On Sum Composite Graphs and Embedding Problems

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Abstract

A sum composite labeling of a (p, q) graph $G = (V, E)$ is an injective function $f : V(G) \to \{1, 2, \ldots, 2p\}$ such that the function $f^+ : E(G) \to C$ is also injective, where C denotes the set of all composite numbers and f^+ is defined by $f^+(uv) = f(u) + f(v)$ for all $uv \in E(G)$. A graph G is sum composite if there exists a sum composite labeling for G. We give some classes of sum composite graphs and some classes of graphs which are not sum composite. We prove that it is possible to embed any graph G with a given property P in a sum composite graph which preserves the property P, where P is the property of being the graph connected, eulerian, hamiltonian or planar. We also discuss the NP-completeness of the problem of determining the chromatic number and the clique number of sum composite graphs.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither multiple edges nor loops. The order and size of G are denoted by p and q respectively. Terms that are not defined here are used in the sense of Harary [1].

Graph labelings where the elements (i.e. vertices and/or edges) of a graph are assigned real values subject to certain conditions, have been motivated by practical problems. There is an enormous amount of literature built up on several kinds of labelings of graphs over the past three decades or so, and for a survey of results on graph labeling problems we refer to Gallian [2].

In this paper we introduce the concept of sum composite labelings of a graph and present several results on graphs which admit such a labeling.

We need the following theorem.

Theorem 1.1. [3] *Given any integer $k \geq 3$, the problem of deciding whether the chromatic number or clique number of a graph are greater than or equal to k is NP complete.*

2 Main Results

Definition 2.1. A sum composite labeling of a (p, q) graph $G = (V, E)$ is an injective function $f : V(G) \to \{1, 2, \ldots, 2p\}$ such that the function $f^+ : E(G) \to C$ is also injective, where C denotes the set of all composite numbers and f^+ is defined by $f^+(uv) = f(u) + f(v)$ for all $uv \in E(G)$. A graph G is sum composite if there exists a sum composite labeling for G.

Following are some immediate observations.

Observation 2.2. If f is a sum composite labeling of G, then $4 \leq f^+(uv) \leq 4p - 1$ for all $uv \in E(G)$.

Observation 2.3. A graph G has a sum composite labeling with all its vertex values odd if and only if G has a sum composite labeling with all its vertex values even.

Observation 2.4. If G has a sum composite labeling with all its vertex labels odd (even), then $q \leq 2p - 3$.

Observation 2.5. The complete graph K_n has a sum composite labeling with all the vertex labels odd (even) if and only if $n \leq 3$.

Theorem 2.6. *Cycles are sum composite.*

Proof. Let $C_n = (v_1v_2, \ldots, v_nv_1)$.
Case i. n is odd.

Define $f : V(C_n) \to \{1, 2, \ldots, 2n\}$ by $f(v_i) = 2i - 1$, $i = 1, 2, \ldots, n$. Then $f^+(v_iv_{i+1}) = 4i$, $i = 1, 2 \ldots, n-1$ and $f^+(v_nv_1) = 2n$ and hence f is a sum composite labeling of C_n.
Case ii. n is even.

Then $f : V(C_n) \to \{1, 2, \ldots, 2n\}$ defined by $f(v_i) = 2i - 1, i = 1, 2, \ldots, n-2$, $f(v_{n-1}) = 2n - 1$ and $f(v_n) = 2n - 3$ is a sum composite labeling of C_n. □

Theorem 2.7. *Any tree T is sum composite.*

Proof. We root the tree T at an arbitrary vertex v_1 and let $v_2, v_3, \ldots, v_n$ be the vertices of T in the order in which they are visited using the BFS algorithm. Then $f : V \to \{1, 2, \ldots, 2n\}$ defined by $f(v_i) = 2i - 1$ is a sum composite labeling of T. □

Theorem 2.8. *The quadrilateral Snake $S_{4,n}$ obtained from the path $P = (u_0, u_1, u_2, \ldots, u_n)$ by replacing the edge u_iu_{i+1} by the cycle $(u_iu_{i+1}v_{i+1}$ $w_{i+1}u_i)$, $0 \leq i \leq n-1$, is a sum composite graph.*

Proof. Define $f : V(S_{4,n}) \to \{1, 2, \ldots, 6n+2\}$, by

$f(u_0) = 1$,
$f(u_i) = 6i - 1$,
$f(v_i) = 1 + 6i$ and
$f(w_i) = 6i - 3$, $\quad i = 1, 2, \ldots, n$.

Clearly f is injective, $f(v)$ is odd for all vertices v, and f^+ is injective, so that f is a sum composite labeling of G. □

Theorem 2.9. *The Grid $L_{m,n} = P_m \times P_n$ is sum composite.*

Proof. Let $V(L_{m,n}) = \{u_{i,j}/1 \le i \le m, 1 \le j \le n\}$. We relabel the vertices as follows. $u_{1,1} = v_1$, $u_{2,1} = v_2$, $u_{1,2} = v_3$, $u_{3,1} = v_4$, $u_{2,2} = v_5$, $u_{1,3} = v_6$, $u_{4,1} = v_7$, and so on. Now if v_iv_j, $v_rv_s \in E(L_{m,n})$, then $i+j \neq r+s$. Hence $f : V \to \{1, 2, \ldots, 2mn\}$ defined by $f(v_i) = 2i-1, 1 \le i \le mn$, is a sum composite labeling for $L_{m,n}$. □

Theorem 2.10. *$K_{2,n}$ is sum composite for all n.*

Proof. Let $V_1 = \{u_1, u_2\}$, $V_2 = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of $K_{2,n}$. Define $f : V(K_{2,n}) \to \{1, 2, \ldots, 2n+4\}$ by $f(u_1) = 2$, $f(u_2) = 2n+4$ and $f(w_i) = 2i+2$, $i = 1, 2, \ldots, n$. Clearly, $f(v)$ is even for all v and f^+ is injective, so that f is a sum composite labeling of $K_{2,n}$. □

The crown $C_n \odot K_1$, is the graph obtained from the cycle C_n by attaching a pendant edge at each vertex of the cycle.

Theorem 2.11. *The crown $C_n \odot K_1$ is sum composite.*

Proof. Let $C_n = (v_1v_2 \ldots v_nv_1)$ and let u_i be the pendant vertex adjacent to v_i.
Case i. n is odd.
Let $n = 2m+1$. Define $f : V(C_{2m+1} \odot K_1) \to \{1, 2, \ldots, 4m+2\}$ by

$$f(v_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ 2m+1+i & \text{if } i \text{ is even} \end{cases}$$

and

$$f(u_i) = \begin{cases} 6m+4+i & \text{if } i \text{ is odd and } 1 \le i \le 2m-1 \\ 4m+3 & \text{if } i = 2m+1 \\ 4m+3+i & \text{if } i \text{ is even.} \end{cases}$$

It can be easily verified that f is a sum composite labeling of $C_n \odot K_1$.

Case ii. n is even.
Let $n = 2m$. Define $f : V(C_{2m} \odot K_1) \to \{1, 2, \ldots, 4m\}$ by

$$f(v_i) = \begin{cases} 2i-1 & \text{if } 1 \le i \le 2m-2 \\ 4m-1 & \text{if } i = 2m-1 \\ 4m-3 & \text{if } i = 2m \end{cases}$$

and

$$f(u_i) = \begin{cases} 4m+2i-1 & \text{if } 1 \le i \le 2m-2 \text{ and } i \neq m-1 \\ 8m-2 & \text{if } i = m-1 \\ 8m-1 & \text{if } i = 2m-1 \\ 8m-3 & \text{if } i = 2m. \end{cases}$$

Then f is a sum composite labeling of $C_n \odot K_1$. □

3 Some graphs which are not sum composite

Theorem 3.1. *$K_{3,3}$ is not sum composite.*

Proof. Let $V_1 = \{u_1, u_2, u_3\}, V_2 = \{w_1, w_2, w_3\}$ be the bipartition of $K_{3,3}$. If possible, suppose that $f : V(K_{3,3}) \to \{1, 2, \ldots, 12\}$ is a sum composite labeling of $K_{3,3}$. Then $4 \leq f^+(uv) \leq 23$. The set of all composite numbers in this collection is $\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22\}$, which contains exactly three odd numbers. Hence the number of edges with $f^+(uv)$ odd is either zero or three. Hence we have the following three possibilities.

i. Exactly one vertex value is odd.

ii. Exactly one vertex value is even.

iii. All the vertex values are of the same parity.

In the first two cases, all the three odd composites should occur as edge values. To get 21, as an edge label, we may assume that $f(u_1) = 9$ and $f(w_1) = 12$. But then we cannot get 9 as an edge value, which is a contradiction.

In the third case all the edge values are even. Suppose $4 \in f^+(E(K_{3,3}))$. Then $1, 3 \in f(V(K_{3,3}))$. So all the vertex values are odd and hence $22 \notin f^+(E(K_{3,3}))$. Therefore $f^+(E(K_{3,3})) = \{4, 6, 8, 10, 12, 14, 16, 18, 20\}$. Without loss of generality let $f(u_1) = 1$ and $f(w_1) = 3$. To get 6 and 8 as edge values, w_2 and w_3 should be given the values 5 and 7 respectively. But we cannot label u_2 or u_3 so as to get 10 as an edge value, which is a contradiction.

Now, suppose $4 \notin f^+(E(K_{3,3}))$. Then $f^+(E(K_{3,3})) = \{6, 8, 10, 12, 14, 16, 18, 20, 22\}$. Since $22 \in f^+(E(K_{3,3}))$, 10 and 12 are vertex values and so all the vertex values are even. Without loss of generality assume that $f(u_1) = 2$ and $f(w_1) = 4$. To get 8 and 10 as edge values, w_2 and w_3 should be given the values 6 and 8 respectively. But then we cannot label u_2 or u_3 so as to get 12 as an edge value, which is a contradiction. □

Remark 3.2. By a similar argument, it can be proved that $K_{4,5}$ is not sum composite.

Theorem 3.3. *The complete graph K_4 is not sum composite.*

Proof. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Suppose f is a sum composite labeling for K_4. Then $f : V(G) \to \{1, 2, \ldots, 8\}$ is injective and $4 \leq f^+(uv) \leq 15$. Hence $f^+(E(K_4)) \subset \{4, 6, 8, 9, 10, 12, 14, 15\}$, so that $f^+(v_iv_j)$ is odd for at most two edges. Clearly $f(v_i)$ cannot be odd for all i, since in this case we get two edges with label 8. Similarly $f(v_i)$ cannot be even for all i. If

$f(v_i)$ is odd for exactly one i or for exactly two values of i, or three values of i, then the number of edges with odd label is at least 3, which is a contradiction. Hence K_4 is not a sum composite graph. □

4 Embedding

Since there are graphs which are not sum composite, the following questions naturally arise.

i. Is it possible to embed every graph in a sum composite graph?

ii. Is it possible to embed a graph with a property P in a sum composite graph with property P?

The following theorem gives an affirmative answer to the first question.

Theorem 4.1. *Every (p, q) graph can be embedded as an induced subgraph of a sum composite graph with 5^p vertices and q edges.*

Proof. Let G be a (p, q) graph and let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let G' be the graph obtained from G by adding $5^p - p$ isolated vertices. Let f be the function which assigns the value 5^i to v_i and the values $1, 2, 3, 4, 6, \ldots 5^2 - 1, 5^2 + 1, \ldots, 5^3 - 1, 5^3 + 1, \ldots, 5^4 - 1, 5^4 + 1, \ldots, 5^p - 1$ to the isolated vertices. Clearly f is injective. Now, suppose $f^+(v_iv_j) = f^+(v_kv_l)$, for distinct edges v_iv_j, v_kv_l. Then we get the equation

$$5^i + 5^j = 5^k + 5^l \tag{1}$$

If $i = k$, then dividing the equation (1) by 5^i, we get $1 + 5^{j-i} = 1 + 5^{l-i}$, and hence $j = l$, which is a contradiction, since the edges v_iv_j and v_kv_l are distinct.

If $i < k$, then dividing the equation (1) by 5^i, we get $1 + 5^{j-i} = 5^{k-i} + 5^{l-i}$, which is a contradiction, since 5 divides the number on the right side but 5 does not divide the number on the left side.

We get a similar contradiction if $i > k$. Thus $f^+ : E(G) \to C$ is injective and hence f is a sum composite labeling of G' and G is an induced subgraph of G'. □

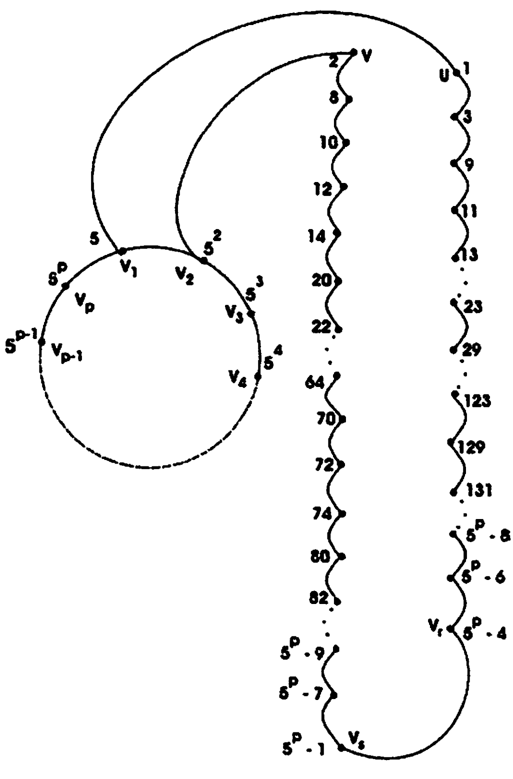
The graph G' constructed in Theorem 4.1 is disconnected. The following theorem shows that a connected graph can be embedded as an induced subgraph of a connected sum composite graph.

Theorem 4.2. *Any connected (p, q) graph G can be embedded as an induced subgraph of a connected sum composite graph with $5^p - p^2 - 4$ vertices and $5^p - p^2 - p + q - 3$ edges.*

Proof. Let G be a connected (p, q) graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$ such that v_1 and v_2 are adjacent vertices. Let S be the set obtained from the set $X = \{1, 2, 3, \ldots, 5^p\}$ by removing the set of numbers $\{5^i, 5^i + 2/1 \leq i \leq p-1\} \bigcup \{\frac{1}{2}(5^i + 5^j) + 1,\ \frac{1}{2}(5^i + 5^j) + 3/1 \leq i < j \leq p\} \bigcup \{4,\ 6,\ \ 5^p,\ 5^p - 2,\ \ 5^p - 3,\ 5^p - 5\}$. Clearly $|S| = 5^p - p^2 - p - 4$.

Let G' be the graph with $V(G') = V(G) \bigcup S$, $< V(G) >= G$, $< S >$ is the path P with origin 1, terminus 2 and with internal vertices consisting of all the odd numbers in S in the increasing order followed by all the even numbers in S in the decreasing order, along with the edges $\{1, v_1\}$ and $\{2, v_2\}$. Clearly G is an induced subgraph of G'. Now f defined on $V(G')$ by $f(v_i) = 5^i$, $1 \leq i \leq p$ and $f(a) = a$ for all $a \in S$ is a sum composite labeling for the connected graph G'. □

The construction is illustrated in the following figure.



Embedding of a graph in a sum composite graph

Corollary 4.3. *Every disconnected graph H can be embedded as an induced subgraph of a connected sum composite graph.*

Proof. The result follows by applying Theorem 4.2 to the connected graph $G = H + v_1$. □

Corollary 4.4. *Every hamiltonian graph G can be embedded as an induced subgraph of a hamiltonian sum composite graph.*

Proof. If G is hamiltonian and v_1v_2 is an edge of a hamiltonian cycle, then the graph G' obtained in Theorem 4.2 is also hamiltonian. □

Corollary 4.5. *Every planar graph G can be embedded as an induced subgraph of a planar sum composite graph.*

Corollary 4.6. *Given any integer $k \geq 3$, the problem of deciding whether the chromatic number or clique number of a graph is greater than or equal to k is NP-complete even when restricted to sum composite graphs.*

Proof. Since $\chi(G') = \chi(G)$ and $\omega(G') = \omega(G)$, the result follows Theorem 1.1. □

Corollary 4.7. *Every eulerian graph G can be embedded as an induced subgraph of an eulerian sum composite graph.*

Proof. In the proof of Theorem 4.2, replace S by $S_1 = (S - \{1, 2, 8\}) \bigcup \{4\}$. Let G' be the graph with $V(G') = V(G) \bigcup S_1, < V(G) >= G, < S_1 \bigcup \{v_1\} >$ is the cycle with origin and terminates 3, consisting of all the odd integers of S_1 in the increasing order, followed by all the even integers of S in the decreasing order and the vertex v_1. Clearly G is an induced subgraph of G'. Now f defined on $V(G')$ by $f(v_i) = 5^i$, $1 \leq i \leq p$ and $f(a) = a$ for all $a \in S$ is a sum composite labeling for the graph G'. Since G is eulerian, G' also is eulerian. □

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