Asteroidal Chromatic Number of a Graph

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Abstract

Let $G = (V, E)$ be a connected graph. A subset A of V is called an asteroidal set if for any three vertices u, v, w in A, there exists a u-v path in G that avoids the neighbourhood of w. The asteroidal chromatic number χ_a of G is the minimum order of a partition of V into asteroidal sets. In this paper we initiate a study of this parameter. We determine the value of χ_a for several classes of graphs, obtain sharp bounds and Nordhaus-Gaddum type results.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. For graph theoretic terminology we refer to Harary [2].

An independent set of three vertices is called an asteroidal triple if between any pair in the triple there exists a path that avoids the neighborhood of the third. A graph G is asteroidal triple free (AT-free) if it contains no asteroidal triple. AT-free graphs were introduced by Lekkerkerker and Boland [4], who showed that a graph is an interval graph if and only if it is chordal and AT-free. AT-free graphs offer a common generalisation of interval graphs, permutation graphs, trapezoid graphs and cocomparability graphs. Corneil et al. [1] observed that the property of being AT-free enforces linearity in these four classes of graphs.

Walter [7] introduced the concept of asteroidal sets and asteroidal number of a graph. A subset A of V is called an asteroidal set if for every $a \in A$ there exists a connected component of $G - N[a]$ containing all the vertices of $A - \{a\}$. The asteroidal number $an(G)$ of G is the maximum cardinality of an asteroidal set in G. It follows immediately from the definition that any asteroidal set is independent and hence $an \leq \beta_0$, where β_0 is the independence number of G. Further if $G - N[v]$ is connected for every vertex v, then every independent set is an asteroidal set and hence $an(G) = \beta_0$.

Since any asteroidal set is independent, the concept of chromatic number can be generalized in a natural way using asteroidal sets. In this paper we introduce the concept of asteroidal chromatic number of a graph and initiate a study of this new parameter.

2 Main Results

Throughout this paper $G = (V, E)$ is a connected graph of order p and size q.

Definition 2.1. *The asteroidal chromatic number $\chi_a(G)$ of G is the minimum order of a partition of V into asteroidal sets.*

Example 2.2.

(i) *Since paths P_n with $n \geq 3$ and complete bipartite graphs $K_{m,n}$ are both AT-free, $\chi_a(P_n) = \lceil \frac{n}{2} \rceil$ and $\chi_a(K_{m,n}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$, where for any real number x, $\lceil x \rceil$ is the least integer greater than or equal to x. This shows that χ_a can be much larger than χ.*

(ii) *If $G - N[v]$ is connected for every $v \in V$, then $\chi_a = \chi$. In particular, for the cycle C_n we have $\chi_a(C_n) = \chi(C_n)$.*

We observe that $\chi_a(G) = p$ if and only if $G = K_p$. The following theorem gives a characterization of graphs for which $\chi_a = p - 1$.

Theorem 2.3. *The asteroidal chromatic number of a graph G with p vertices is $p-1$ if and only if $G = K_p - \{e_1, e_2, \ldots, e_j\}$ where $1 \leq j \leq p-2$ and the subgraph induced by the edges $e_1, e_2, \ldots, e_j$ is a star.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $\chi_a(G) = p-1$. Let $\{\{v_1, v_2\}, \{v_3\}, \ldots, \{v_p\}\}$ be an asteroidal chromatic partition of G. Then the subgraph induced by $\{v_3, v_4, \ldots, v_p\}$ is complete, v_1 is not adjacent to v_2 and all non-adjacent pairs of vertices $\{u, v_i\}$ with $u \in \{v_1, v_2\}$ and $v_i \in \{v_3, v_4, \ldots, v_p\}$

are such that $u = v_1$ for all v_i or $u = v_2$ for all v_i. Hence $G = K_p - \{e_1, e_2, \dots, e_i\}$ where $1 \le j \le p-2$ and the subgraph induced by the edges $e_1, e_2, \dots, e_j$ is a star. The converse is obvious. □

Though the asteroidal chromatic number of a complete bipartite graph is larger than its chromatic number, there are k-regular bipartite graphs G with $\chi_a = \chi = 2$, as shown in the following theorem.

Theorem 2.4. *For any positive integer k, there exists a k-regular bipartite graph G with $\chi_a(G) = 2$.*

Proof. If $k = 1$, we take $G = K_2$. If $k \ge 2$, let $X = \{u_0, u_1, u_2, \dots, u_{n-1}\}$, $Y = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ where n is any positive integer with $n > k$. Let $N(u_i) = \{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ where addition in the suffices is taken modulo n. This gives a k-regular bipartite graph G such that $N(v_i) = \{u_i, u_{i-1}, \dots, u_{i-k+1}\}$. Further $G - N[v]$ is connected for every vertex v and hence $\chi_a(G) = \chi(G) = 2$. □

We now proceed to obtain sharp bounds for χ_a.

Theorem 2.5. *Let G be any connected graph of order p. Then $\chi_a \le p - \left\lfloor \frac{\beta_0}{2} \right\rfloor$. Further $\chi_a(G) = p - \left\lfloor \frac{\beta_0}{2} \right\rfloor$ if and only if G is complete or the following hold.*

(*i*) *G is a split graph with split partition (K, S) where $G[K]$ is complete and S is independent.*

(*ii*) *If $|S|$ is odd, every vertex in S is adjacent to every vertex in K.*

(*iii*) *If $|S|$ is even and if there exist nonadjacent vertices u, v with $u \in K$ and $v \in S$, then any other nonadjacent pair of vertices a, b with $a \in K, b \in S$ and $a \ne u$ is such that $b = v$.*

Proof. Let S be a maximum independent set in G so that $|S| = \beta_0$. Since any two-element subset of S is an asteroidal set, S can be partitioned into $\left\lceil \frac{\beta_0}{2} \right\rceil$ asteroidal sets. This partition together with the collection of all singleton subsets of $V - S$ is an asteroidal chromatic partition of G and hence $\chi_a(G) \le p - \beta_0 + \left\lceil \frac{\beta_0}{2} \right\rceil = p - \left\lfloor \frac{\beta_0}{2} \right\rfloor$.

Now, if $\chi_a(G) = p - \left\lfloor \frac{\beta_0}{2} \right\rfloor$, then the subgraph induced by $K = V - S$ is complete and hence G is a split graph. If $|S|$ is odd and if there exists a pair of nonadjacent vertices u, v with $u \in S$ and $v \in K$, then a partition of $S - \{u\}$ into two element subsets together with $\{u, v\}$ and singleton subsets of $K - \{v\}$ form an asteroidal partition of cardinality $p - \left\lfloor \frac{\beta_0}{2} \right\rfloor - 1$, which is a contradiction. This proves (i) and (ii) and the proof of (iii) is similar.

Conversely, if G is a graph satisfying the conditions (i), (ii) and (iii), then G is AT-free and hence $\chi_a(G) = p - \left\lfloor \frac{\beta_0}{2} \right\rfloor$. □

Theorem 2.6. *Let G be a connected graph of order p. Then $p/\beta_0 \leq p/an \leq \chi_a \leq p - an + 1$.*

Proof. Let $\chi_a(G) = n$ and let $\{V_1, V_2, \ldots, V_n\}$ be an asteroidal chromatic partition of G. Since $|V_i| \leq an$ and $\sum_{i=1}^{n} |V_i| = p$, we have $p/an \leq \chi_a$. Also if S is a maximum asteroidal set in G, then $\{S\} \cup \{\{v\}/v \in V - S\}$ is an asteroidal chromatic partition of G and hence $\chi_a \leq p - an + 1$. Further $an \leq \beta_0$, and hence $p/\beta_0 \leq p/an$. □

Remark 2.7. *The bounds given in Theorem 2.6 are sharp. For any even cycle C_n, we have $p/an = p/\beta_0 = \chi_a = 2$. The class of all graphs for which $\chi_a = p - an + 1$ is given in the following theorem.*

Theorem 2.8. *Let G be a connected graph. Then $\chi_a = p - an + 1$ if and only if G is complete or the following hold.*

(i) *G is a split graph with split partition (K, S), where $G[K]$ is complete and S is independent.*

(ii) *$N(u) \not\subseteq N(v)$ for all $u, v \in S$.*

(iii) *If H is the subgraph of G induced by the set of all edges of the form uv with $u \in S$ and $v \in K$, then there does not exist a matching in H that saturates all the vertices of S.*

Proof. If G is complete, $\chi_a = p - an + 1 = p$. Suppose G satisfies the conditions (i), (ii) and (iii). Let $u, v, w \in S$. By (ii), there exist vertices v' and w' such that $v' \in N(v) - N(u)$ and $w' \in N(u) - N(v)$. Now (v, v', w', w) is a v-w path in $G - N[u]$ and hence S is an asteroidal set. Further $S \cup \{x\}$, where $x \in K$, is not an asteroidal set. Thus S is a maximum asteroidal set and $|S| = an$ and $|K| = p - an$. Now, let ζ be any asteroidal chromatic partition of G. Since any element of ζ can cover at most one element of K, it follows that $|\zeta| \geq p - an$. Further it follows from (iii) that ζ contains at least one set A such that $A \subseteq S$ and hence $|\zeta| \geq p - an + 1$. Thus $\chi_a \geq p - an + 1$ and hence by Theorem 2.6, we have $\chi_a = p - an + 1$. Conversely, let G be a connected graph with $\chi_a = p - an + 1$. Suppose G is not complete. Let S be a maximum asteroidal set in G. Then $K = V - S$ is a clique in G, so that G is a split graph. Since S is an asteroidal set, $N(u) \not\subseteq N(v)$ for all $u, v \in S$. Also if there exists a matching M in H that saturates all vertices in S, then $\{\{u, u'\} : u \in S, uu' \in M\} \cup \{\{v\} : v \in K$ and v is not M-saturated $\}$ is an asteroidal chromatic partition of G so that $\chi_a \leq p - an$, which is a contradiction. □

Nordhaus and Gaddum [5] obtained bounds for the sum and product of the chromatic number of a graph and its complement. In the following theorem we obtain similar results for χ_a.

Theorem 2.9. *Let χ_a and $\overline{\chi}_a$ denote respectively the asteroidal chromatic number of G and its complement $\overline{G}$. Then*

(i) $2\sqrt{p} \leq \chi_a + \overline{\chi}_a \leq p + \lceil \frac{p}{2} \rceil$ *and*

(ii) $p \leq \chi_a \overline{\chi}_a \leq (\frac{3p+1}{2})^2$.

Proof. Let $\chi_a = n$ and let $\{V_1, V_2, \ldots, V_n\}$ be an asteroidal chromatic partition of G. Then $\max |V_i| \geq p/n$ and since each V_i induces a complete graph in $\overline{G}$, we have $\overline{\chi}_a \geq \max |V_i| \geq p/n = p/\chi_a$. Hence $\chi_a \overline{\chi}_a \geq p$. Also $\chi_a + \overline{\chi}_a \geq 2\sqrt{(\chi_a \overline{\chi}_a)} \geq 2\sqrt{p}$. We now prove the inequality $\chi_a + \overline{\chi}_a \leq p + \lceil \frac{p}{2} \rceil$ by induction on p. The inequality is obvious when $p = 1$ or 2. Suppose the result is true for any graph with at most $p - 1$ vertices. Let G be any graph with p vertices and $p \geq 3$. Let u, v be any two adjacent vertices in G and let $H = G - \{u, v\}$. By induction hypothesis $\chi_a(H) + \chi_a(\overline{H}) \leq p - 2 + \lceil \frac{p-2}{2} \rceil$. Also $\chi_a(G) \leq \chi_a(H) + 2$ and $\chi_a(\overline{G}) \leq \chi_a(\overline{H}) + 1$. Hence it follows that $\chi_a(G) + \chi_a(\overline{G}) \leq p + \lceil \frac{p}{2} \rceil$. Now $\chi_a \overline{\chi}_a \leq \frac{(\chi_a + \overline{\chi}_a)^2}{4} \leq (\frac{3p+1}{2})^2$. □

Remark 2.10. *The bounds given in Theorem 2.9 are sharp. For the cycle $C_4, \chi_a + \overline{\chi}_a = 4 = 2\sqrt{p}$ and $\chi_a \overline{\chi}_a = 4 = p$.*

The class of all graphs for which $\chi_a + \overline{\chi}_a = p + \lceil \frac{p}{2} \rceil$ is given in the following theorem.

Theorem 2.11. $\chi_a + \overline{\chi}_a = p + \lceil \frac{p}{2} \rceil$ *if and only if $G = K_p$ or $\overline{K}_p$.*

Proof. Let G be a graph with $\chi_a + \overline{\chi}_a = p + \lceil \frac{p}{2} \rceil$. We prove by induction on p that if G has no isolated vertices, then G is complete. The result is obviously true when $p = 2$ or 3. We now assume that the result is true for any graph with at most $p - 1$ vertices. Let G be a graph with $p \geq 4$. Let $u, v \in V(G)$. Let w_1, w_2 be two vertices distinct from u and v and let $H = G - \{w_1, w_2\}$. Since $\chi_a(G) + \chi_a(\overline{G}) \leq \chi_a(H) + \chi_a(\overline{H}) + 3$ and $\chi_a(G) + \chi_a(\overline{G}) = p + \lceil \frac{p}{2} \rceil$, it follows that $\chi_a(H) + \chi_a(\overline{H}) = p - 2 + \lceil \frac{p-2}{2} \rceil$. By induction hypothesis H is complete, so that u and v are adjacent in H. Hence u and v are adjacent in G also, so that G is complete. The converse is obvious. □

We now prove that the problem of deciding whether $\chi_a(G) \leq 3$ is NP-complete even when restricted to the family of line graphs.

Theorem 2.12. *Deciding whether $\chi_a(G) \leq 3$ is NP-complete.*

Proof. The problem of deciding whether a 3-edge connected cubic graph G is 3-edge colourable is NP-complete [3].

Let G be a 3-edge connected cubic graph and let $L(G)$ be its line graph. Let $x = uv \in E(G)$. We claim that $L(G) - N[x]$ is connected. Let e_1, e_2 be the other two edges incident at u and let e_3, e_4 be the other two edges incident at v.

Since G is 3-edge connected, $G - \{x, e_1, e_2, e_3, e_4\}$ has at most one non-trivial component, since otherwise at most two of the edges e_1, e_2, e_3, e_4 form an edge-cut. Hence $L(G) - N[x]$ is connected. For any vertex v of $L(G)$, we have $L[G] - N[v]$ is connected and hence $\chi_a(L(G)) = \chi(L(G)) = \chi'(G)$, so that $\chi_a(L(G)) \leq 3$ if and only if $\chi'(G) \leq 3$.

Since the problem of deciding whether $\chi'(G) \leq 3$ is NP-complete [6], it follows that the problem of deciding whether $\chi_a(L(G)) \leq 3$ is also NP-complete. □

We conclude with the following problems for further investigation.

Problem 2.13. *Characterize the class of graphs for which* $\chi_a = \chi$.

Problem 2.14. *Characterize the class of graphs for which* $\chi_a = \frac{p}{an}$.

References

[1] D.G. Corneil, S. Olariu and L. Stewart, Asteroidal triple-free graphs, *SIAM J. Discrete Math.*, **10**(1997), 399–430.

[2] F. Harary, *Graph Theory*, Addition Wesley, Reading Mass., 1969.

[3] I. Holyer, The NP-completeness of edge-colouring, *SIAM J. Computing,* **10**(1981), 718–720.

[4] C.G. Lekkerkerker and J.C. Boland, Representation of a finite graph by a set of intervals on the real line, *Fund. Math.*, **51**(1962), 45–64.

[5] E.A. Nordhaus and J.W. Gaddum, On complementary graphs, *Amer. Math. Monthly,* **63**(1956), 175–177.

[6] L. Stockmeyer, Planar 3-colourability is polynomial complete, *ACM SIGACT News,* **5**(1973), 19–25.

[7] J.R. Walter, Representations of Chordal graphs as subtrees of a tree, *J. Graph Theory* , **2** (1978), 265–267.