On the Connected Geodetic Number of a Graph

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Abstract

For a connected graph G of order $p \geq 2$, a set $S \subseteq V(G)$ is a geodetic set of G if each vertex $v \in V(G)$ lies on an x-y geodesic for some elements x and y in S. The minimum cardinality of a geodetic set of G is defined as the geodetic number of G, denoted by $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set of G. A connected geodetic set of G is a geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the connected geodetic number of G and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a g_c-set of G. Connected graphs of order p with connected geodetic number 2 or p are characterized. It is shown that for positive integers r, d and $n \geq d+1$ with $r \leq d \leq 2r$, there exists a connected graph G of radius r, diameter d and $g_c(G) = n$. Also, for integers p, d and n with $2 \leq d \leq p-1$, $d+1 \leq n \leq p$, there exists a connected graph G of order p, diameter d and $g_c(G) = n$.

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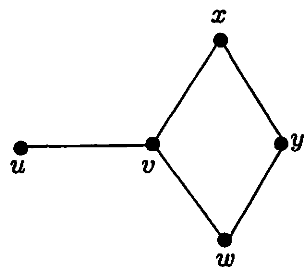
1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an *x-y geodesic*. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. For any vertex u of G, the *eccentricity* of u is $e(u) = \max\{d(u, v) : v \in V\}$. The *radius rad* G and *diameter diam* G are defined by $rad\ G = \min\{e(v) : v \in V\}$ and $diam\ G = \max\{e(v) : v \in V\}$ respectively. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [1, 5] and further studied in [2]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. Geodetic concepts were first studied from the point of view of domination by Chartrand, Harary, Swart, and Zhang in [3], where a pair x, y of vertices in a nontrivial connected graph G is said to *geodominate a vertex* v of G if $v \in I[x, y]$, that is, v lies on an x-y geodesic of G. In [3], geodetic sets and the geodetic number were referred to as *geodominating sets* and *geodomination number*.

The concept of connected geodomination number was introduced by Mojdeh and Rad in [6]. A *connected geodominating set* of G is a geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodominating set of G is the *connected geodomination number* of G and is denoted by $g_c(G)$. A connected geodominating set of cardinality $g_c(G)$ is called a g_c-set of G. We refer to a connected geodominating set and the connected geodomination number of a graph G as a connected geodetic set and the connected geodetic number of G.

Consider the graph G of Figure 1.1. For the vertices u and y in G, $d(u, y) = 3$ and every vertex of G lies on an u-y geodesic in G. Thus $S = \{u, y\}$ is the unique minimum geodetic set of G and so $g(G) = 2$. Here the induced subgraph $G[S]$ is not connected so that S is not a connected geodetic set of G. Now it is clear that $T = \{u, v, x, y\}$ is a minimum connected geodetic set of G and so $g_c(G) = 4$. The following theorems will be used in the sequel.



G

Figure 1.1

Theorem 1.1. [2] *Every extreme vertex of a connected graph G belongs to every geodetic set of G.*

Theorem 1.2. [5] *For any tree T with k end vertices, $g(T) = k$.*

Throughout the following G denotes a connected graph with at least two vertices.

2 Some Results on the Connected Geodetic Number

Theorem 2.1. *Every extreme vertex of a connected graph G belongs to every connected geodetic set of G. In particular, every end vertex of G belongs to every connected geodetic set of G.*

Proof. Since every connected geodetic set is also a geodetic set, the result follows from Theorem 1.1. □

Corollary 2.2. *For the complete graph $K_p (p \geq 2), g_c(K_p) = p$.*

Theorem 2.3. *Let G be a connected graph with cut vertices and let S be a connected geodetic set of G. If v is a cut vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component B of G at a cut vertex v such that B contains no vertex of S. Let $u \in V(B)$. Since S is a connected geodetic set, there exists a pair of vertices x and y in S such that u lies in some x-y geodesic $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in G. Since v is a cut vertex of G, the x-u subpath of P and the u-y subpath of P both contain v, it follows that P is not a path, contrary to assumption. □

Corollary 2.4. *Let G be a connected graph with cut vertices and let S be a connected geodetic set of G. Then every branch of G contains an element of S.*

Theorem 2.5. *Every cut vertex of a connected graph G belongs to every connected geodetic set of G.*

Proof. Let v be any cut vertex of G and let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - \{v\}$. Let S be any connected geodetic set of G. Then by Theorem 2.3, S contains at least one element from each $G_i (1 \leq i \leq r)$. Since $G[S]$ is connected, it follows that $v \in S$. □

Corollary 2.6. *For a connected graph G with k extreme vertices and l cut vertices, $g_c(G) \geq \max\{2, k + l\}$.*

Proof. This follows from Theorems 2.1 and 2.5. □

Corollary 2.7. *For any non-trivial tree T of order p, $g_c(T) = p$.*

Proof. This follows from Corollary 2.6. □

Theorem 2.8. *For a connected graph G of order p, $2 \leq g(G) \leq g_c(G) \leq p$.*

Proof. Any geodetic set needs at least two vertices and so $g(G) \geq 2$. Since every connected geodetic set is also a geodetic set, it follows that $g(G) \leq g_c(G)$. Also since $V[G]$ induces a connected geodetic set of G, it is clear that $g_c(G) \leq p$. □

Remark 2.9. *The bounds in Theorem 2.8 are sharp. For any non-trivial path P, $g(P) = 2$. For any tree T, $g_c(T) = p$ by Corollary 2.7. For the complete graph K_p, $g(K_p) = g_c(K_p)$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2.1, $g(G) = 3$, $g_c(G) = 5$ and $p = 6$ so that $2 < g(G) < g_c(G) < p$.*

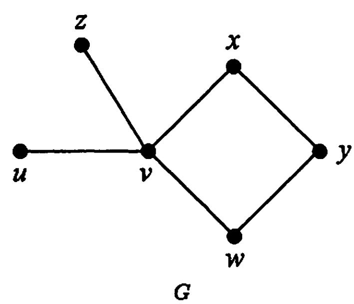


Figure 2.1

Corollary 2.10. *Let G be any connected graph. If $g_c(G) = 2$, then $g(G) = 2$.*

The following Theorems 2.11 and 2.12 characterize graphs for which $g_c(G) = 2$ and $g_c(G) = p$ respectively.

Theorem 2.11. *Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $g_c(G) = 2$.*

Proof. If $G = K_2$, then $g_c(G) = 2$. Conversely, let $g_c(G) = 2$. Let $S = \{u, v\}$ be a minimum connected geodetic set of G. Then uv is an edge. If $G \neq K_2$, then there exists a vertex w different from u and v. Then w can not lie on any u-v geodesic, so that S is not a g_c-set, which is a contradiction. Thus $G = K_2$. □

Theorem 2.12. *Let G be a connected graph. Then every vertex of G is either a cut vertex or an extreme vertex if and only if $g_c(G) = p$.*

Proof. Let G be a connected graph with every vertex of G is either a cut vertex or an extreme vertex. Then the result follows from Theorem 2.1 and Theorem 2.5.

Conversely, suppose $g_c(G) = p$. Suppose that there is a vertex x in G which is neither a cut vertex nor an extreme vertex. Since x is not an extreme vertex, $N(x)$ does not induce a complete subgraph and hence there exist u and v in $N(x)$ such that $d(u, v) = 2$. Clearly x lies on a u-v geodesic in G. Also, since x is not a cut vertex of G, $G - x$ is connected. Thus $V - \{x\}$ is a connected geodetic set of G and so $g_c(G) \leq |V - \{x\}| = p - 1$, which is a contradiction. □

We leave the following problem as an open question.

Problem 2.13. *Characterize graphs G for which $g_c(G) = g(G)$.*

We denote the vertex connectivity of a connected graph G by $\kappa(G)$ or κ.

Theorem 2.14. *If G is a non-complete connected graph such that it has a minimum cut set, then $g_c(G) \leq p - \kappa(G) + 1$.*

Proof. Since G is non-complete, it is clear that $1 \leq \kappa(G) \leq p - 2$. Let $U = \{u_1, u_2, \ldots, u_\kappa\}$ be a minimum cut set of G. Let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - U$ and let $S = V(G) - U$. Then every vertex $u_i (1 \leq i \leq \kappa)$ is adjacent to at least one vertex of G_j for every j $(1 \leq j \leq r)$. It is clear that S is a geodetic set of G and $G[S]$ is not connected. Also, it is clear that $G[S \cup \{x\}]$ is a connected geodetic set for any vertex x in U so that $g_c(G) \leq p - \kappa(G) + 1$. □

Remark 2.15. *The bound in Theorem 2.14 is sharp. For the cycle* $G = C_4$, $g_c(G) = 3$. *Also* $\kappa(G) = 2$, $p - \kappa(G) + 1 = 3$. *Thus* $g_c(G) = p - \kappa(G) + 1$.

Corollary 2.16. *If* G *is a connected non-complete graph having no cut vertices, then* $g_c(G) \leq p - 1$.

Proof. Since $\kappa(G) \geq 2$, the result follows from Theorem 2.14. □

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [7] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Mojdeh and Rad [6] showed that $g_c(G) \geq diam\ G + 1$. Ostrand's theorem can be extended so that the connected geodetic number can be prescribed when $g_c(G) \geq diam\ G + 1$.

Theorem 2.17. *For positive integers* r, d *and* $n \geq d+1$ *with* $r \leq d \leq 2r$, *there exists a connected graph* G *with* $rad\ G = r$, $diam\ G = d$ *and* $g_c(G) = n$.

Proof. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_n$. Then by Corollary 2.2, $g_c(G) = n$. If $d = 2$, let $G = K_{1,n-1}$. Then by Corollary 2.7, $g_c(G) = n$. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows:

Case 1. Suppose $r = d$. For $n = d + 1$, let $G = C_{2r}$. Then it is clear that $r = d$. It is easily seen that $g_c(G) = d + 1 = n$. Now, let $n \geq d + 2$. Let K_{n-r+1} be the complete graph with $V(K_{n-r+1}) = \{x_1, x_2, \ldots, x_{n-r+1}\}$ and let C_{2r} be the even cycle with $V(C_{2r}) = \{u_1, u_2, \ldots, u_{2r}\}$. Let G be the graph obtained from K_{n-r+1} and C_{2r} by identifying the edge x_1x_2 in K_{n-r+1} with u_1u_2 in C_{2r}. The graph G is shown in Figure 2.2.

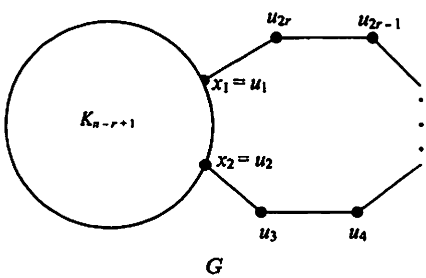


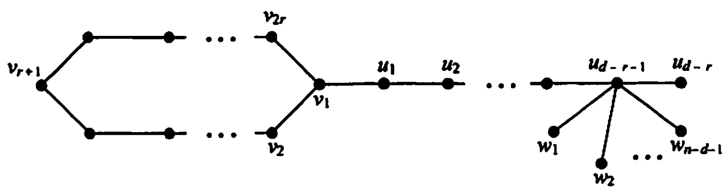
Figure 2.2

It is easily verified that the eccentricity of each vertex of G is r so that $rad\ G = diam\ G = r$. Let $S = \{x_3, x_4, \ldots, x_{n-r+1}\}$. Then S is the set of all extreme vertices of G with $|S| = n - r - 1$. It is clear that S is not a connected geodetic set of G. Let $T = S \cup \{u_1, u_2, u_3, \ldots, u_{r+1}\}$. It is clear that T is a connected geodetic set of G and so $g_c(G) \leq |T| = n$. Now, if $g_c(G) < n$, then there exists a connected geodetic set M of G such that $|M| < n$. By Theorem 2.1, M contains S and since $|M| < n$, M contains at most r vertices of C_{2r}. Since M is a connected geodetic set of G, x_1 or x_2 must belong to M. We consider two cases.

Case a. Suppose $x_1 \in M$ and $x_2 \notin M$. Since M is a connected geodetic set of G and $|M| < n$, M contains at most the vertices $x_1 = u_1, u_{2r}, u_{2r-1}, \ldots, u_{r+2}$ of C_{2r}. Then u_{r+1} does not lie on any geodesic joining a pair of vertices of M and so M is not a connected geodetic set of G, which is a contradiction.

Case b. Suppose $x_1, x_2 \in M$. Now we may assume without loss of generality that M contains at most the vertices $x_1 = u_1$, $x_2 = u_2, u_3, \ldots, u_r$ of C_{2r}. Then u_{r+1} does not lie on any geodesic joining a pair of vertices of M and so M is not a connected geodetic set of G, which is a contradiction. Thus $g_c(G) = n$.

Case 2. Suppose $r < d \leq 2r$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Now, we add $n-d-1$ new vertices $w_1, w_2, \ldots, w_{n-d-1}$ to the graph H and join each vertex $w_i (1 \leq i \leq n-d-1)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 2.3.



G
Figure 2.3

Then *rad* $G = r$ and *diam* $G = d$. Let $S = \{v_1, u_1, u_2, \ldots, u_{d-r}, w_1, w_2, \ldots, w_{n-d-1}\}$ be the set of all cut vertices and extreme vertices of G. By Theorems 2.1 and 2.5, every connected geodetic set of G contains S. It is clear that S is not a connected geodetic set of G. Let $T = S \cup \{v_2, v_3, \ldots, v_{r+1}\}$. It is clear that T is a connected geodetic set of G and so $g_c(G) \leq |T| = n$. Then by an argument similar to that given in the proof of Case 1 of this theorem, it can be proved that $g_c(G) = n$. □

Theorem 2.18. *If p, d and n are integers such that $2 \leq d \leq p - 1$ and $d + 1 \leq n \leq p$, then there exists a connected graph G of order p, diameter d and $g_c(G) = n$.*

Proof. We prove this theorem by considering two cases.

Case 1. Let $d = 2$. If $n = d + 1$, then $n = 3$. Let $P_3 : u_1, u_2, u_3$ be the path of order 3. Now, add $p - 3$ new vertices $w_1, w_2, \ldots, w_{p-3}$ to P_3. Let G be the graph obtained by joining each w_i $(1 \leq i \leq p - 3)$ to u_1 and u_3. The graph G is shown in Figure 2.4. Then G has order p and diameter $d = 2$. Clearly, $S = \{u_1, u_2, u_3\}$ is a minimum connected geodetic set of G so that $g_c(G) = 3 = n$.

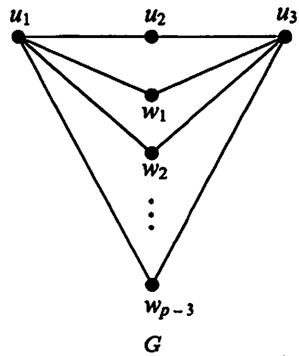


Figure 2.4

Now, let $d + 2 \leq n \leq p$. Let K_{p-1} be the complete graph with the vertex set $\{w_1, w_2, \ldots, w_{p-n+1}, v_1, v_2, \ldots, v_{n-2}\}$. Now add the new vertex x to K_{p-1}. Let G be the graph obtained by joining x with each $w_i (1 \leq i \leq p - n + 1)$. The graph G is shown in Figure 2.5. Then G has order p and diameter $d = 2$.

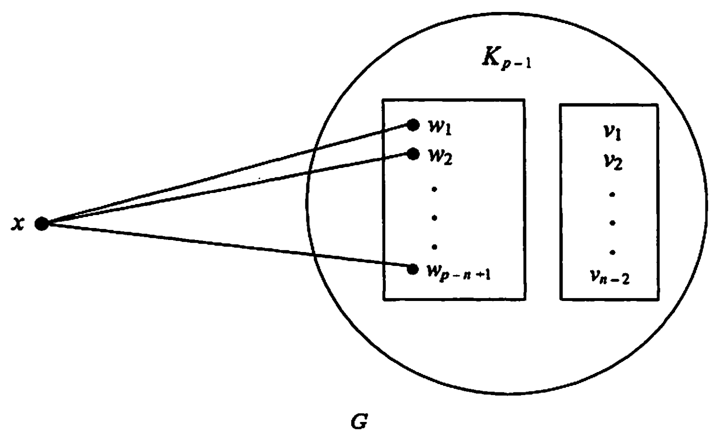


Figure 2.5

By Theorem 2.1, every connected geodetic set of G contains $S = \{v_1, v_2, \ldots, v_{n-2}\}$ of all extreme vertices of G. It is clear that S is not a connected geodetic set of G. Now, let $S' = S \cup \{x\}$. Clearly, $I[S'] = V(G)$ and $I[S \cup \{w_i\}] \neq V(G)$ for $i = 1, 2, \ldots, p-n+1$. Since the induced subgraph $G[S']$ is not connected, $g_c(G) > n-1$. Let $T = S' \cup \{w_1\}$. Clearly, T is a connected geodetic set of G and so $g_c(G) = n$.

Case 2. Let $3 \leq d \leq p-2$. Let $P_{d+1} : u_0, u_1, u_2, \ldots, u_d$ be a path of length d. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-n}, v_1, v_2, \ldots, v_{n-d-1}$ to P_{d+1} and join $w_1, w_2, \ldots, w_{p-n}$ to both u_0 and u_2 and join $v_1, v_2, \ldots, v_{n-d-1}$ to u_{d-1}, there by producing the graph G of Figure 2.6. Then G has order p and diameter d. Let $S = \{u_2, u_3, \ldots, u_d, v_1, v_2, \ldots, v_{n-d-1}\}$ be the set of all cut vertices and extreme vertices of G. By Theorems 2.1 and 2.5, every connected geodetic set of G contains S. It is clear that S is not a connected geodetic set of G. Now, let $S' = S \cup \{u_0\}$. Clearly, $I[S'] = V(G)$ and $I[S \cup \{y\}] \neq V(G)$ for $y \in \{u_1, w_1, w_2, \ldots, w_{p-d-1}\}$. Since the induced subgraph $G[S']$ is not connected, $g_c(G) > n-1$. Let $T = S' \cup \{u_1\}$. Clearly, T is a connected geodetic set of G and so $g_c(G) = n$.

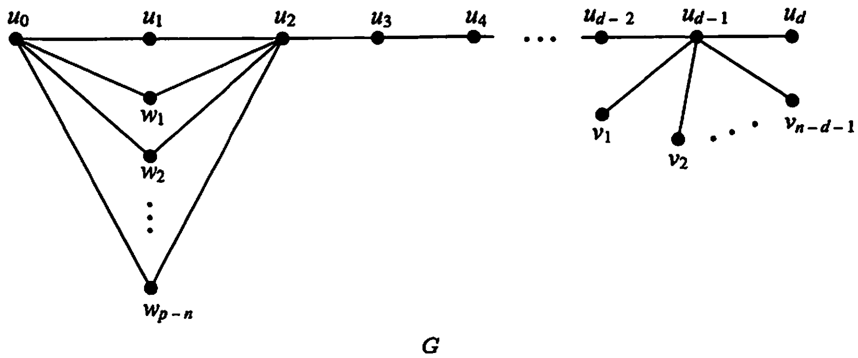


Figure 2.6

Case 3. Let $d = p-1$. Then $n = p$. Let G be the path of order n. Then, by Corollary 2.7, $g_c(G) = n$. □

We proved (Theorem 2.8) that $2 \leq g(G) \leq g_c(G) \leq p$. The following theorem gives a realization for these parameters when $2 \leq a < b \leq p$.

Theorem 2.19. *If p, a and b are positive integers such that $2 \leq a < b \leq p$, then there exists a connected graph G of order p, $g(G) = a$ and $g_c(G) = b$.*

Proof. We prove this theorem by considering two cases.

Case 1. $2 \leq a < b = p$. Let G be any tree with a pendent vertices. Then by Theorem 1.2, $g(G) = a$ and by Corollary 2.7, $g_c(G) = p$.

Case 2. $2 \leq a < b < p$. Let $P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2}$ be a path of length $b-a+1$. Add $p-b+a-2$ new vertices $w_1, w_2, \ldots, w_{p-b}, v_1, v_2, \ldots, v_{a-2}$ to P_{b-a+2} and join $w_1, w_2, \ldots, w_{p-b}$ to both u_1 and u_3 and join $v_1, v_2, \ldots, v_{a-2}$ to u_{b-a+1}, there by producing the graph G of Figure 2.7. Then G has order p and $S = \{u_{b-a+2}, v_1, v_2, \ldots, v_{a-2}\}$ is the set of all extreme vertices of G. It is clear that S is not a geodetic set of G. On the other hand, $S \cup \{u_1\}$ is a geodetic set of G and it follows from Theorem 1.1 that $g(G) = a$. By an argument exactly similar to the one given in Case 2 of Theorem 2.18, it can be proved that $g_c(G) = b$. □

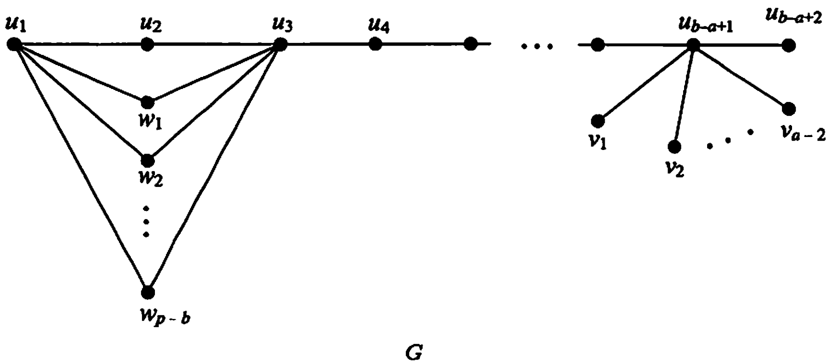


Figure 2.7

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