The Connected Detour Number of a Graph

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Abstract

For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. A set $S \subseteq V$ is called a *detour set* of G if every vertex in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is a *detour basis* of G. A set $S \subseteq V$ is called a *connected detour set* of G if S is detour set of G and the subgraph $G[S]$ induced by S is connected. The *connected detour number* $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a *connected detour basis* of G. Certain general properties of these concepts are studied. The connected detour numbers of certain classes of graphs are determined. The relationship of the connected detour number with the detour diameter is discussed and it is proved that for each triple D, k, p of integers with $3 \leq k \leq p - D + 1$ and $D \geq 4$, there is a connected graph G of order p with detour diameter D and $cdn(G) = k$. A connected detour set S, no proper subset of which is a connected detour set, is a *minimal connected detour set*. The *upper connected detour number* $cdn^+(G)$ of a graph G is the maximum cardinality of a minimal connected detour set of G. It is shown that for every pair a, b of integers with $5 \leq a \leq b$, there is a connected graph G with $cdn(G) = a$ and $cdn^+(G) = b$.

Keywords. detour, connected detour set, connected detour basis, connected detour number.
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1 Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1, 4].

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. It is known that the detour distance is a metric on the vertex set V. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand et al. [2].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied by Chartrand et al. [3].

For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. Thus every end-block is a branch of G at the cut-vertex v of G. The following theorem is used in the sequel.

Theorem 1.1. [3] *Every end-vertex of a non-trivial connected graph G belongs to every detour set of G. Also if the set S of all end-vertices of G is a detour set, then S is the unique detour basis for G.*

Throughout this paper G denotes a connected graph with at least two vertices.

2 Connected Detour Sets of a Graph

Definition 2.1. *Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V$ is called a connected detour set of G if S is a detour set of G and the subgraph $G[S]$ induced by S is connected. The connected detour number $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a connected detour basis of G.*

Example 2.2. *For the graph G given in Figure 2.1, the sets $S_1 = \{v_1, v_3\}$, $S_2 = \{v_1, v_5\}$ and $S_3 = \{v_1, v_4\}$ are the three detour bases of G so that $dn(G) = 2$. It is clear that no two element subset of V is a connected detour set of G. However the set $S_4 = \{v_1, v_2, v_3\}$ is a connected detour basis of G so that $cdn(G) = 3$. Also the set $S_5 = \{v_1, v_2, v_5\}$ is another connected detour basis of G. Thus there can be more than one connected detour basis for a graph G.*

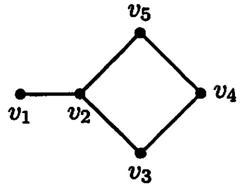


Figure 2.1: G

Example 2.3. *For the graph G given in Figure 2.2, the set $S = \{v_1, v_2\}$, is a connected detour basis for G so that $cdn(G) = dn(G) = 2$.*

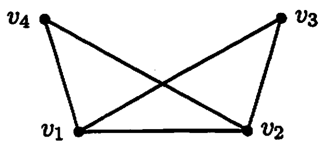


Figure 2.2: G

Theorem 2.4. *For any graph G of order $p \geq 2$, $2 \leq dn(G) \leq cdn(G) \leq p$.*

Proof. A detour set needs at least two vertices so that $dn(G) \geq 2$. Since a connected detour set is also a detour set, $dn(G) \leq cdn(G)$. Also, since the graph G is connected, the set of all vertices of G is a connected detour set of G so that $cdn(G) \leq p$. Thus $2 \leq dn(G) \leq cdn(G) \leq p$. □

Corollary 2.5. *For any connected graph G, if $cdn(G) = 2$, then $dn(G) = 2$.*

We leave the following question as an open problem.

Problem 2.6. *Characterize graphs G for which $dn(G) = cdn(G)$.*

Remark 2.7. *The bounds in Theorem 2.4 are sharp. For the path P_3, $cdn(P_3) = p$. For the graph G given in Figure 2.1, $dn(G) = 2$. For the graph G given in Figure 2.2, $dn(G) = cdn(G)$. Also all the inequalities in Theorem 2.4 can be strict. For the graph G given in Figure 2.3, $p = 7$, $dn(G) = 3$ and $cdn(G) = 5$ so that $2 < dn(G) < cdn(G) < p$.*

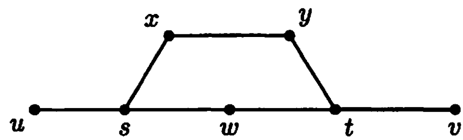


Figure 2.3: G

Definition 2.8. *A vertex v in a graph G is a connected detour vertex if v belongs to every connected detour basis of G. If G has a unique connected detour basis S, then every vertex in S is a connected detour vertex of G.*

Example 2.9. *For the graph G given in Figure 2.4, $S_1 = \{u, w, x\}$, $S_2 = \{u, w, y\}$, $S_3 = \{v, w, x\}$ and $S_4 = \{v, w, y\}$ are the only four connected detour bases. Thus the vertex w is the unique connected detour vertex of G. For the graph G given in Figure 2.1, S_4 and S_5 are the two connected detour bases of G as in Example 2.2 so that v_1 and v_2 are the two connected detour vertices of G.*

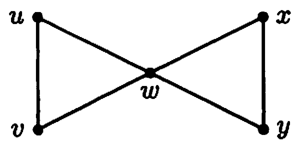


Figure 2.4: G

In the following we show that there are certain vertices in a non-trivial connected graph G that are connected detour vertices of G.

Theorem 2.10. *Every end-vertex of a connected graph G belongs to every connected detour set of G.*

Proof. Since every connected detour set of G is also a detour set of G the result follows from Theorem 1.1 □

Theorem 2.11. *Let G be a connected graph with cut-vertices and S a connected detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Proof. Let v be a cut-vertex of G such that one of the components, say C of $G - v$ contains no vertex of S. Let $u \in V(C)$. Since S is a connected detour set of G, there exist vertices $x, y \in S$ such that the vertex u lies on some x–y detour $P : x = u_0, u_1, \ldots, u, \ldots, u_t = y$ in G. Let P_1 be the x–u subpath of P and P_2 be the u–y subpath of P. Since v is a cut-vertex of G both P_1 and P_2 contain v so that P is not a detour, which is a contradiction. Thus every component of $G - v$ contains an element of S. □

Corollary 2.12. *Let G be a connected graph with cut-vertices and S a connected detour set of G. Then every branch of G contains an element of S.*

Theorem 2.13. *Let G be a connected graph with cut-vertices. Then every cut-vertex of G belongs to every connected detour set of G.*

Proof. Let G be a connected graph and v be a cut-vertex of G. Let G_1, G_2, ..., G_k ($k \geq 2$) be the components of $G - v$. Let S be any connected detour set of G. Then by Theorem 2.11, S contains at least one element from each component G_i ($1 \leq i \leq k$) of $G - v$. Since $G[S]$ is connected, it follows that $v \in S$. □

Corollary 2.14. *All the end-vertices and the cut-vertices of a connected graph G belong to every connected detour set of G.*

Proof. This follows from Theorems 2.10 and 2.13. □

Corollary 2.15. *If G is a graph of order $p \geq 2$ such that every vertex v of G is either an end-vertex or a cut-vertex, then $cdn(G) = p$.*

Proof. This follows from Corollary 2.14. □

Corollary 2.16. *If T is a tree of order $p \geq 2$, then $cdn(T) = p$.*

Proof. This follows from Corollary 2.15 □

Remark 2.17. *By Corollary 2.12, if S is a connected detour set of a connected graph G, then every end-block of G must contain at least one element of S. However, it is possible that the blocks of G that are not end-blocks must contain an element of S that are not cut-vertices as well. For example, consider the graph G of Figure 2.5, where the cycle $C_4 : v_2, v_3, v_4, v_6, v_2$ is a block of G that is not an end-block. By Corollary 2.14, every connected detour set of G must contain v_1, v_2, v_4 and v_5. It is clear that the set $\{v_1, v_2, v_4, v_5\}$ is not a connected detour set of G. Thus every connected detour set of G must contain at least one vertex from the block C_4 that is not a cut-vertex.*

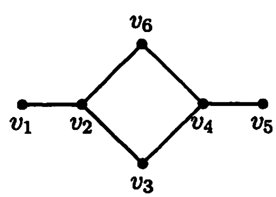


Figure 2.5: G

Corollary 2.18. *For any connected graph G with k end-vertices and l cut-vertices, $cdn(G) \geq max\{2, k+l\}$.*

Proof. This follows from the Theorem 2.4 and Corollary 2.14 □

For the graph H and an integer $k \geq 1$, we write kH for the union of the k disjoint copies of H.

Theorem 2.19. *Let $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 4$ such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \ldots + n_r + k = p - 1$. Then $cdn(G) = r + k + 1$.*

Proof. Let $u_1, u_2, \ldots, u_k$ be the end-vertices of G. Let S be any connected detour set of G. Then by Corollary 2.14, $v \in S$ and $u_i \in S (1 \leq i \leq k)$. Also by Theorem 2.11, S contains a vertex from each component $K_{n_i} (1 \leq i \leq r)$. Now, choose exactly one vertex v_i from each K_{n_i} such that $v_i \in S$. Then $|S| \geq r + k + 1$. Let $T = \{v, v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_k\}$. Since every vertex of G lies on a detour joining a pair of vertices of T, it follows that T is a detour basis of G. Also, since $G[T]$ is connected, $cdn(G) = r + k + 1$. □

Remark 2.20. *If the blocks of the graph G in Theorem 2.19 are not complete, then the theorem is not true. For the graph G given in Figure 2.6, there are two blocks and $\{v_4, v_9, v_5, v_7\}$ is a connected detour basis so that $cdn(G) = 4$.*

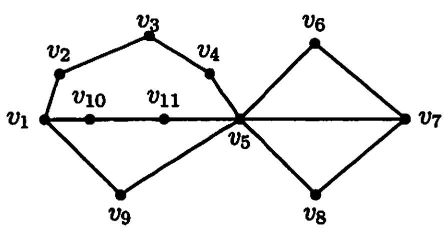


Figure 2.6: G

In the following theorems we list down for certain classes of graphs G for which $cdn(G) = 2$.

Theorem 2.21. *Let G be a Hamilton graph of order $p \geq 3$. Then $cdn(G) = 2$.*

Proof. Let $S = \{u, v\}$ be any set of two adjacent vertices in a Hamilton cycle of G. Then clearly S is a connected detour set of G so that $cdn(G) = 2$. □

Remark 2.22. *The converse of Theorem 2.21 is not true. For the graph G given in Figure 2.7, $cdn(G) = 2$ but G is not a Hamilton graph.*

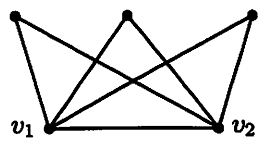


Figure 2.7: G

Theorem 2.23. *Let G be the complete graph K_p ($p \geq 2$) or the cycle C_p or the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$). Then a set S of vertices is a connected detour basis if and only if S consists of two adjacent vertices of G.*

Proof. If G is the complete graph K_p ($p \geq 2$) or the cycle C_p, then it is clear that any set of two adjacent vertices is a connected detour basis of G. Let G be the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$). Let X and Y be the bipartite sets of $K_{m,n}$ ($2 \leq m \leq n$) with $X = \{x_1, x_2, \ldots, x_m\}$. Let $u \in X$ and $v \in Y$. It is clear that $D(u, v) = 2m-1$. Let $y \in Y-\{v\}$. Then the vertex y lies on a u–v detour $P : u = x_1, y, x_2, y_1, x_3, y_2, \ldots, x_{m-1}, y_{m-2}, x_m, v$, where $y_1, y_2, \ldots, y_{m-2} \in Y - \{v, y\}$. Thus the set $\{u, v\}$ is a connected detour basis of $K_{m,n}$.

Now, let S be a connected detour basis of G. Let S' be any set consisting of two adjacent vertices of G. Then as in the first part of this theorem S' is a connected detour basis of G. Hence $|S| = |S'| = 2$ and it follows that the two vertices of S are adjacent. The converse is obvious. □

Corollary 2.24. *If G is the complete graph K_p ($p \geq 2$) or the cycle C_p or the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$), then $cdn(G) = 2$.*

Proof. This follows from Theorem 2.23. □

Theorem 2.25. *For each positive integer $k \geq 2$, there exists a connected graph G and a vertex v of degree k in G such that v belongs to a connected detour basis of G and $cdn(G) = k$.*

Proof. For $k = 2$, let $G = K_2 + v$. Then $deg_G v = 2 = k$, $cdn(G) = 2 = k$ by Corollary 2.24 and v belongs to a connected detour basis of G. For $k \geq 3$, let $G = (K_2 \cup (k-2)K_1) + v$. Then clearly $deg_G v = k$ and by Theorem 2.13, v belongs to every connected detour basis of G. Also, by Theorem 2.19, $cdn(G) = 1 + k - 2 + 1 = k$. □

It is proved in Theorem 2.4 that $2 \leq dn(G) \leq cdn(G) \leq p$ for any connected graph G. Now, the following theorem gives a realization result when $2 \leq a < b \leq p$.

Theorem 2.26. *For any three integers a, b and p with $2 \leq a < b \leq p$, there exists a connected graph G of order p with $dn(G) = a$ and $cdn(G) = b$.*

Proof. We consider two cases.
Case 1. $2 \leq a < b = p$. Let P be a path of order $p - a + 2$ and let G be the graph obtained from P by adding $a - 2$ new vertices to P and joining them to any cut-vertex of P. Then G is a tree of order p so that by Theorem 1.1, $dn(G) = a$ and by Corollary 2.16, $cdn(G) = p = b$.
Case 2. $2 \leq a < b < p$. Let $C : z_1, z_2, \ldots, z_{p-b+2}, z_1$ be a cycle of order $p - b + 2$ and let $P : y_1, y_2, \ldots, y_{b-a+1}$ be a path of order $b - a + 1$. Let H be the graph obtained from C and P by identifying z_1 of C with y_1 of P. Let G be the graph obtained from H by adding $a - 2$ new vertices $x_1, x_2, \ldots, x_{a-2}$ to H and joining each $x_i (1 \leq i \leq a - 2)$ to the vertex y_{b-a}. Then the graph G is connected of order p and is shown in Figure 2.8.

Let $X = \{x_1, x_2, \ldots, x_{a-2}, y_{b-a+1}\}$ be the set of all end-vertices of G and let $Y = \{y_1, y_2, \ldots, y_{b-a}\}$ be the set of all cut-vertices of G. Now, we show that $dn(G) = a$ and $cdn(G) = b$. It is clear that the set X is not a detour set of G so that $dn(G) \geq |X| + 1 = a$. Let $S = X \cup \{z_2\}$. Since every vertex of G lies on a detour joining a pair of vertices of S, S is a detour set of G and hence it follows from Theorem 1.1 that S is a detour basis of G so that $dn(G) = a$. By Corollary 2.14, every connected detour set of G contains $X \cup Y$. Since $X \cup Y$ is not a detour set of G and since $T = X \cup Y \cup \{z_2\}$ is a connected detour set of G, it follows that $cdn(G) = |T| = b$. □

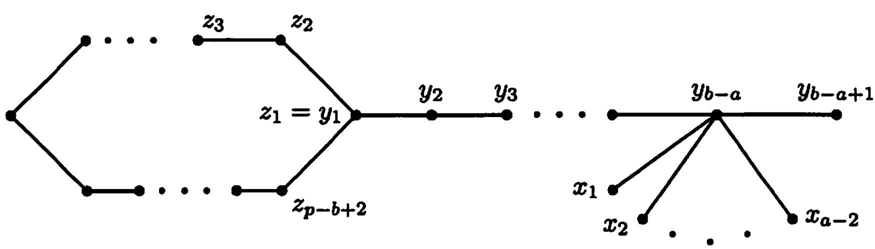


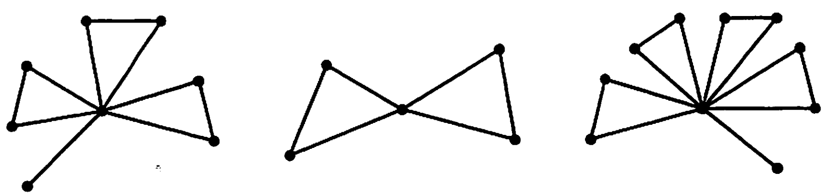
Figure 2.8: G

3 Connected Detour Number and Detour Diameter of a Graph

In [3], an upper bound for the detour number of a graph is given in terms of its order and detour diameter D as follows:

Proposition A [3] *If G is a nontrivial connected graph of order p and detour diameter D, then $dn(G) \leq p - D + 1$.*

Remark 3.1. *In the case of connected detour number $cdn(G)$ of a graph G, there are graphs G for which $cdn(G) = p - D + 1$, $cdn(G) > p - D + 1$ and $cdn(G) < p - D + 1$. For the graph G in Figure* 3.1(a), $p = 8$, $D = 4$. *Also by Theorem* 2.19, $cdn(G) = 5$ *so that* $cdn(G) = p - D + 1$. *Similarly for the graphs G in Figure* 3.1(b) *and Figure* 3.1(c), $cdn(G) > p - D + 1$ *and* $cdn(G) < p - D + 1$ *respectively.*



(a) (b) (c)

Figure 3.1: G

In the following we give conditions for the graph G so that $cdn(G) \geq p - D + 1$.

Theorem 3.2. *Let G be a graph of order $p \geq 2$. If $D = p - 1$, then $cdn(G) \geq p - D + 1$.*

Proof. For any graph G, $cdn(G) \geq 2$. Since $D = p - 1$, we have $p - D + 1 = 2$ and so $cdn(G) \geq p - D + 1$. □

Remark 3.3. *The converse of Theorem* 3.2 *is not true. For the graph G given in Figure* 3.2, $p = 7$ *and* $D = 3$ *so that* $p - D + 1 = 5$. *Also by Theorem* 2.19, $cdn(G) = 6$. *Thus* $dn(G) > p - D + 1$, *but* $D \neq p - 1$.

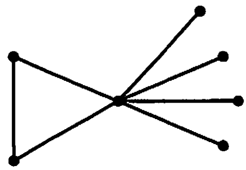


Figure 3.2: G

Theorem 3.4. *Let G be a graph of order $p \geq 2$ such that every vertex v of G is either an end-vertex or a cut-vertex, then $cdn(G) \geq p - D + 1$.*

Proof. By Corllary 2.15, $cdn(G) = p$. Since $D \geq 1$, it follows that $cdn(G) \geq p - D + 1$. □

The following theorem gives a realization result.

Theorem 3.5. *For each triple D, k, p of integers with $3 \leq k \leq p - D + 1$ and $D \geq 4$, there exists a connected graph G of order p with detour diameter D and $cdn(G) = k$.*

Proof. Let G be the graph obtained from the cycle C_D: $u_1, u_2, \ldots, u_D, u_1$ of order D by adding $k - 2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ and joining each vertex v_i $(1 \leq i \leq k-2)$ to u_1 and adding $p - D - k + 2$ new vertices $w_1, w_2, \ldots, w_{p-D-k+2}$ and joining each vertex w_i $(1 \leq i \leq p - D - k + 2)$ to both u_1 and u_3. The graph G is connected of order p and detour diameter D and is shown in Figure 3.3.

Now, we show that $cdn(G) = k$. Let $S = \{u_1, v_1, v_2, \ldots, v_{k-2}\}$ be the set of all end-vertices together with the cut-vertex u_1 of G. It is clear that S is not a detour set of G. Let $T = S \cup \{u_D\}$. Then every vertex of G lies on a detour joining a pair of vertices of T and also $G[T]$ is connected so that T is a connected detour set of G. Now, it follows from Corollary 2.14 that T is a connected detour basis of G and so $cdn(G) = k$. □

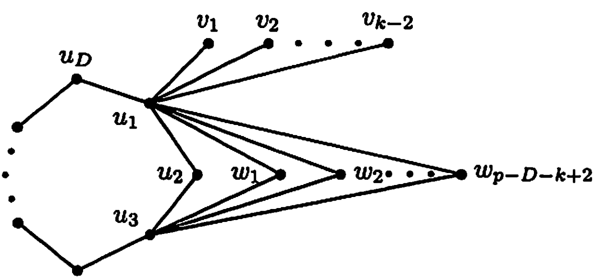


Figure 3.3: G

4 Minimal Connected Detour Sets in a Graph

Definition 4.1. *A connected detour set S in a connected graph G is called a minimal connected detour set of G if no proper subset of S is a connected detour set of G. The upper connected detour number $cdn^+(G)$ of G is the maximum cardinality of a minimal connected detour set of G.*

Example 4.2. *For the graph G given in Figure 2.3, $S_1 = \{u, s, w, t, v\}$ and $S_2 = \{u, s, x, y, t, v\}$ are the minimal connected detour sets of G so that $cdn(G) = 5$ and $cdn^+(G) = 6$.*

Remark 4.3. *Every minimum connected detour set is a minimal connected detour set, but the converse is not true. For the graph G given in Figure 2.3, $S_2 = \{u, s, x, y, t, v\}$ is a minimal connected detour set of G but not a minimum connected detour set of G.*

Theorem 4.4. *For any connected graph G, $cdn(G) \leq cdn^+(G)$.*

Proof. Let S be any connected detour basis of G. Then S is also a minimal connected detour set of G and hence the result follows. □

Corollary 4.5. *Let G be any connected graph. If $cdn(G) = p$, then $cdn^+(G) = p$.*

Remark 4.6. *The bound in Theorem 4.4 is sharp. For the path P_3, $cdn(P_3) = cdn^+(P_3) = 3$. Also for the graph G given in Figure 2.3, $cdn(G) < cdn^+(G)$.*

Now, we proceed to determine $cdn^+(G)$ for some classes of graphs.

Theorem 4.7. *Let G be the complete graph $K_p (p \geq 2)$ or the cycle C_p or the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$. Then a set S of vertices is a minimal connected detour set of G if and only if S consists of two adjacent vertices of G.*

Proof. If S consists of any two adjacent vertices of G, then by Theorem 2.23, S is a connected detour set of G so that S is minimal. Conversely assume that $S \subseteq V$ be a minimal connected detour set of G with $|S| \geq 3$. Since $G[S]$ is connected, there exists a subset $S_1 = \{u, v\}$ of S such that u and v are adjacent vertices in G. Then by Theorem 2.23, S_1 is a connected detour set of G so that S is not a minimal connected detour set of G, which is a contradiction. □

Theorem 4.8. *a) If G is the complete graph $K_p (p \geq 2)$ or the the cycle C_p or the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$, then $cdn(G) = cdn^+(G) = 2$.*
b) If G is any nontrivial tree of order p, then $cdn(G) = cdn^+(G) = p$.

Proof. a) This follows from Corollary 2.24 and Theorem 4.7.
b) By Corollary 2.15, the set of all end-vertices and cut-vertices of G is the unique detour basis of G and so the result follows. □

Problem 4.9. *Characterize graphs G for which $cdn(G) = cdn^+(G)$.*

The following theorem gives a realization result.

Theorem 4.10. *For every pair a, b of integers with $5 \leq a \leq b$, there exists a connected graph G with $cdn(G) = a$ and $cdn^+(G) = b$.*

Proof. Let $5 \leq a = b$. Then by Theorem 4.8(b), $cdn(T) = cdn^+(T) = a$ for any tree T with a vertices. Let $5 \leq a < b$. Let G be the graph obtained from the cycle C: $v_1, v_2, \ldots, v_{b-a+4}, v_1$ of order $b - a + 4$ by adding $a - 3$ new vertices $u_1, u_2, \ldots, u_{a-3}$ and joining u_1 to v_1 and each $u_i (2 \leq i \leq a-3)$ to v_{b-a+3} of C. The graph G is connected of order $b + 1$ and is shown in Figure 4.1. Let $X = \{v_2, v_3, \ldots, v_{b-a+2}\}$, $Y = \{u_1, u_2, \ldots, u_{a-3}, v_1, v_{b-a+3}\}$ and $Z = \{v_{b-a+4}\}$.

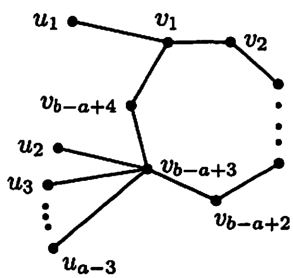


Figure 4.1: G

First, we show that $cdn(G) = a$. By Corollary 2.14, every connected detour set of G contains Y. Clearly Y is not a connected detour set of G and so $cdn(G) \geq |Y| + 1 = a$. On the other hand, it is clear that the set $S = Y \cup Z$ is a connected detour set of G and so $cdn(G) \leq a$. Therefore $cdn(G) = a$.

Now, we show that $cdn^+(G) = b$. Let $S' = X \cup Y$. Then it is clear that S' is a connected detour set of G. We show that S' is a minimal connected detour set of G. Assume, to the contrary, that S' is not a minimal connected detour set of G. Then there is a proper subset T of S' such that T is a connected detour set of G. Since T is a proper subset of S', there exists a vertex $v \in S'$ and $v \notin T$. By Corollary 2.14, every connected detour set contains Y and so we must have $v = v_i \in X$ for some i $(2 \leq i \leq b - a + 2)$. Then it is clear that $G[T]$ is not connected and so T is not a connected detour set of G, which is a contradiction. Thus S' is a minimal connected detour set of G and so $cdn^+(G) \geq |S'| = b$. Now, if $cdn^+(G) > b$, then let M be a minimal connected detour set of G with $|M| \geq b+1$. Since G has $b+1$ elements and S' is a minmal connected detour set of G, it follows that M is not a minimal connected detour set of G, which is a contradiction. Therefore, $cdn^+(G) = b$. □

Remark 4.11. *The graph G in Figure 4.1 contains exactly 2 minimal connected detour sets namely $X \cup Y$ and $Y \cup Z$. Hence this example shows that there is no "Intermediate Value Theorem" for minimal connected detour sets, that is, if k is an integer such that $cdn(G) < k < cdn^+(G)$, then there need not exist a minimal connected detour set of cardinality k in G.*

Using the structure of the graph G constructed in the proof of Theorem 4.10, we can obtain a graph H_n of order n with $cdn(G) = 5$ and $cdn^+(G) = n-1$ for all $n \geq 6$. Thus we have the following.

Theorem 4.12. *There is an infinite sequence $\{H_n\}$ of connected graphs H_n of order $n \geq 6$ such that $cdn(H_n) = 5$, $cdn^+(H_n) = n-1$, $\lim_{n\to\infty} \frac{cdn(H_n)}{n} = 0$ and $\lim_{n\to\infty} \frac{cdn^+(H_n)}{n} = 1$.*

Proof. Let H_n be the graph obtained from the cycle C: $v_1, v_2, \ldots, v_{n-2}, v_1$ of order $n-2$ by adding two new vertices u_1, u_2 and joining u_1 to v_1 and u_2 to v_{n-3} of C. The graph H_n is connected and is shown in Figure 4.2.

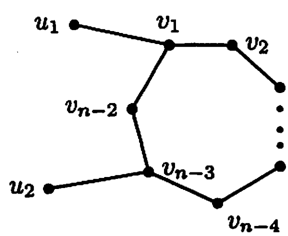


Figure 4.2: H_n

Let $X = \{v_2, v_3, \ldots, v_{n-4}\}$, $Y = \{u_1, u_2, v_1, v_{n-3}\}$ and $Z = \{v_{n-2}\}$. It is clear from the proof of Theorem 4.10 that the graph H_n contains exactly 2 minimal connected detour sets namely $X \cup Y$ and $Y \cup Z$ so that $cdn^+(H_n) = n-1$ and $cdn(H_n) = 5$. Hence the theorem follows. □

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