Independent Domination Number of Cayley Graphs on $\mathbb{Z}_n$

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Abstract

A Cayley graph is a graph constructed out of a group Γ and its generating set A. In this paper, we determine the independent domination number, perfect domination number and independent dominating sets of $Cay(\mathbb{Z}_n, A)$, for a specified generating set A of $\mathbb{Z}_n$.

Keywords. Cayley graphs, dominating sets, independent domination number.
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1 Introduction

Let Γ be a finite group with e as the identity. A generating set of the group Γ is a subset A such that every element of Γ can be expressed as a product of finitely many elements of A. We assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The *graph* $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a : x, y \in V(G)$ and there exists $a \in A$ such that $y = xa\}$ is called the Cayley graph associated with the pair (Γ, A) and it is denoted by $Cay(\Gamma, A)$ [5]. Clearly $Cay(\Gamma, A)$ is a connected simple, regular graph and degree of any vertex in G is $|A|$.

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighbourhood $N(v)$ of v is the set of all vertices adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood $N(S)$ is defined to be $\cup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$ [4]. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or adjacent to an element of S [4]. A dominating set S is a minimal dominating set if no proper subset is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets in G [4] and the corresponding dominating set is called a γ-set. A dominating set S is an *independent dominating set* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ of a graph G is the minimum cardinality taken over all independent dominating sets in G [1]. A dominating set S is called a *perfect dominating set* if for every vertex $u \in V - S$, $|N(u) \cap S| = 1$. The *perfect domination number* $\gamma_p(G)$ of a graph G is the minimum cardinality of a perfect dominating set. The *domatic number* $d(G)$ of a graph G is the maximum number of elements in a partition of $V(G)$ into dominating sets. The *independent domatic number* $d_i(G)$ of a graph G is the maximum number of elements in a partition of $V(G)$ into independent dominating sets [4]. Similarly the *perfect domatic number* $d_p(G)$ is the maximum number of elements in a partition of $V(G)$ into perfect dominating sets of G.

The minimum cardinality of the disjoint union of a dominating set S and an independent dominating set I, is denoted by $\gamma i(G)$ and such a pair of dominating sets (S, I) is called a γi-pair. The *disjoint domination number* $\gamma\gamma(G)$ is defined as follows: $\gamma\gamma(G) = \min\ \{|S_1| + |S_2| : S_1, S_2$ are disjoint dominating sets of G$\}$. The two disjoint dominating sets, whose union has cardinality $\gamma\gamma(G)$, is a $\gamma\gamma$-pair of G [2]. The *disjoint independent domination number* $ii(G)$ is the minimum cardinality of the union of two disjoint independent dominating sets in a graph G [3].

Throughout this paper, $n \geq 3$ is a fixed positive integer, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and $G = Cay(\mathbb{Z}_n, A)$, where A is a generating set for $\mathbb{Z}_n$. Unless otherwise specified A stands for the set $\{1, n-1, 2, n-2, \ldots, k, n-k\}$ where $1 \leq k \leq (n-1)/2$. Hereafter $+$ stands for modulo n addition in $\mathbb{Z}_n$.

In [6] we have proved the following theorem, which determines the domination number of $Cay(\mathbb{Z}_n, A)$.

Theorem 1.1. [6] *Let $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$ and n, k are positive integers with $1 \leq k \leq (n-1)/2$. Then $\gamma(G) = \lceil \frac{n}{|A|+1} \rceil$. Further $D = \{0, (2k+1), 2(2k+1), 3(2k+1), \ldots, (\ell - 1)(2k+1)\}$, where $\ell = \lceil \frac{n}{|A|+1} \rceil$ is a γ-set of G.*

In this paper we determine the independent domination number and perfect domination number of $Cay(\mathbb{Z}_n, A)$.

2 Main Results

Theorem 2.1. *Let n and k be positive integers such that $k \leq \frac{n-1}{2}$ and $(2k+1)$ divides n. Then $i(G) = \gamma_p(G) = \frac{n}{2k+1}$, where $G = Cay(\mathbb{Z}_n, A)$.*

Proof. By Theorem 1.1, $D = \{0,\ (2k+1),\ 2(2k+1),\ 3(2k+1),\ \ldots,\ (\ell-1)(2k+1)\}$ where $\ell = \lceil \frac{n}{2k+1} \rceil$ is a γ-set of G. Since $(2k+1)$ divides n we have $\ell = n/(2k+1)$. Note that, for any j, $N[(2k+1)j] = \{(2k+1)j,\ (2k+1)j+1,\ (2k+1)j+(n-1),\ (2k+1)j+2,\ (2k+1)j+(n-2), \ldots, (2k+1)j+k,\ (2k+1)j+(n-k)\}$.

Now, let $v_1, v_2 \in D$ and $v_1 \neq v_2$. We claim that $N[v_1] \cap N[v_2] = \emptyset$. Let $v_1 = (2k+1)i$ and $v_2 = (2k+1)j$ with $0 \le i < j \le (\ell-1)$. Suppose $N[v_1] \cap N[v_2] \neq \phi$ and let $x \in N[v_1] \cap N[v_2]$. Then $x = (2k+1)i + t_1$ or $(2k+1)i + (n-t_1)$ and $x = (2k+1)j + t_2$ or $(2k+1)j + (n-t_2)$ where $0 \le t_1, t_2 \le k$. Clearly $1 \le j-i \le (\ell-1)$, $-k \le t_1 - t_2 \le k$, $0 \le t_1 + t_2 \le 2k$ and $2k+1 \le (2k+1)(j-i) \le (2k+1)(\ell-1)$.

If $(2k+1)i + t_1 = (2k+1)j + t_2$, then $(2k+1)(j-i) = t_1 - t_2$. If $(2k+1)i + (n-t_1) = (2k+1)j + t_2$, then $(2k+1)(j-i) = n - (t_1+t_2)$. If $(2k+1)i + t_1 = (2k+1)j + (n-t_2)$, then $(2k+1)(j-i) = t_1 + t_2$. If $(2k+1)i + (n-t_1) = (2k+1)j + (n-t_2)$, then $(2k+1)(j-i) = n - (t_1 - t_2)$. In each of these cases, the left hand side is a multiple of $2k+1$, whereas the right hand side is not so, which is a contradiction. Hence $N[v_1] \cap N[v_2] = \phi$. Thus D is independent and $|D| = \ell = \frac{n}{2k+1}$ and hence $i(G) = \frac{n}{2k+1}$. Also every element of $V - D$ is adjacent to exactly one element in D and hence D is perfect. Therefore $\gamma_p(G) = n/(2k+1)$. □

Corollary 2.2. *Let n and k be positive integers such that $k \le \frac{n-1}{2}$ and $(2k+1)$ divides n. Then for each h, $1 \le h \le 2k$, $D+h$ is both an independent and perfect dominating set of G with minimum cardinality, where $G = Cay(\mathbb{Z}_n, A)$ and $D = \{0,\ (2k+1),\ 2(2k+1),\ 3(2k+1),\ \ldots,\ (\lceil \frac{n}{2k+1} \rceil - 1)(2k+1)\}$.*

Corollary 2.3. *Let n and k be positive integers such that $k \le \frac{n-1}{2}$ and $(2k+1)$ divides n. Then $d_i(G) = d_p(G) = 2k+1$, where $G = Cay(\mathbb{Z}_n, A)$.*

Proof. Any element of V is of the form $(2k+1)t + h$, $0 \le t \le (\lceil \frac{n}{2k+1} \rceil - 1)$, $0 \le h \le 2k$. Also if $D = \{0,\ (2k+1),\ 2(2k+1),\ 3(2k+1),\ \ldots,\ (\lceil \frac{n}{2k+1} \rceil - 1)(2k+1)\}$, then $(D+h_1) \cap (D+h_2) = \phi$ for $0 \le h_1, h_2 \le 2k$ and $h_1 \neq h_2$. If not, let $x \in (D+h_1) \cap (D+h_2)$. Then $h_1 - h_2 = (2k+1)(j-i)$ for some i and j with $i < j$, and $0 \le i < j \le (\ell-1)$. Since $0 < h_1 - h_2 < 2k$, it cannot be a multiple of $2k+1$ and hence a contradiction. Hence $V = \bigcup_{h=0}^{2k} (D+h)$ and each $D+h$ is both an independent and perfect dominating set of minimum cardinality. Therefore $d_i(G) = d_p(G) = 2k+1$. □

Remark 2.4. For any $v \in G = Cay(\mathbb{Z}_n, A)$, $|N[v]| = 2k+1$. In view of Theorem 1.1 and Theorem 2.1, the perfect domination number of G exists only when $2k+1$ divides n.

Theorem 2.5. *If n, k are positive integers such that $k \le \frac{n-1}{2}$ and $(2k+1)$ does not divide n, then $i(G) = \lceil \frac{n}{2k+1} \rceil$.*

Proof. Let $\ell = \lceil \frac{n}{2k+1} \rceil$ and t be the least positive integer satisfying $n \equiv t(mod\ (2k+1))$. Take $D = \{0,\ 2k+1,\ 2(2k+1), \ldots, (\ell-2)(2k+1),\ n-(k+\lceil \frac{t}{2} \rceil)\}$. Let $v_1, v_2 \in D$ and such that $v_1 \neq v_2$. Without loss of generality, one can assume that $v_1 = (2k+1)i$ and $v_2 = (2k+1)j$ or $n-(k+\lceil \frac{t}{2} \rceil)$ for some $i,\ j$ with $0 \leq i,\ j \leq (\ell-2)$ and $i \neq j$. Suppose $v_1 \in N(v_2)$. Then $v_1 = v_2+s$ or $v_1 = v_2+(n-s)$ for some s with $1 \leq s \leq k$.

Now the following cases arise:

Case 1. Let $v_1 = (2k+1)i$ and $v_2 = (2k+1)j$. The non-adjacency of v_1 and v_2 can be proved as in Theorem 2.1.

Case 2. Let $v_1 = (2k+1)i$ and $v_2 = n-(k+\lceil \frac{t}{2} \rceil)$. We have the following sub cases:

Sub case 2.1. Suppose $v_1 = v_2+s$. Then $(2k+1)i = n-(k+\lceil \frac{t}{2} \rceil)+s$ and so $n = (2k+1)i+(k+\lceil \frac{t}{2} \rceil)-s$. Since $(k+\lceil \frac{t}{2} \rceil)-s < (2k+1)$, we get that $n = (2k+1)i+(k+\lceil \frac{t}{2} \rceil)-s < n$, which is a contradiction.

Sub case 2.2. Suppose $v_1 = v_2+n-s$. Then $(2k+1)i = n-(k+\lceil \frac{t}{2} \rceil)+(n-s)$ and so $n-s = (2k+1)i+(k+\lceil \frac{t}{2} \rceil) < n-s$, which is a contradiction.

Hence in both the cases we have $v_1 \notin N[v_2]$. Therefore D is independent. It follows from Theorem 1.1 that D is a dominating set and hence $i(G) \leq \lceil \frac{n}{2k+1} \rceil$. Now the reverse inequality follows from $i(G) \geq \gamma(G) \geq \lceil \frac{n}{2k+1} \rceil$. □

Corollary 2.6. *For a positive integer $n \geq 3$, $1 \leq i(G) \leq \lceil \frac{n}{3} \rceil$.*

Proof. When $n \geq 3$, we have $2 \leq |A| \leq (n-1)$. When $|A| = n-1$, $i(G) = 1$. Further, when $|A| = 2$, by Theorem 2.1 and Theorem 2.5, we have $i(G) = \lceil \frac{n}{3} \rceil$. □

Corollary 2.7. *Let n, k be positive integers such that $k \leq \frac{n-1}{2}$ and $(2k+1)$ does not divide n. If D is an independent dominating set of G, then $D+u$ is an independent dominating set for any u with $1 \leq u \leq n-1$.*

Theorem 2.8. *Let n and k be positive integers such that $k \leq \frac{n-1}{2}$. Then $ii(G) = \gamma i(G) = \gamma\gamma(G) = 2\lceil \frac{n}{2k+1} \rceil$, where $G = Cay(\mathbb{Z}_n, A)$.*

Proof. Let $\ell = \lceil \frac{n}{2k+1} \rceil$. When $2k+1$ divides n, the result follows from Corollary 2.2. When $2k+1$ does not divide n, by Theorem 2.5, $D = \{0, 2k+1, 2(2k+1), \ldots, (\ell-2)(2k+1), n-(k+\lceil \frac{t}{2} \rceil)\}$ is an independent dominating set. When the points of $\mathbb{Z}_n$ are represented as n equi-distant points on the circumference of a circle, then D is simply a successive set of points starting from 0 and distance $2k+1$, except the last point, whose circular distance from 0 is less than $2k+1$. Actually this circular distance is simply the remainder t when n is divided by $2k+1$. If $t > 1$, then D and $D+1$ are disjoint dominating sets and if $t = 1$, then D and $D+2$ are disjoint dominating sets. Hence $ii(G) = i(G)+i(G) = 2\ell, \gamma i(G) = \gamma(G)+i(G) = 2\ell$ and $\gamma\gamma(G) = \gamma(G)+\gamma(G) = 2\ell$. □

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