An Application of a Graph Labelling to Root Systems

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Abstract

The main result: If the vertices of a connected graph are labelled by positive real numbers such that the number assigned to any vertex is half of the sum of the numbers assigned to the vertices of its neighbourhood, then each label is an integral multiple of the minimum of all labels. Using this, a result proved earlier in [7] is derived: If V is a linearly dependent subset of a root system in which all roots have same norm, then one of the roots in V is an integral combination of the other roots in V.

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In this paper, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{R}^\infty$ is the *Euclidean space* of countably infinite dimension over the set of all real numbers, with the usual innerproduct $\langle\cdot,\cdot\rangle$—for all $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$ and for all $y = (y_1, y_2, \ldots)$, $\sum_{i=1}^\infty x_i^2 < \infty$ and $\langle x, y\rangle = \sum_{i=1}^\infty x_i y_i$. A linearly dependent subset V of $\mathbb{R}^\infty$ is called *critical* if every proper subset of V is linearly independent. If $v_1, v_2, \ldots, v_n$ are vectors in $\mathbb{R}^\infty$ and $t_1, t_2, \ldots, t_n$ are integers, then $t_1v_1 + t_2v_2 + \cdots + t_nv_n$ is called an *integral combination* of $v_1, v_2, \ldots, v_n$.

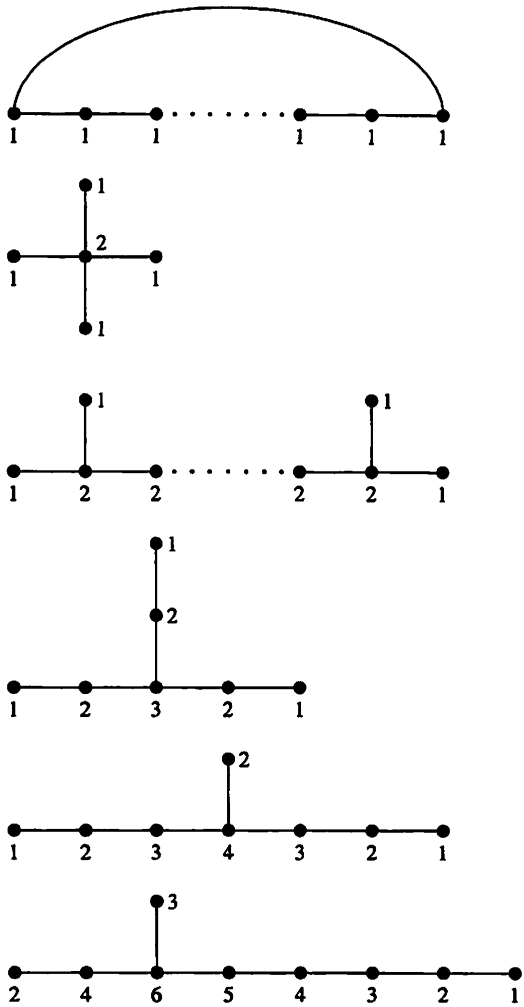
All graphs considered in this paper are finite; they have neither loops nor multiple edges. For graph theoretic terms and notation employed in this paper we follow [8].

Let G be a graph and V be its vertex set. A map θ from V to the set of all positive real numbers is called a *2-fold labelling* of G if the following holds.

(**) For all $v \in V$, $2\theta(v) = \sum_{x\in N(v)} \theta(x)$ where $N(v)$ is the neighbourhood of v.

If a graph has a 2-fold labelling then it is said to be a *2-fold graph*. The main result of this paper is the following.

Theorem 1. *If θ is a 2-fold labelling of a connected graph G whose vertex set is V and m is the minimum of $\{\theta(v) : v \in V\}$, then G is one of the graphs in the figure and each label is an integral multiple of m—i.e., for each $v \in V$, $\frac{\theta(v)}{m} \in \mathbb{Z}$.*



The family of all 2-fold graphs.

Proof. First suppose that G contains a cycle $C = (v_1, v_2, \ldots, v_n)$. Assume that $\theta(v_1)$ is the minimum of $\{\theta(v_i) : i = 1, \ldots, n\}$. Now for $i = 1$,

$\ldots, n$, by $(**)$, $2m = 2\theta(v_i) \geqslant \theta(v_{i-1}) + \theta(v_{i+1}) \geqslant 2m$ (here $v_0 = v_n$ and $v_{n+1} = v_1$) implying $\deg v_i = 2$ and $\theta(v_{i+1}) = m$. Consequently, being connected, G has to be the cycle C and the conclusion holds. So, henceforth we assume that G is a tree. Now, suppose $\Delta(G) \geqslant 4$. Then, there is a vertex a whose neighbourhood has at least 4 vertices, say a_1, a_2, a_3 and a_4. By $(**)$, for each $i \in \{1,2,3,4\}$, $\theta(a_i) \geqslant \frac{1}{2}\theta(a)$ and $2\theta(a) \geqslant \theta(a_1) + \theta(a_2) + \theta(a_3) + \theta(a_4) \geqslant 2\theta(a)$ yielding $\theta(a_1) = \theta(a_2) = \theta(a_3) = \theta(a_4) = \frac{1}{2}\theta(a)$. Therefore for each $i \in \{1,2,3,4\}$, $\deg a_i = 1$ and $\deg a = 4$. Consequently, by connectivity of G, $V = \{a, a_1, a_2, a_3, a_4\}$ and the conclusion holds. So, we assume that $\Delta(G) \leqslant 3$. Let $X = \{v \in V : \deg v = 3\}$ and $Y = \{v \in V : \deg v = 1\}$. Now summing the relations given by $(**)$ for all $v \in V$ and simplifying we get,

$$\sum_{x \in X} \theta(x) = \sum_{y \in Y} \theta(y) \tag{1}$$

Therefore $|X| \neq 0$; suppose $|X| \geqslant 2$. Then there is a path $(b_0, b_1, \ldots, b_{n-1}, b_n)$ such that $\deg b_0 = \deg b_n = 3$ and the degree of each internal vertex is 2. Let $N(b_0) - \{b_1\} = \{a_1, a_2\}$ and $N(b_n) - \{b_{n-1}\} = \{c_1, c_2\}$. Now summing the relations given by $(**)$ for $v = b_0, \ldots, b_n$ and simplifying, we get $\theta(b_0) + \theta(b_n) = \theta(a_1) + \theta(a_2) + \theta(c_1) + \theta(c_2)$. Since $\theta(a_1), \theta(a_2) \geqslant \frac{1}{2}\theta(b_0)$ and $\theta(c_1), \theta(c_2) \geqslant \frac{1}{2}\theta(b_n)$, it follows that $\theta(a_1) = \theta(a_2) = \frac{1}{2}\theta(b_0)$ and $\theta(c_1) = \theta(c_2) = \frac{1}{2}\theta(b_n)$. Therefore $\deg a_1 = \deg a_2 = \deg c_1 = \deg c_2 = 1$ and by connectivity of G, $V = \{a_1, a_2\} \cup \{b_0, \ldots, b_n\} \cup \{c_1, c_2\}$; additionally, for $k = 0, \ldots, n-1$, $v = b_k$ in $(**)$ yields $\theta(b_{k+1}) = \theta(b_0)$; therefore the conclusion holds. Now, assume that $|X| = 1$. This implies that G is formed by three paths which can be expressed as $(a_1, \ldots, a_p = w)$, $(b_1, \ldots, b_q = w)$ and $(c_1, \ldots, c_r = w)$ where w is the vertex of degree 3. By $(**)$, for $k = 2, \ldots, p$, $\theta(a_k) = k\theta(a_1)$; we have similar relations for other two paths also; thus $\theta(w) = p\theta(a_1) = q\theta(b_1) = r\theta(c_1)$; now by (1), $\theta(a_1) + \theta(b_1) + \theta(c_1) = \theta(w)$ yielding $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Let us assume that $p \geqslant q \geqslant r$. If $r \geqslant 3$, then $\frac{1}{p}, \frac{1}{q}, \frac{1}{r} \leqslant \frac{1}{3}$ implying $p = q = r = 3$ and therefore the conclusion holds. So, let us assume that $r = 2$. Then $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. If $q \geqslant 4$, then it can be seen that $p = q = 4$ and it follows that for each $v \in V$, $\theta(v)$ is an integral multiple of $\theta(a_1) = \theta(b_1)$ and G is the second graph from below in the figure. So assume that $q = 3$. Then $p = 6$ and it can be verified that for each $v \in V$, $\frac{\theta(v)}{\theta(a_1)} \in \mathbb{Z}$ and G is the last graph in the figure. □

Corollary 4, an application of the above result, is the main tool for proving Theorem 5; before deriving that, let us digress a little to observe an important property of 2-fold graphs which can yield an alternative proof for the above result. Let the adjacency matrix and the largest eigenvalue of a graph G be denoted by A and Λ respectively—sometimes $\Lambda(G)$ in place of Λ to avoid ambiguity. If θ is a 2-fold labelling of a graph whose vertex set is $\{v_1, \ldots, v_n\}$, then $A\alpha^\top = 2\alpha^\top$ where $\alpha = (\theta(v_1), \ldots, \theta(v_n))$ and $\alpha^\top$ is the transpose of α; thus it follows that 2 is an eigenvalue of any 2-fold graph. In fact, by combining Theorem 1 and a result of [6] which says that *for a connected graph, $\Lambda = 2$ if and only if it is one of the graphs in the figure*, we have the following result.

Proposition 2. *The following hold for a connected graph G.*

(1) *G has a 2-fold labelling.*
(2) *Its largest eigenvalue is 2.*
(3) *It is one of the graphs in the figure.*

By using a simple method found in [4] for showing the equivalence of (2) and (3), we can give a short proof for the above proposition; this proof is based on the following result. (Each part of this result is implied by Perron-Frobenius Theorem; for details, see [5, Theorem 31.11] and [2, Theorem 8.8.1].)

Theorem 3. *For a connected graph G, the following hold.*

(a) *There exists an eigenvector of A corresponding to Λ such that all of its coordinates are positive and every eigenvector of A, all of whose coordinates are nonnegative, is a scalar multiple of the former.*

(b) *Any eigenvalue of any proper subgraph of G is less than $\Lambda(G)$.*

Derivation of Proposition 2. If G is one of the graphs in the figure, then it is easy to verify that the labels of the vertices form a 2-fold labelling; i.e., (3) $\Rightarrow$ (1). If G has a 2-fold labelling, then, the numbers assigned by this labelling form a vector α satisfying $A\alpha^\top = 2\alpha^\top$ whence by (a) it can be verified that $\Lambda = 2$; i.e., (1) $\Rightarrow$ (2).

Now, suppose $\Lambda = 2$. It can be seen that there exists a graph H in the figure such that either G or H is a subgraph of the other. Noting that $\Lambda(H) = 2$ because of (3) $\Rightarrow$ (1) $\Rightarrow$ (2), and using (b) we get $G = H$. Thus, it follows that (2) $\Rightarrow$ (3).

Alternative Proof of Theorem 1. The labelling given by θ yields a vector α satisfying $A\alpha^\top = 2\alpha^\top$. Since G is one of the graphs in the figure by Proposition 2, we have $A\beta^\top = 2\beta^\top$ where β is the vector formed by the labels assigned to the vertices of G in the figure. Since both vectors have positive coordinates only, by (a) either is a scalar multiple of the other; since all the coordinates of β are integers and one of them is 1, the conclusion follows.

Corollary 4. *If V is a finite subset of $\mathbb{R}^\infty$ such that the following hold, then there exists a vector in V which is an integral combination of the other vectors in V.*

(A) *For all distinct* $u, v \in V$, $\langle u, u\rangle = 2$ *and* $\langle u, v\rangle = -1$ or 0.
(B) *There are positive real numbers* t_v, $v \in V$ *satisfying* $\sum_{v\in V} t_v v = 0$.

Proof. Define a graph as follows: Its vertex set is V and two vertices u, v are joined, if $\langle u, v\rangle = -1$; now for any $v \in V$, from $\langle \sum_{x\in V} t_x x, v\rangle = 0$, we have $2t_v = \sum_{x\in N(v)} t_x$. Let U be a subset of V such that the subgraph induced on U is a connected component; then by Theorem 1, there is a vector $a \in U$ such that $\frac{t_x}{t_a} \in \mathbb{Z}$ for all $x \in U$. Since for all $u \in U$ and $v \in V - U$, $\langle u, v\rangle = 0$, by (B) we get $\sum_{x\in U} t_x x = 0$ whence a serves our purpose. □

In [1], the following question has been raised. If V is a finite subset of $\mathbb{R}^n$ such that for all distinct $u, v \in V$, $\langle u, u\rangle = 2$ and $\langle u, v\rangle = 0$ or 1, does there exist a vector in V which is an integral combination of the rest in V? A process to settle a generalization of this question has been found in [7] by using the properties of Ω defined below; the current proof involves a simpler variation of that process.

Theorem 5. *If X is a finite linearly dependent subset of $\mathbb{R}^\infty$ such that for all $x, y \in X$, $\langle x, x\rangle = 2$ and $\langle x, y\rangle \in \mathbb{Z}$, then one of the vectors in X is an integral combination of the rest in X.*

Proof. Let Ω be the set of all integral combinations of vectors in X, which are of norm $\sqrt{2}$; for all $a, b \in \Omega$, we have the following.

$\langle a, b\rangle \in \{-2, -1, 0, 1, 2\}$ for $|\langle a, b\rangle| \leqslant \|a\|\,\|b\| = 2$.
If $\langle a, b\rangle = 2$, then $\|a - b\|^2 = 0$ and therefore $a = b$.
If $\langle a, b\rangle = 1$, then $\|a - b\|^2 = 2$ and therefore $a - b \in \Omega$.

Let $U = \{a_1, a_2, \ldots, a_k\}$ be a linearly dependent subset of Ω and M be the $k \times k$ matrix whose rows are $a_1, a_2, \ldots, a_k$—here we assume that $U \subset \mathbb{R}^k$. Since M is singular and the entries of $MM^\top$ are integers, there exists a nonzero vector $\sigma = (s_1, s_2, \ldots, s_k)$ in $\mathbb{Z}^k$ such that $\sigma MM^\top = 0$ implying $\sigma MM^\top\sigma^\top = 0$; i.e., $(\sigma M)(\sigma M)^\top = 0$; therefore $\sigma M = 0$. Thus we have the following.

(1) If $\{a_1, a_2, \ldots, a_k\}$ is a linearly dependent subset of Ω, then there exist integers $s_1, s_2, \ldots, s_k$, not all zero, such that $s_1a_1 + s_2a_2 + \cdots + s_ka_k = 0$.

When U is critical, choosing the above integers without any common divisor, define $\rho(U) = -(|s_1|+|s_2|+\cdots+|s_k|)$. [Note that $\rho(U)$ is independent of the choice of k-tuple $(s_1, s_2, \ldots, s_k)$ for, the only other choice is $(-s_1, -s_2, \ldots, -s_k)$.]

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a critical subset of Ω. It is enough to show that the conclusion holds with V in place of X. We can assume the following.

(2) The conclusion holds for any critical subset of Ω with cardinality $< n$.

Let $X = \{x_1, x_2, \ldots, x_m\}$; for any $x \in \Omega$ define $\eta(x) = (\langle x, x_1\rangle, \langle x, x_2\rangle, \ldots, \langle x, x_m\rangle)$. Then for any $x, y \in \Omega$, $\eta(x) = \eta(y) \Rightarrow x = y$; therefore $|\Omega| = |\{\eta(x) : x \in \Omega\}|$; consequently Ω is finite; so we can assume the following.

(3) If U is any critical subset of Ω such that $|U| = n$ and $\rho(U) < \rho(V)$, then the conclusion holds for U.

By (1), there exist integers $t_1, t_2, \ldots, t_n$ having no common divisors and satisfying

(4) $$t_1 v_1 + t_2 v_2 + \cdots + t_n v_n = 0.$$

For any $i \leqslant n$, we can assume that $t_i > 0$, for otherwise t_i and v_i can be replaced by $-t_i$ and $-v_i$ respectively. Thus (B) of Corollary 4 holds; so we can assume the existence of two vectors in V, say v_1 and v_2, such that their innerproduct is positive, for otherwise (A) of Corollary 4 also holds and the conclusion follows from that corollary. Then, $v_1 - v_2 \in \Omega$. We can assume $\{v_1 - v_2, v_3, \ldots, v_n\}$ to be linearly independent for otherwise it is critical and by (2) the conclusion holds for this subset and therefore for V also; then $S := \{v_1 - v_2, v_2, v_3, \ldots, v_n\}$ is critical. Now by (4), we have

(5) $$t_1(v_1 - v_2) + (t_1 + t_2)v_2 + t_3 v_3 + \cdots + t_n v_n = 0.$$

Since $\rho(S) < \rho(V)$, by (3) the conclusion holds for S and therefore for V also because one of the coefficients in the equation (5) is 1. □

Corollary 6. *Let Φ be a root system in which all roots have same norm and V be a linearly dependent subset of Φ. Then one of the roots in V is an integral combination of the other roots in V.*

Proof. We use one of the (defining) properties of Φ: for all $u, v \in \Phi$, $\frac{2\langle u,v\rangle}{\langle v,v\rangle} \in \mathbb{Z}$. (cf. [3].) Let $X = \left\{\frac{\sqrt{2}}{\|v\|}v : v \in V\right\}$. It is easy to verify that for all x, $y \in X$, $\langle x, x\rangle = 2$ and $\langle x, y\rangle \in \mathbb{Z}$. Therefore from Theorem 5, the result follows. □

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