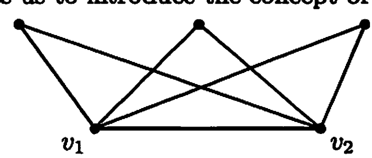

Figure 2.1: G

Definition 2.1. *Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V$ is called an edge detour set of G if every edge in G lies on a detour joining a pair of vertices of S. The edge detour number $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an edge detour basis of G. A graph G is called an edge detour graph if it has an edge detour set.*

Example 2.2. *For the graph G given in Figure 2.2, it is clear that no two element subset of V is an edge detour set of G. It is easily seen that $S_1 = \{v_1, v_2, v_4\}$ is an edge detour set of G so that S_1 is an edge detour basis of G and so $dn_1(G) = 3$. Thus G is an edge detour graph. Also $S_2 = \{v_1, v_2, v_5\}$ is another edge detour basis of G. Thus there can be more than one edge detour basis for a graph G.*

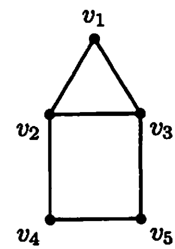

Figure 2.2: G

The graph G given in Figure 2.1 does not contain an edge detour set and so G is not an edge detour graph.

Remark 2.3. *For the graph G given in Figure 2.3, the sets $S_1 = \{u, x\}$ and $S_2 = \{u, v, x, y\}$ are detour basis and edge detour basis of G respectively and hence $dn(G) = 2$ and $dn_1(G) = 4$. Thus the detour number and the edge detour number of a graph G are different.*

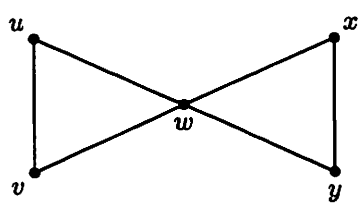

Figure 2.3: G

Theorem 2.4. *For any edge detour graph G of order $p \geq 2$, $2 \leq dn_1(G) \leq p$.*

Proof. An edge detour set needs at least two vertices so that $dn_1(G) \geq 2$ and the set of all vertices of G is an edge detour set of G so that $dn_1(G) \leq p$. Thus $2 \leq dn_1(G) \leq p$. □

Remark 2.5. *The bounds in Theorem 2.4 are sharp. For the complete graph K_p ($p = 2$ or 3), $dn_1(K_p) = p$. The set of two end-vertices of a path $P_n (n \geq 2)$ is its unique edge detour set so that $dn_1(P_n) = 2$. Thus the complete graph K_p ($p = 2$ or 3) has the largest possible edge detour number p and the non-trivial paths have the smallest edge detour number 2.*

This suggests the following question.

Problem 2.6. *Is the upper bound in Theorem 2.4 sharp if $p \geq 4$?*

Definition 2.7. *A vertex v in an edge detour graph G is an edge detour vertex if v belongs to every edge detour basis of G. If G has a unique edge detour basis S, then every vertex in S is an edge detour vertex of G.*

Example 2.8. *For the graph G given in Figure 2.2, $S_1 = \{v_1, v_2, v_4\}$, $S_2 = \{v_1, v_2, v_5\}$, $S_3 = \{v_1, v_3, v_4\}$, $S_4 = \{v_1, v_3, v_5\}$ and $S_5 = \{v_1, v_4, v_5\}$ are the only edge detour bases for G so that the vertex v_1 is the unique edge detour vertex in G. For the graph G given in Figure 2.4 (a), $S = \{u, v, w\}$ is the unique edge detour basis so that every vertex of S is an edge detour vertex in G. For the graph G given Figure 2.4 (b), $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, w\}$ are the only edge detour bases of G so that u and v are edge detour vertices in G.*

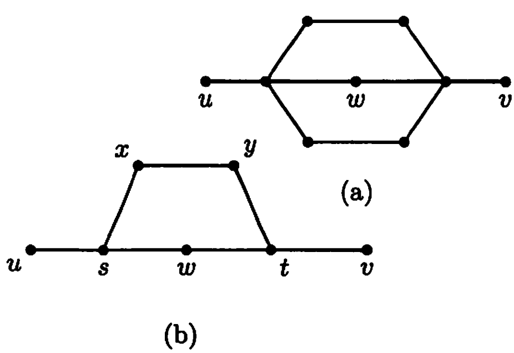

Figure 2.4: G

Theorem 2.9. *If G is an edge detour graph of order $p \geq 3$ such that $\{u, v\}$ is an edge detour basis of G, then u and v are not adjacent.*

Proof. Suppose that u and v are adjacent. Let e be an edge such that $e \neq uv$. If $D(u, v) = 1$, then e cannot lie on any u–v detour. If $D(u, v) \geq 2$, then the edge uv cannot be on any u–v detour. Thus $\{u, v\}$ is not an edge detour set of G, which is a contradiction. Hence u and v are not adjacent. □

In the following theorem we show that there are certain vertices in a non-trivial edge detour graph G that are edge detour vertices of G.

Theorem 2.10. *Every end-vertex of an edge detour graph G belongs to every edge detour set of G. Also if the set S of all end-vertices of G is an edge detour set, then S is the unique edge detour basis for G.*

Proof. Let v be an end-vertex of G and uv an edge in G incident with the end-vertex v. Then uv is either the initial edge or the terminal edge of any detour containing the edge uv. Hence it follows that v belongs to every edge detour set of G. If S is the set of all end-vertices of G, then by the first part of this Theorem $dn_1(G) \geq |S|$. If S is an edge detour set of G, then $dn_1(G) \leq |S|$. Hence $dn_1(G) = |S|$ and S is the unique edge detour basis for G. □

Corollary 2.11. *If T is a tree with k end-vertices, then $dn(T) = dn_1(T) = k$.*

Proof. It is easy to see that the set of all end-vertices of T is the unique edge detour basis and so T is an edge detour graph. Now the result follows from Theorems 1.2 and 2.10. □

Corollary 2.12. *Every end-vertex of an edge detour graph G is a detour vertex and an edge detour vertex of G.*

Proof. This follows from Theorems 1.1, and 2.10. □

Corollary 2.13. *For any edge detour graph G with k end-vertices, $max\{2, k\} \leq dn_1(G) \leq p$.*

Proof. This follows from the Theorems 2.4 and 2.10. □

Theorem 2.14. *Let G be an edge detour graph with cut-vertices and S an edge detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Proof. Let v be a cut-vertex of G such that one of the components, say C of $G - v$ contains no vertex of S. Then by Theorem 2.10, C does not contain any end-vertex of G. Hence C contains at least one edge, say uw. Since S is an edge detour set, there exist vertices $x, y \in S$ such that uw lies on some x–y detour $P : x = u_0, u_1, u_2, \ldots, u, w, \ldots, u_t = y$ in G. Let

P_1 be the x–u subpath of P and P_2 be the u–y subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contain v so that P is not a detour, which is a contradiction. Thus every component of $G - v$ contains an element of S. □

Corollary 2.15. *Let G be an edge detour graph with cut-vertices and S an edge detour set of G. Then every branch of G contains an element of S.*

Remark 2.16. *By Corollary* 2.15, *if S is an edge detour set of an edge detour graph G, then every end-block of G must contain at least one element of S. However, it is possible that some blocks of G that are not end-blocks must contain an element of S as well. For example, consider the graph G of Figure* 2.4 (b), *where the cycle C_5: x, y, t, w, s, x is a block of G that is not an end-block. By Theorem 2.10, every edge detour set of G must contain u and v. Since the u–v detour does not contain the edges sw and wt, it follows that $\{u, v\}$ is not an edge detour set. Thus every edge detour set of G must contain at least one vertex from the block C_5.*

Corollary 2.17. *If G is an edge detour graph with $k \geq 2$ end-blocks, then $dn_1(G) \geq k$.*

Corollary 2.18. *If G is an edge detour graph with a cut-vertex v and the number of components of $G - v$ is r, then $dn_1(G) \geq r$.*

Theorem 2.19. *Let G be an edge detour graph with cut-vertices. Then no cut-vertex of G belongs to any edge detour basis.*

Proof. Suppose that S is an edge detour basis that contains a cut-vertex v of G. Let $G_1, G_2, \ldots, G_k$ ($k \geq 2$) be the components of $G - v$ and let B_1, $B_2, \ldots, B_k$ be the branches of G at v such that B_i contains G_i ($1 \leq i \leq k$). Then v is adjacent to at least one vertex of G_i for each i ($1 \leq i \leq k$). Also by Theorem 2.14, each component G_i contains an element of S, say u_i. Let $S' = S - \{v\}$. We show that S' is an edge detour set of G. Let uw be an edge of G which lies on a detour P joining a pair of vertices, say x and v of S. We may assume that $x \in V(G_1)$ and so $V(P) \subseteq V(B_1)$. Let $P_1 : x = x_0$, $x_1, \ldots, x_m = v$ be a detour containing the edge uw and $P_2 : v = v_0$, v_1, $\ldots, v_n = u_2$ any v–u_2 detour in G. Then since v is a cut-vertex of G, the path Q: $x = x_0, x_1, \ldots, x_m = v = v_0, v_1, \ldots, v_n = u_2$ is an x–u_2 detour containing the edge uw. We have $x \neq v$ and $u_2 \neq v$. Thus we have shown that every edge that lies on a detour joining a pair of vertices x and v of S also lies on a detour joining a pair of vertices of S'. Hence it follows that every edge of G is contained in a detour joining a pair of vertices of S' so that S' is an edge detour set of G. Since $|S'| = |S| - 1$, this contradicts that S is an edge detour basis of G. Therefore $v \notin S$ and hence no cut-vertex of G belongs to any edge detour basis of G. □

For the graph H and an integer $k \geq 1$, we write kH for the union of the k disjoint copies of H.

Theorem 2.20. *Let $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 5$ such that $r \geq 2$, each $n_i \geq 2$ and $n_1 + n_2 + \cdots + n_r + k = p - 1$. Then G is an edge detour graph and $dn_1(G) = 2r + k$.*

Proof. **Step 1.** We prove that G is an edge detour graph. Let S be the set formed by taking exactly two vertices v_{i_1}, v_{i_2} from each component K_{n_i} of $G - v$ $(1 \leq i \leq r)$ and the k end-vertices $v_1, v_2, \ldots, v_k$ of G so that $|S| = 2r + k$. We show that S is an edge detour set of G. Let uw be any edge in G. Then uw lies in any one of the branches, say B_i of G at v containing K_{n_i} $(1 \leq i \leq r)$ or $uw = vv_j$ $(1 \leq j \leq k)$. If $uw \neq v_{i_1}v_{i_2}$, since B_i is complete uw lies on some v_{i_1}–v_{i_2} detour in B_i itself. If $uw = v_{i_1}v_{i_2}$, let $P_1 : v_{i_1}, v_{i_2}, \ldots, v$ be a v_{i_1}–v detour in B_i and P_2 any v–v_{j_1} detour in a branch B_j $(i \neq j)$, where $v_{j_1} \in S$. Then since v is a cut-vertex of G, $P_1 \cup P_2$ is a v_{i_1}–v_{j_1} detour in G which contains the edge uw. Now, if $uw = vv_j$ $(1 \leq j \leq k)$, then uw lies on v_j–v_{i_1} $(1 \leq i \leq r)$ detour. Hence S is an edge detour set of G and so G is an edge detour graph.

Step 2. We prove that S is an edge detour basis of G. Let T be any set of vertices of G such that $|T| < |S|$. Then $|T| \leq 2r + k - 1$.

Case 1. T contains all the end-vertices of G. Then since $|T| \leq 2r + k - 1$, there is a component of $G - v$, say K_{n_i} $(i \leq i \leq r)$ such that T contains at most one vertex of K_{n_i}. If T contains no vertex from the component K_{n_i}, then no edge of the K_{n_i} lies on any x–y detour, where $x, y \in T$. If T contains exactly one vertex, say u of K_{n_i}, then since $n_i \geq 2$, the edge uv does not lie on any x–y detour, where x, $y \in T$. Thus T is not an edge detour set of G.

Case 2. If T does not contain at least one end-vertex, say v_1. Then clearly the edge vv_1 does not lie on any x–y detour for x, $y \in T$ so that T is not an edge detour set of G. Thus it follows that S is an edge detour basis of G and hence $dn_1(G) = 2r + k$. □

Remark 2.21. *If the blocks of the graph G in Theorem 2.20 are not complete, then the theorem is not true. For the graph G given in Figure 2.5, $\{v_3, v_4, v_7, v_9\}$ is an edge detour basis so that $dn_1(G) = 4$.*

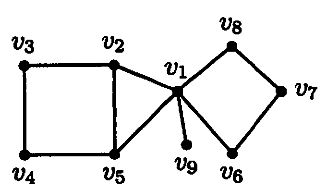

Figure 2.5: G

In the following theorem we give certain graphs G for which $dn_1(G) = 2$ and $dn_1(G) = 3$.

Theorem 2.22. *If G is the complete graph K_2 or $K_p - e$ $(p \geq 3)$ or an even cycle C_n or a non-trivial path P_n or a complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$, then G is an edge detour graph and $dn_1(G) = 2$.*

Proof. It is clear that the vertex set of K_2, those two vertices of degree $p-2$ in $K_p - e$ $(p \geq 3)$, any set of two antipodal vertices in an even cycle C_n, the two end-vertices of a non-trivial path P_n and any set of two non-adjacent vertices of $K_{m,n}$ $(m, n \geq 2)$ are edge detour bases in K_2, $K_p - e$ $(p \geq 3)$, an even cycle C_n, P_n and $K_{m,n}$ $(m, n \geq 2)$ respectively. Hence the result follows. □

Theorem 2.23. *If G is the complete graph K_p $(p \geq 3)$ or an odd cycle C_n, then G is an edge detour graph and $dn_1(G) = 3$.*

Proof. For any two element subset $\{u, v\}$ of $V(K_p)$ $(p \geq 3)$, all the edges of K_p other than uv lie on a u–v detour. Hence it follows that no two element subset of $V(K_p)$ $(p \geq 3)$ is an edge detour set and any three element subset of $V(K_p)$ $(p \geq 3)$ is an edge detour set of K_p $(p \geq 3)$. Thus K_p is an edge detour graph and $dn_1(K_p) = 3$ for $p \geq 3$.

If $\{u, v\}$ is any set of two vertices of an odd cycle C_n, then no edge of the u–v geodesic lie on the u–v detour in C_n and so no two element subset of $V(C_n)$ is an edge detour set of C_n. Let $S = \{u, v, w\} \subseteq V$ be any set of three vertices of C_n .Then every edge in C_n lies on any one of the u–v, v–w or u–w detours so that S is an edge detour basis of C_n. Hence the result follows. □

The following theorems give realization results.

Theorem 2.24. *For each pair of integers k and p with $2 \leq k < p$, there exists an edge detour graph G of order p with $dn_1(G) = k$.*

Proof. For $2 \leq k < p$, let P be a path of order $p - k + 2$. Then the graph G obtained from P by adding $k - 2$ new vertices to P and joining them to any cut-vertex of P is a tree of order p. Then G is an edge detour graph and so by Corollary 2.11, $dn_1(G) = k$. □

Theorem 2.25. *For each positive integer $k \geq 2$, there exists an edge detour graph G and a vertex v of degree k in G such that v belongs to an edge detour basis of G and $dn_1(G) = k$.*

Proof. For $k = 2$, let G be the cycle C_4. Then a set of two antipodal vertices of G satisfies the requirements of the theorem. For $k \geq 3$, let G be the graph obtained from the complete graph K_{k+1}, where

$V(K_{k+1}) = \{v_1, v_2, \ldots, v_k, v_{k+1}\}$, by adding $k-2$ new vertices u_1, u_2, $\ldots$, u_{k-2} and joining each u_i $(1 \le i \le k-2)$ to v_1. Then $\deg v_2 = k$. Let $S = \{u_1, u_2, \ldots, u_{k-2}\}$. Then neither S nor $S \cup \{v_i\}$ $(1 \le i \le k+1)$ is an edge detour set of G. However, $S \cup \{v_2, v_3\}$ is an edge detour set of G and hence by Theorem 2.10, $S \cup \{v_2, v_3\}$ is an edge detour basis of G so that $dn_1(G) = k$. □

3 Minimal Edge Detour Sets in a Graph

Definition 3.1. *An edge detour set S in an edge detour graph G is called a minimal edge detour set of G if no proper subset of S is an edge detour set of G.*

Example 3.2. *For the graph G given in Figure 3.1, $S_1 = \{v_1, v_2, v_7, v_{10}\}$, $S_2 = \{v_1, v_2, v_7, v_4, v_8\}$, $S_3 = \{v_1, v_2, v_7, v_4, v_9\}$, $S_4 = \{v_1, v_2, v_7, v_5, v_8\}$ and $S_5 = \{v_1, v_2, v_7, v_5, v_9\}$ are the minimal edge detour sets of G.*

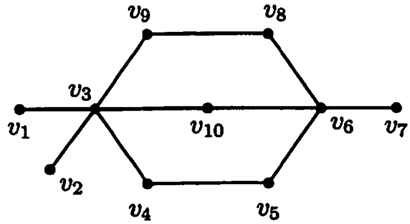

Figure 3.1: G

Remark 3.3. *Every minimum edge detour set is a minimal edge detour set, but the converse is not true. For the graph G given in Figure 3.1, $S_2 = \{v_1, v_2, v_7, v_4, v_8\}$ is a minimal edge detour set of G but not a minimum edge detour set of G.*

Definition 3.4. *For an edge detour graph G, the upper edge detour number $dn_1^+(G)$ of G is defined to be the maximum cardinality of a minimal edge detour set of G.*

Example 3.5. *For the graph G given in Figure 3.1, it follows from Example 3.2 that $dn_1(G) = 4$ and $dn_1^+(G) = 5$.*

Theorem 3.6. *For any edge detour graph G, $dn_1(G) \le dn_1^+(G)$.*

Proof. Let S be any edge detour basis of an edge detour graph G. Then S is also a minimal edge detour set of G and hence the result follows. □

Remark 3.7. *The bound in Theorem* 3.6 *is sharp. For any non-trivial path P,* $dn_1(P) = dn_1^+(P) = 2$. *Also for the graph G given in Figure* 3.1, $dn_1(G) < dn_1^+(G)$.

Theorem 3.8. *If S is a minimal edge detour set in an edge detour graph G, then no cut-vertex of G belongs to S.*

Proof. Proof is similar to that of Theorem 2.19. □

In the following theorem, we give a class of graphs for which these two parameters are equal.

Theorem 3.9.

(a) *If G is the complete graph* $K_p (p \geq 3)$, *then* $dn_1(G) = dn_1^+(G) = 3$.

(b) *If G is the complete bipartite graph* $K_{m,n}$ $(m, n \geq 2)$, *then* $dn_1(G) = dn_1^+(G) = 2$.

(c) *If G is an odd cycle* C_p $(p \geq 3)$, *then* $dn_1(G) = dn_1^+(G) = 3$.

(d) *If G is a tree with k end-vertices, then* $dn_1(G) = dn_1^+(G) = k$.

Proof. a) By Theorem 2.23, $dn_1(G) = 3$ and so $dn_1^+(G) \geq 3$. Let $S \subseteq V$ be an edge detour set of G with $|S| \geq 4$. As in the proof of Theorem 2.23, any set of three vertices is an edge detour set of G so that S cannot be a minimal edge detour set of G and hence the result follows.
b) By Theorem 2.22, $dn_1(G) = 2$ and so $dn_1^+(G) \geq 2$. Let $S \subseteq V$ be an edge detour set of G such that $|S| \geq 3$. Then there exists a subset $S_1 = \{u, v\}$ of S such that u and v are nonadjacent and hence as in the proof of Theorem 2.22, S_1 is an edge detour set of G. Thus S is not a minimal edge detour set of G and hence the result follows.
c) If $G = K_3$, then the result follows from Theorem 3.9 (a). If $p \geq 5$, then by Theorem 2.23, $dn_1(G) = 3$ and so $dn_1^+(G) \geq 3$. Let $S \subseteq V$ be an edge detour set of G with $|S| \geq 4$. Then as in the proof of Theorem 2.23, any set of three vertices is an edge detour set of G so that S cannot be a minimal edge detour set of G and hence the result follows.
d) The set of all end-vertices of G is the unique edge detour basis of G and so the result follows from Theorem 2.10 and Corollary 2.11. □

Theorem 3.10. *Let G be an even cycle of order* $p \geq 4$. *A set* $S = \{u, v\}$ *is an edge detour set of G if and only if u and v are antipodal vertices in G.*

Proof. If u and v are antipodal, then every edge e of G lies on a u–v detour in G. Thus S is an edge detour set of G. Conversely, assume that S is an edge detour set of G. If u and v are not antipodal, then the edges of u–v geodesic do not lie on the u–v detour in G so that S is not an edge detour set of G, which is a contradiction. □

Theorem 3.11. *If G is an even cycle C_p, then $dn_1(G) = 2$, $dn_1^+(G) = 2$ for $p = 4$ and $dn_1^+(G) = 3$ for $p \geq 6$.*

Proof. By Theorem 2.22, $dn_1(G) = 2$. Now, we show that $dn_1^+(G) = 2$ for $p = 4$ and $dn_1^+(G) = 3$ for $p \geq 6$. We consider two cases.
Case 1. $p = 4$. Since $G = K_{2,2}$, by Theorem 3.9 (b), $dn_1^+(G) = 2$.
Case 2. $p \geq 6$. Let $S = \{u, v, w\} \subseteq V$ be such that no two vertices of S are pairwise antipodal in G. If w lies on the u–v detour, then all the edges of the cycle that constitute the u–v detour lie on the u–v detour and the edges on the u–v geodesic lie either on the w–u detour or w–v detour. Similarly, if w lies on the u–v geodesic, then all the edges of the cycle that constitute the u–v detour lie on the u–v detour and the edges on the u–v geodesic lie either on the w–u detour or w–v detour and hence it follows that S is an edge detour set of G. Since no two vertices of S are pairwise antipodal, it follows from Theorem 3.10 that no proper subset of two vertices of S is an edge detour set of G. Thus S is a minimal edge detour set of G and so $dn_1^+(G) \geq |S| = 3$.

Now, if $dn_1^+(G) > 3$, then let M be a minimal edge detour set of G with $|M| \geq 4$. Let S be any subset of M such that $|S| = 3$. If S contains a pair of antipodal vertices, then by Theorem 3.10, M is not a minimal edge detour set of G, which is a contradiction. Otherwise, no two vertices of S are pairwise antipodal. Then, as in the first part of Case 2 of this theorem, S is an edge detour set of G so that M is not a minimal edge detour set of G, which is a contradiction. Hence $dn_1^+(G) = 3$. □

The Theorems 3.9 and 3.11 give a partial answer to the following problem.

Problem 3.12. *Characterize graphs G for which $dn_1(G) = dn_1^+(G)$.*

Theorem 3.13. *For every pair a, b of integers with $2 \leq a \leq b$, there exists an edge detour graph G with $dn_1(G) = a$ and $dn_1^+(G) = b$.*

Proof. Let $a = b$. Then for any tree T with a end-vertices $dn_1(G) = dn_1^+(G) = a$, by Theorem 3.9 (d). So, assume that $2 \leq a < b$. Let C: $v_1, v_2, v_3, v_4, v_5, v_6, v_1$ be the cycle of length 6. The graph G is obtained from C by adding $b + 1$ new vertices $z_1, z_2, \ldots, z_{a-1}, w, x_1, x_2, \ldots, x_{b-a+1}$ and joining each z_i $(1 \leq i \leq a - 1)$ to v_2, w to v_1, v_3 and v_5 and each x_i $(1 \leq i \leq b - a + 1)$ to both v_1 and v_3. The graph G is shown in Figure 3.2.

Let $X = \{x_1, x_2, \ldots, x_{b-a+1}\}$, $Y = \{v_1, v_2, v_3\}$, $W = \{v_4, v_5, v_6, w\}$ and $Z = \{z_1, z_2, \ldots, z_{a-1}\}$.

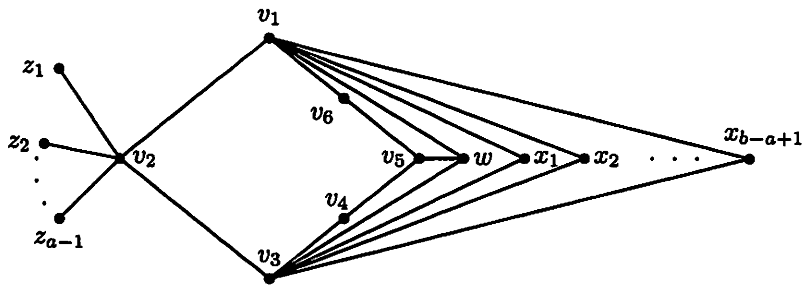

Figure 3.2: G

First, we show that G is an edge detour graph. Let $S = Z \cup \{v\}$ where $v \in W$. Then, for $v \in W$, $D(z_i, v) = 6$ if $v = v_5$ and $D(z_i, v) = 7$ if $v \neq v_5$ $(1 \leq i \leq a-1)$. Since every edge of G lies on some z_i–v $(1 \leq i \leq a-1, v \in W)$ detour, S is an edge detour set of G and so G is an edge detour graph.

Now, we show that $dn_1(G) = a$. By Theorem 2.10, every edge detour set of G contains Z. Clearly, Z is not an edge detour set of G and so $dn_1(G) \geq |Z| + 1 = a$. Also, as above S is an edge detour set of G and so $dn_1(G) \leq |S| = a$. Therefore, $dn_1(G) = a$.

Next, we show that $dn_1^+(G) = b$. Let $S = X \cup Z$. Since $D(z_i, x_j) = 7$ $(1 \leq i \leq a-1, 1 \leq j \leq b-a+1)$ and every edge of G lies on some z_i–x_j detour, S is an edge detour set of G. We claim that S is a minimal edge detour set of G. Assume, to the contrary, that S is not a minimal edge detour set of G. Then there is a proper subset T of S such that T is an edge detour set of G. Since T is a proper subset of S, there exists a vertex $s \in S$ and $s \notin T$. Since every edge detour set contains all end-vertices of G, we must have $s = x_i$ for $1 \leq i \leq b-a+1$, say $s = x_1$. Since the edge x_1v_1 does not lie on any x–y detour for $x, y \in T$, it follows that T is not an edge detour set of G, which is a contradiction. Thus S is a minimal edge detour set of G and so $dn_1^+(G) \geq |S| = a - 1 + b - a + 1 = b$. Assume, to the contrary, that $dn_1^+(G) > b$. Let M be a minimal edge detour set of G with $|M| > b$. Then there exists at least one vertex, say $v \in M$ such that $v \notin S = X \cup Z$. Thus $v \in W \cup Y = \{v_1, v_2, v_3, v_4, v_5, v_6, w\}$.

Claim 1. $M \cap W = \phi$. Assume, to the contrary, that $M \cap W \neq \phi$. Then there exists a vertex $v \in M$ and $v \in W$. Clearly, $Z \cup \{v\}$ is a proper subset of M and an edge detour set of G by the first part of the proof of the theorem. This is a contradiction to the fact that M is a minimal edge detour set of G.

Claim 2. $X \not\subseteq M$. Assume, the contrary, that $X \subseteq M$. Then $X \cup Z$ is a proper subset of M and an edge detour set of G, which is a contradiction to M a minimal edge detour set of G.

Claim 3. $M \cap X \neq \phi$. Assume, to the contrary, that $M \cap X = \phi$. Then $M = Z \cup T$, $T \subseteq Y$ and $T \neq \phi$. Then the edge v_1x_i (or v_3x_i) $(1 \leq i \leq b-a+1)$ does not lie on any x–y detour for x, $y \in M$. Hence M is not an edge detour set of G, which is a contradiction. Thus we conclude that $M = Z \cup T \cup X'$, where $T \subseteq Y, T \neq \phi$ and X' is a proper subset of X. Therefore, there exists a vertex $v \in X$ such that $v \notin M$, say $v = x_1$. Then the edge x_1v_1 does not lie on any x–y detour in G for x, $y \in M$. Hence M is not an edge detour set of G, which is a contradiction. Therefore, $dn_1^+(G) = b$. □

Remark 3.14. *The graph G of Figure 3.2 contains exactly five minimal edge detour sets namely $Z \cup \{v\}$, where $v \in \{v_4, v_5, v_6, w\}$ and $X \cup Z$. Hence this example shows that there is no "Intermediate Value Theorem" for minimal edge detour sets in edge detour graphs, that is, if k is an integer such that $dn_1(G) < k < dn_1^+(G)$, then there need not exist a minimal edge detour set of cardinality k in G.*

Using the structure of the graph G constructed in the proof of Theorem 3.13, we can obtain a graph H_n of order n with $dn_1(G) = 2$ and $dn_1^+(G) = n - 7$ for all $n \geq 9$. Thus we have the following.

Theorem 3.15. *There is an infinite sequence $\{H_n\}$ of edge detour graphs H_n of order $n \geq 9$ such that $dn_1(H_n) = 2$, $\lim_{n\to\infty} \frac{dn_1(H_n)}{n} = 0$ and $\lim_{n\to\infty} \frac{dn_1^+(H_n)}{n} = 1$.*

Proof. Let $n \geq 9$ and C: $v_1, v_2, v_3, v_4, v_5, v_6, v_1$ be the cycle of length 6. Then the graph H_n is obtained from C by adding $n - 6$ new vertices z, w, $x_1, x_2, \ldots, x_{n-8}$ and joining z to v_2, w to eachv_1, v_3 and v_5 and each x_i $(1 \leq i \leq n-8)$ to both v_1 and v_3. The graph H_n is shown in Figure 3.3.

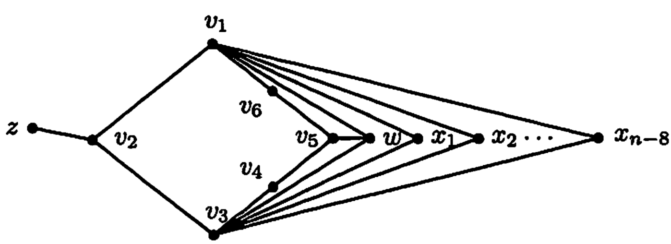

Figure 3.3: H_n

Let $X = \{x_1, x_2, \ldots, x_{n-8}\}$, $Y = \{v_1, v_2, v_3\}$, $W = \{v_4, v_5, v_6, w\}$ and $Z = \{z\}$. It is clear from the proof of Theorem 3.13 that the graph G contains exactly five minimal edge detour sets namely $Z \cup \{v\}$, where $v \in W$ and $X \cup Z$ so that $dn_1(H_n) = 2$ and $dn_1^+(H_n) = n - 7$. Hence the theorem follows. □

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