Trees and Unicyclic Graphs are γ-graphs*

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Abstract

A subset D of the vertex set $V(G)$ of a graph G is said to be a dominating set of G, if each $v \in V - D$ is adjacent to at least one vertex of D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D with cardinality $\gamma(G)$ is called a γ-set of G. Given a graph G, a new graph, denoted by $\gamma \cdot G$ and called γ-graph of G, is defined as follows: $V(\gamma \cdot G)$ is the set of all γ-sets of G and two sets D and S of $V(\gamma \cdot G)$ are adjacent in $\gamma \cdot G$ if and only if $|D \cap S| = \gamma(G) - 1$. A graph G is said to be γ-connected if $\gamma \cdot G$ is connected. A graph G is said to be a γ-graph if there exists a graph H such that γ. H is isomorphic to G. In this paper we show that trees and unicyclic graphs are γ-graphs. Also we obtain a family of graphs which are not γ-graphs.

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1 Introduction

We consider only simple graphs. For all graph theoretic terminology we refer to Bondy and Murty [1]. If G is a graph, a subset D of $V(G)$ is called

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dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set with minimum cardinality is called a γ-set of G. Two γ-sets D and S of G are said to be γ-exchangeable if $|D \cap S| = \gamma(G) - 1$, or equivalently, there is a vertex $u \in D$ and a vertex $v \in S$ such that $D - \{u\} = S - \{v\}$.

In [2], we have introduced the concept of γ-graph of a graph G.

The vertex set $V(\gamma \cdot G)$ of the γ-graph $\gamma \cdot G$ is the set of all γ-sets of G and for two sets $D, S \in V(\gamma \cdot G)$, D and S are adjacent in $\gamma \cdot G$ if and only if the γ-sets D and S of G are γ-exchangeable.

In [2], we have obtained γ-graphs of some standard graphs. A graph G is said to be γ-connected if the γ-graph $\gamma \cdot G$ is connected. If G is γ-connected, then given γ-sets D and S, there exist γ-sets $D = D_1, D_2, \ldots, D_k = S$ (for some k) such that $|D_i \cap D_{i+1}| = \gamma(G) - 1$ for $i = 1, 2, \ldots, k - 1$. In [3] it was shown that every tree is γ-connected.

The following question naturally arises.

"For which graphs G, does there exist a graph H such that $\gamma \cdot H = G$?"

In this paper we prove that trees and unicyclic graphs are γ-graphs. If S is a γ-set of the graph G and $x \in S$, then $pn(S, x) = \{y \in V(G) : N[y] \cap S = \{x\}\}$, is called the private neighbourhood of x with respect to the γ-set S.

2 Main Results

Theorem 2.1. *For every tree T, there exists a graph G such that $\gamma \cdot G = T$.*

Proof. We prove the theorem by induction on $n = |V(T)|$. If $n = 1, 2$ or 3, we take G to be $K_{1,m}$ with $m \geq 2$ or P_2 or P_5. We observe that in all these case if S is a γ-set of G and $x \in S$, then $pn(S, x) \neq \{x\}$. We now assume that, each tree of order $\leq k$ is $\gamma \cdot G$ for some graph G and for every γ-set S of G and to each $x \in S, pn(S, x) \neq \{x\}$. Let T be a tree of order $k + 1$. Let u be a pendant vertex of T and w be the vertex of T adjacent to u.

Let $T' = T - \{u\}$. By induction hypothesis, there exists a graph G with $\gamma \cdot G = T'$. For each $x \in V(T')$, let S_x be the corresponding γ-set of G. Note that $pn(S_x, a) \neq \{a\}$ for all $a \in S_x$. Let $\gamma(G) = m$, let $S_w = \{v_1, v_2, v_3, \ldots, v_m\}$ be the γ-set of G corresponding to the vertex w to T'.

We now construct a graph H as follows:

1. $V(H) = V(G) \cup \{x, y, a_1, a_2, \ldots, a_{2m}\}$ where the set $\{x, y, a_1, a_2, \ldots, a_{2m}\}$ is disjoint from the set $V(G)$.

2. $E(H) = E(G) \cup \{xy, xa_i : 1 \le i \le 2m\} \cup \{v_i a_{2i}, v_i a_{2i-1}, 1 \le i \le m\}$.
It can be easily verified that H has the following properties

(i) $\gamma(H) = \gamma(G) + 1 = m + 1$ and $\{y, v_1, v_2, \ldots, v_m\}$ is a γ-set of H.

(ii) For every $D \subseteq V(G), D$ is γ-set of G if and only if $D \cup \{x\}$ is a γ-set of H.

(iii) The number of γ-sets of H is $|V(T)|$.

(iv) If S is a γ-set of H, then either $S = \{y, v_1, v_2, \ldots, v_m\}$ or $S = S_a \cup \{x\}$ for some $a \in V(T')$.
(This follows from the fact that $pn(S_a, b) \neq \{b\}$ in T', for all $b \in S_a$).

In $\gamma \cdot H$, the vertex representing the γ-set $\{y, v_1, v_2, \ldots, v_m\}$ is adjacent only to the vertex representing $\{x, v_1, v_2, \ldots, v_m\}$. Thus $\gamma \cdot H \cong T$, and for every γ-set S of H, $pn(S, u) \neq \{u\}$ for all $u \in S$.

□

Remark 2.2. *Using the proof technique of Theorem 2.1, we obtain the following:*

Let G be a graph such that $\gamma \cdot H = G$ for some graph H and for every γ-set S of H, $pn(S, u) \neq \{u\}$ for all $u \in S$. Let G^* be the graph obtained from G by attaching a pendant vertex to any vertex of G. Then there exists a graph H^* such that $\gamma \cdot H^* = G^*$.

We now proceed to prove that unicyclic graphs are γ-graphs.

The Harary graph $H_{m,2r}$ is defined as follows:

Let $V(H_{m,2r}) = \{0, 1, \ldots, m-1\}$ and $E(H_{m,2r}) = \{i(i \pm j) : i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, r\}$. Here $i \pm j$ is computed modulo m.

Theorem 2.3. $\gamma \cdot H_{4n+1,2n} = C_{4n+1}$.

Proof. Clearly $\{\{i, i+2n\}, \{i, i+2n+1\} : i = 0, 1, 2, \ldots, 2n\}$ is the set of all γ-sets of $H_{4n+1,2n}$ and hence it follows that $\gamma \cdot H_{4n+1,2n} \cong C_{4n+1}$. □

Theorem 2.4. *For every cycle C_n, there exists a graph G such that $\gamma \cdot G = C_n (n \ge 3)$.*

Proof. **Case i.** C_n is an odd cycle.

By Theorem 2.3, if $G = H_{4n+1,2n}$ then $\gamma \cdot H_{4n+1,2n} = C_{4n+1}$. Thus it is enough to prove that for the odd cycle C_{4n+3} there exists G such that $\gamma \cdot G = C_{4n+3}$. Consider the graph $G_0 = H_{4n+1,2n}$ with $V(G_0) = \{v_0, v_1, \ldots, v_{4n}\}$ and $E(G_0) = \{v_i, v_{i+r} : r = \pm 1, \pm 2, \ldots, \pm n; 0 \le i \le 4n\}$ (where addition is done under modulo $(4n+1)$).

Let us construct a graph G from G_0 as follows:

$V(G) = V(G_0) \cup \{a, a', b, b', c, c', d, d'\}$ and
$E(G) = E(G_0) \cup \{aa', ab, ab', ad, ad'\} \cup \{a'b, a'b', a'c, a'c'\}$
$\cup\{v_ic, v_ic' : 0 \leq i \leq 2n-1\} \cup \{v_id, v_id' : v_i = v_{2n}, v_{4n}\}$.
Clearly $\gamma(G) = 3$.

Note that if a γ-set S of G contains a' then $a \in S$ and either v_{2n} or $v_{4n} \in S$. If $a' \in S$ then $v_i \in S$ for some $i, 0 \leq i \leq 2n-1$. The γ-sets of G are $\{0, v_{2n}, a'\}, \{v_{2n}, v_{4n}, a'\}$; $\{v_{2n-1}, v_{4n}, a'\}$ and $\{v_i, v_{i+2n}, a\}$; $\{v_i, v_{i+2n+1}, a\}$ for all i, $0 \leq i \leq 2n-1$.

Hence $\gamma \cdot G = C_{4n+3}$.

Case ii. C_n is an even cycle.

We construct a graph G as follows:
$V(G) = \{a_i, b_i, u_i, v_i : 1 \leq i \leq n\}$
$\cup\{x_{ij}, y_{ij} : i = 1, 2, \ldots, n-2, i < j < 2\}$ and
$E(G) = \{a_ib_i, a_iu_i, a_iv_i, b_iu_i, b_iv_i\}$
$\cup\{a_ix_{ij}, b_iy_{ij} : 1 \leq i < j < n\}$
$\cup\{b_jx_{ij}, a_{j+1}x_{ij}, a_jy_{ij}, b_{j+1}y_{ij} : 1 \leq i < j < n\}$.

If S is a γ-set of G, then $S \cap \{a_i, b_i, u_i, v_i\} \neq \emptyset$ for each $i, 1 \leq i \leq n$ and hence $\gamma(G) \geq n$. Also $\{a_i : 1 \leq i \leq n\}$ is a dominating set of G and hence $\gamma(G) = n$.

Now, let S be any γ-set of G. If $S \cap \{a_i, b_i\} = \emptyset$ for some i, then both $u_i, v_i \in S$ and hence $|S| > n$, which is a contradiction. Hence $S \cap \{a_i, b_i\} \neq \emptyset$ and in fact $|S \cap \{a_ib_i\}| = 1$ for all i. Hence S can be represented by a vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where

$$\alpha_i = \begin{cases} 0 & \text{if } a_i \in S \\ 1 & \text{if } b_i \in S. \end{cases}$$

Note that the γ-sets $\{a_i : 1 \leq i \leq n\}$ and $\{b_i : 1 \leq i \leq n\}$ are respectively represented by $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. We claim that in the representation $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of a γ-set S of G, zeros will appear together and ones will appear together.

Suppose $\alpha_1 = 0$ and $\alpha_i = 1$ for some $i > 1$. Then $a_1 \in S$, $b_1 \notin S$ and $b_i \in S$. Since S dominates the vertex y_{1i}, it follows that $b_{i+1} \in S$ and hence $\alpha_{i+1} = 1$. Thus $\alpha_j = 1$ for all $j \geq i$. By a similar argument we can prove that if $\alpha_1 = 1$ and $\alpha_i = 0$ for some $i > 1$, then $\alpha_j = 0$ for all $j \geq i$.

Further any such vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ represents a γ-set of S. Hence there are exactly $2n$ γ-sets of G, namely, $(0, 0, \ldots, 0), (1, 0, 0, \ldots, 0)$, $(1, 1, 0, \ldots, 0), \ldots,$ $(1, 1, \ldots, 1, 0)$, $(1, 1, 1, \ldots, 1)$, $(0, 1, 1, \ldots, 1)$, $(0, 0, 1, 1, \ldots, 1), \ldots, (0, 0, \ldots, 0, 1)$ and $\gamma \cdot G = C_{2n}$. □

Remark 2.5. *Note that if G is the graph constructed in the proof of Theorems 2.1 and 2.3, then whenever S is a γ-set of G and $x \in S$, we have $pn(S, x) \neq \{x\}$.*

Using these observation and also Theorem 2.1 and 2.3 we have the following:

Theorem 2.6. *If G is either a tree or a unicylic graph then there exists a graph H such that $\gamma \cdot H = G$.*

We now present a class of graphs which are not γ-graphs.

Theorem 2.7. *Let Δ_3 be the graph given in Figure 1. If Δ_3 is an induced subgraph of a graph H, then there exists no graph G such that $\gamma \cdot G = H$.*

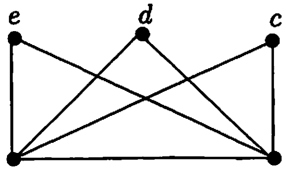

Figure 1

Proof. Assume that Δ_3 is an induced subgraph of H. Let $k = \gamma(H)$. Suppose there exists a graph G such that $\gamma \cdot G = H$. For each vertex u of H, let S_u be the γ-set of G. Now, let abu be a triangle in H and let $S_a = \{w_1, v_2, v_3, \dots, v_k\}$ and $S_b = \{w_2, v_2, v_3, \dots, v_k\}$ where $w_1 \notin S_b$, and $w_2 \notin S_a$.

If $w_2 \notin S_u$, then $S_u = \{w_3, v_2, \dots, v_k\}$ for some $w_3 \neq w_1, w_2$. If $w_2 \in S_u$, then $v_i \notin S_u$ for some $i \geq 2$ and hence $S_u = \{w_2, w_4, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_k\}$ for some $w_4 \neq v_i$. Since $|S_a \cap S_u| = k - 1$ and $w_2 \notin S_a$, we get $w_4 \in S_a$, so that $w_4 = w_1$. Hence $S_u = \{w_1, w_2, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_k\}$.

Hence S_u is either $(S_a - \{v_i\}) \cup \{w_2\}$ for some $i, i \geq 2$ or $(S_a - \{w_1\}) \cup \{w_3\}$ where $w_3 \neq w_1, w_2$. The set S_u is said to be of type I if $S_u = (S_a - \{w_1\}) \cup \{w_3\}$ where $w_3 \neq w_1, w_2$ and is said to be of type II if $S_u = (S_a - \{v_i\}) \cup \{w_2\}$ for some $i \geq 2$. Since Δ_3 is an induced subgraph of H; abc, abd and abe are triangles in H. Hence at least two of S_c, S_d, S_e, say S_c and S_d are the same type.

If S_c and S_d are of type I then $S_c = (S_a - \{w_1\}) \cup \{w_3\}$ and $S_d = (S_a - \{w_1\}) \cup \{w_4\}$ for some $w_3, w_4 \neq w_1, w_2$. Then $S_c - \{w_3\} = S_d - \{w_4\}$ and hence cd is an edge in H, a contradiction. Also if S_c and S_d are of type II, then $S_c = (S_a - \{v_{1i}\}) \cup \{w_2\}$ and $S_d = (S_a - \{v_{i2}\}) \cup \{w_2\}$ for $i_1 \neq i_2 \geq 2$. Then $S_c - \{v_{i1}\} \neq S_d - \{v_{i2}\}$ and hence cd is an edge in H, a contradiction.

Thus, there is no graph of G such that $H = \gamma \cdot G$. □

Theorem 2.7 shows that H is a forbidden subgraph for γ-graphs. Hence the following problems arise naturally.

Problem 2.8. *Does there exist other forbidden subgraphs for γ-graphs?*

Problem 2.9. *Obtain a characterization of γ-graphs.*

References

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