Graphs with Unique Minimum Simple Acyclic Graphoidal Cover *

S. ARUMUGAM AND I. SAHUL HAMID
Core Group Research Facility (CGRF)
National Centre for Advanced Research in Discrete Mathematics
(n-CARDMATH)
Kalasalingam University
Anand Nagar,Krishnankoil-626 190.
Tamil Nadu, INDIA
e-mail: *s_arumugam_akce@yahoo.com*

Abstract

A simple acyclic graphoidal cover of a graph G is a collection ψ of paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ, every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of G and is denoted by $\eta_{as}(G)$. A simple acyclic graphoidal cover ψ of G with $|\psi| = \eta_{as}$ is called a *minimum simple acyclic graphoidal cover* of G. Two minimum simple acyclic graphoidal covers ψ_1 and ψ_2 of G are said to be *isomorphic* if there exists an automorphism α of G such that $\psi_2 = \{\alpha(P) : P \in \psi_1\}$. In this paper we characterize trees, unicyclic graphs and wheels in which any two minimum simple acyclic graphoidal covers are isomorphic.

Keywords. Simple acyclic graphoidal cover.
2000 Mathematics Subject Classification: 05C

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q

*Research supported by NBHM Project 48/2/2004/R&D-II/7372.

respectively. For graph theoretic terminology we refer to Harary [7]. All graphs in this paper are assumed to be connected and non-trivial.

If $P = (v_0, v_1, v_2, \ldots, v_n)$ is a path or a cycle in a graph G, then $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P and v_0, v_n are called external vertices of P. If $P = (v_0, v_1, v_2, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, w_2, \ldots, w_m)$ are two paths in G, then the walk obtained by concatenating P and Q at v_n is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_2, v_1, v_0)$ is denoted by P^{-1}.

Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an *interior* vertex of ψ it is an internal vertex of some path in ψ; otherwise it is called an *exterior* vertex of ψ.

The concepts of graphoidal cover and acyclic graphoidal cover were introduced by Acharya and Sampathkumar [1] and Arumugam et al. [6].

Definition 1.1. [1] *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.*

(i) *Every path in ψ has at least two vertices.*

(ii) *Every vertex of G is an internal vertex of at most one path in ψ.*

(iii) *Every edge of G is in exactly one path in ψ.*

If further no member of ψ is a cycle in G, then ψ is called an acyclic graphoidal cover of G. The minimum cardinality of a (acyclic) graphoidal cover of G is called the (acyclic) graphoidal covering number of G and is denoted by $(\eta_a(G))$ $\eta(G)$. A graphoidal cover ψ of G with $|\psi| = \eta(G)$ is called a minimum graphoidal cover of G.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in Arumugam et al. [2].

For any graph $G = (V, E)$, $\psi = E(G)$ is trivially a graphoidal cover of G and has the interesting property that any two paths in ψ have at most one vertex in common. Motivated by this observation we introduced the concepts of simple graphoidal covers [4] and simple acyclic graphoidal covers [5] in graphs.

Definition 1.2. [4] *A simple graphoidal cover of a graph G is a graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of G and is denoted by $\eta_s(G)$. A simple graphoidal cover ψ of G with $|\psi| = \eta_s(G)$ is called a minimum simple graphoidal cover of G. Similarly we define the simple acyclic graphoidal covering number $\eta_{as}(G)$ and a minimum simple acyclic graphoidal cover of G.*

Theorem 1.3. [5] *If there exists a simple acyclic graphoidal cover ψ of G such that every vertex v of G with $\deg v > 1$ is an internal vertex of a path in ψ, then ψ is a minimum simple acyclic graphoidal cover of G and $\eta_{as}(G) = q - p + n$, where n is the number of pendant vertices of G.*

Theorem 1.4. [5] *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m be the number of vertices of degree greater than 2 on C. Then*

$$\eta_{as}(G) = \begin{cases} 3 & \text{if } m = 0 \\ n+2 & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n & \text{if } m \geq 3. \end{cases}$$

Theorem 1.5. [5] *For the wheel $W_n = K_1 + C_{n-1}$, we have*

$$\eta_{as}(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ n+1 & \text{if } n \geq 5. \end{cases}$$

Arumugam et al. [3] introduced the concept of isomorphism between graphoidal covers of a graph G and studied the properties of graphs in which any two minimum graphoidal covers are isomorphic. Arumugam and Suresh Suseela [6] extended the notion of isomorphism between graphoidal covers to acyclic graphoidal covers.

Definition 1.6. [3] *Two graphoidal covers ψ_1 and ψ_2 of a graph G are said to be isomorphic if there exists an automorphism α of G such that $\psi_2 = \{\alpha(P) : P \in \psi_1\}$. A graph G is said to have a unique minimum graphoidal cover if any two minimum graphoidal covers of G are isomorphic.*

Theorem 1.7. [3] *A tree T has a unique minimum graphoidal cover if and only if there exists at most one vertex v with $\deg v > 2$ and the distance from v to all the pendant vertices of T are equal.*

In this paper we extend the notion of isomorphism between graphoidal covers of a graph to simple acyclic graphoidal covers and characterize trees, unicyclic graphs and wheels having a unique minimum simple acyclic graphoidal cover.

2 Main Results

Definition 2.1. *Two simple acyclic graphoidal covers ψ_1 and ψ_2 of a graph G are said to be isomorphic if there exists an automorphism α of G such that $\psi_2 = \{\alpha(P) : P \in \psi_1\}$. A graph G is said to have a unique minimum simple acyclic graphoidal cover if any two minimum simple acyclic graphoidal covers of G are isomorphic.*

Remark 2.2. *Let ψ_1 be a simple acyclic graphoidal cover of G and let $P \in \psi_1$. Then $\psi_2 = (\psi_1 - \{P\}) \cup \{P^{-1}\}$ is also a simple acyclic graphoidal cover of G and we adopt the convention that the simple acyclic graphoidal covers ψ_1 and ψ_2 are isomorphic.*

Remark 2.3. Clearly any two isomorphic simple acyclic graphoidal covers of a (p, q)-graph G give rise to the same partition of the integer q. However the converse is not true. For example consider the graph G given in Figure 1.

Let $\psi_1 = \{(v_5, v_1, v_2, v_6), (v_1, v_4, v_8), (v_2, v_3, v_7), (v_4, v_3)\}$ and
$\psi_2 = \{(v_5, v_1, v_2, v_3)(v_1, v_4, v_8), (v_4, v_3, v_7), (v_6, v_2)\}$.

Then ψ_1 and ψ_2 are two minimum simple acyclic graphoidal covers of G which are not isomorphic. However ψ_1 and ψ_2 determine the same partition of q.

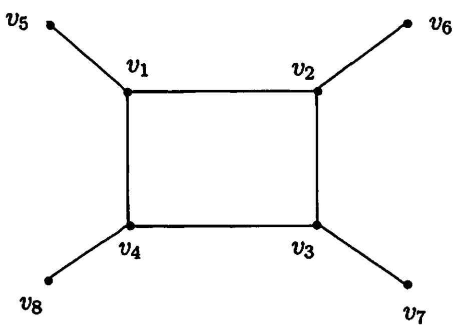

Figure 1

Remark 2.4. Since every member of a simple acyclic graphoidal cover of a graph is an induced path, for a complete graph G, $\psi = E(G)$ is the only simple acyclic graphoidal cover.

Theorem 2.5. *Let $W_n = K_1 + C_{n-1}$ be a wheel on n vertices. Then W_n has a unique minimum simple acyclic graphoidal cover if and only if $n = 4$ or 5.*

Proof. Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\}$.

If $n = 4$, then $W_4 = K_4$ and hence $\psi = E(W_4)$ is the only simple acyclic graphoidal cover.

If $n = 5$, then $\psi = \{(v_1, v_2, v_3), (v_2, v_0, v_4), (v_0, v_1), (v_0, v_3), (v_1, v_4), (v_3, v_4)\}$ is a minimum simple acyclic graphoidal cover of W_5. Clearly, any minimum simple acyclic graphoidal cover of W_5 is isomorphic to ψ. Hence W_5 has a unique minimum simple acyclic graphoidal cover.

Suppose $n \geq 6$. Let $P_1 = (v_1, v_2, \ldots, v_{n-2})$, $P_2 = (v_{n-3}, v_0, v_{n-1})$, $Q_1 = (v_1, v_2, \ldots, v_{n-3})$, $Q_2 = (v_{n-3}, v_{n-2}, v_{n-1})$ and $Q_3 = (v_{n-2}, v_0, v_1)$. Let S_1 be the set of edges of W_n not covered by the paths P_1 and P_2. Let S_2 be the set of edges of W_n not covered by the paths Q_1, Q_2 and Q_3. Then $\psi_1 = \{P_1, P_2\} \cup S_1$ and $\psi_2 = \{Q_1, Q_2, Q_3\} \cup S_2$ are two minimum simple acyclic graphoidal covers of W_n. Since $n \geq 6$, both ψ_1 and ψ_2 determine two different partitions of the integer $q = 2(n-1)$ and hence ψ_1 and ψ_2 are not isomorphic. □

We now proceed to characterize trees and unicyclic graphs having a unique minimum simple acyclic graphoidal cover.

Theorem 2.6. *A tree T has a unique minimum simple acyclic graphoidal cover if and only if there exists at most one vertex v with $\deg v > 2$ and all the pendant vertices of T are at the same distance from v.*

Proof. Since every graphoidal cover of a tree T is a simple acyclic graphoidal cover of T, the result follows from Theorem 1.7. □

Theorem 2.7. *A unicyclic graph G has a unique minimum simple acyclic graphoidal cover if and only if G is either C_3 or C_4 or a graph obtained by attaching a path to a vertex of a triangle.*

Proof. Let $C = (v_1, v_2, \ldots, v_k, v_1)$ be the cycle in G. Let m denote the number of vertices of degree greater than 2 on C. Let n be the number of pendant vertices in G. Suppose G has a unique minimum simple acyclic graphoidal cover.

We claim that $m \leq 1$. Suppose $m \geq 3$. Let v_1, v_i and v_j, where $1 < i < j \leq k$, be vertices of degree greater than 2 on C. Let $P = (v_1, w_1, w_2, \ldots, w_r)$ be the longest path in G such that $V(P) \cap V(C) = \{v_1\}$. Let $P' = (v_i, u_1, \ldots, u_s)$ be the longest path in G such that $V(P') \cap V(C) = \{v_i\}$. Let $P'' = (v_j, z_1, z_2, \ldots, z_t)$ be the longest path in G such $V(P'') \cap V(C) = \{v_j\}$. Let

$P_1 = (w_r, w_{r-1}, \ldots, w_1, v_1, v_k, \ldots, v_j, z_1, z_2, \ldots, z_t)$,
$P_2 = (v_j, v_{j-1}, \ldots, v_i, u_1, u_2, \ldots, u_s)$,
$P_3 = (v_1, v_2, \ldots, v_i)$,
$Q_1 = (w_r, w_{r-1}, \ldots, w_1, v_1, v_k, \ldots, v_j)$,
$Q_2 = (z_t, z_{t-1}, \ldots, z_1, v_j, v_{j-1}, \ldots, v_i)$ and
$Q_3 = (u_s, u_{s-1}, \ldots, u_1, v_i, v_{i-1}, \ldots, v_1)$.

Suppose $m = 2$. Let v_{i_1} and v_{i_2}, where $1 \leq i_1 < i_2 \leq k$, be the vertices of degree greater than 2 on C. Let S_1 and S_2 denote the (v_{i_1}, v_{i_2})-section and (v_{i_2}, v_{i_1})-section of the cycle C respectively and let v_r be an internal vertex of S_1(say). Let R_1 and R_2 denote respectively the (v_{i_1}, v_r) and (v_r, v_{i_2})-sections of S_1. Let Q be the longest path in G with origin v_{i_1} such that $V(Q) \cap V(C) = \{v_{i_1}\}$ and let Q' be the longest path in G with origin v_{i_2} such that $V(Q') \cap V(C) = \{v_{i_2}\}$. Now, let $P_1 = Q^{-1} \circ S_2 \circ Q'$, $P_2 = R_1$, $P_3 = R_2$, $Q_1 = Q^{-1} \circ S_2$, $Q_2 = R_1$ and $Q_3 = R_2 \circ Q'$. Let $\mathcal{P}_1 = \{P_1, P_2, P_3\}$ and $\mathcal{P}_2 = \{Q_1, Q_2, Q_3\}$. In both cases the two collections of paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in G satisfy the following conditions:

(i) Both $\mathcal{P}_1$ and $\mathcal{P}_2$ cover the same set of edges and these edges cannot be covered by a fewer number of paths.

(ii) Both $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same set of vertices as internal vertices.

(iii) The paths in $\mathcal{P}_1$ and $\mathcal{P}_2$ cannot be extended to cover more edges of G.

(iv) Any two paths in each of $\mathcal{P}_1$ and $\mathcal{P}_2$ have at most one vertex in common.

(v) Both $\mathcal{P}_1$ and $\mathcal{P}_2$ cover all the edges of the cycle C.

Hence we can find two minimum simple acyclic graphoidal covers ψ_1 and ψ_2 of G such that $\mathcal{P}_1 \subseteq \psi_1$ and $\mathcal{P}_2 \subseteq \psi_2$. It follows from condition (v) that any automorphism of G maps $\mathcal{P}_1$ to $\mathcal{P}_2$. However, in $\mathcal{P}_1$, both end vertices of P_1 are pendant vertices, whereas no path in $\mathcal{P}_2$ has this property. Hence there is no automorphism of G which maps ψ_1 to ψ_2 so that ψ_1 and ψ_2 are not isomorphic, which is a contradiction.

Thus $m \leq 1$.

Case 1. $m = 0$.

Then $G = C$ and $\eta_{as}(G) = 3$. Suppose $k \geq 5$. Let $P_1 = (v_1, v_k)$, $P_2 = (v_k, v_{k-1}, \ldots, v_2)$, $P_3 = (v_1, v_2)$, $Q_1 = (v_1, v_k, v_{k-1})$ and $Q_2 = (v_{k-1}, v_{k-2}, \ldots, v_2)$. Then $\psi_1 = \{P_1, P_2, P_3\}$ and $\psi_2 = \{Q_1, Q_2, P_3\}$ are two minimum simple acyclic graphoidal covers of G which determine two different partitions of q, which is a contradiction. Thus $k \leq 4$. Hence G is either C_3 or C_4.

Case 2 $m = 1$.

Then $\eta_{as}(G) = n + 2$. Let v_1 be the vertex of degree greater than 2 on C.

Claim 1. Every vertex not on C has degree either 1 or 2.

Suppose there exists a vertex w not on C such that $deg\ w \geq 3$. Let $P = (w, w_1, w_2, \ldots, w_l, v_1)$. Let $Q_1 = (u_1, u_2, \ldots, u_r, w, u_{r+1}, \ldots, u_s)$ be the longest path such that $V(Q_1) \cap V(P) = \{w\}$. Let v_i and v_j, where $1 < i < j \leq k$ be two vertices on C. Let

$Q = (v_i, v_{i+1}, \ldots, v_j)$,

$R = (v_j, v_{j+1}, \ldots, v_k, v_1)$,

$P_1 = (u_s, u_{s-1}, \dots u_{r+1}, w, w_1, w_2, \dots, w_l, v_1, v_2, \dots, v_i)$,
$P_2 = (u_1, u_2, \dots, u_r, w)$ and
$Q_2 = (w, w_1, \dots, w_l, v_1, v_2, \dots, v_i)$.

Then $S_1 = \{P_1, P_2, Q, R\}$ and $S_2 = \{Q_1, Q_2, Q, R\}$ are two collections of paths satisfying the conditions (i) - (v) as stated earlier. Hence we can find paths $P_5, P_6, \dots, P_{n+2}$ such that $\psi_1 = \{Q, R, P_1, P_2, P_5, \dots, P_{n+2}\}$ and $\psi_2 = \{Q, R, Q_1, Q_2, P_5, P_6, \dots, P_{n+2}\}$ are two minimum simple acyclic graphoidal covers of G. Since P_1 and Q_2 are respectively the paths in ψ_1 and ψ_2 having v_1 as an internal vertex, any automorphism α of G maps P_1 to Q_2. However the path P_1 has a pendant vertex, whereas the path Q_2 has no pendant vertex. Hence there is no automorphism of G which maps ψ_1 to ψ_2 so that ψ_1 and ψ_2 are not isomorphic, which is a contradiction.Thus every vertex not on C has degree either 1 or 2.

Claim 2. $deg\ v_1 = 3$.

Suppose $deg\ v_1 = r \geq 4$. Let $u_1, u_2, \dots, u_{r-2}$ be the pendant vertices of G. Let P_i, where $1 \leq i \leq r-2$, be the u_i-v_1 path in G. Let v_{i_1} and v_{i_2}, where $1 < i_1 < i_2 \leq k$, be two vertices on C. Let
$Q_1 = (v_1, v_2, \dots, v_{i_1})$,
$Q_2 = (v_{i_1}, v_{i_1+1}, \dots, v_{i_2})$ and
$Q_3 = (v_{i_2}, v_{i_2+1}, \dots, v_k, v_1)$.
Then $\psi_1 = \{P_1 \circ P_2^{-1}, P_3, \dots, P_{r-2}, Q_1, Q_2, Q_3\}$ and $\psi_2 = \{P_1 \circ Q_1, P_2, P_3, \dots, P_{r-2}, Q_2, Q_3\}$ are two minimum simple acyclic graphoidal covers of G. Now, the path in ψ_1 having v_1 as an internal vertex has two pendant vertices, whereas the path in ψ_2 having v_1 as an internal vertex has exactly one pendant vertex. Hence ψ_1 and ψ_2 are not isomorphic, which is a contradiction. Thus $deg\ v_1 = 3$.

Claim 3. $k = 3$.

By Claim 2, G contains exactly one pendant vertex, say u_1. Let P be the u_1-v_1 path of length $l > 0$.

Suppose $k \geq 5$. Then $\psi_1 = \{P \circ (v_1, v_2), (v_2, v_3), (v_3, v_4, \dots, v_k, v_1)\}$ and $\psi_2 = \{P \circ (v_1, v_2), (v_2, v_3, v_4), (v_4, v_5, \dots, v_k, v_1)\}$ are two minimum simple acyclic graphoidal covers of G giving rise to the following partitions of q respectively.

(i) $l+1, 1, k-2$

(ii) $l+1, 2, k-3$.

Since $l > 0$ and $k \geq 5$, ψ_1 and ψ_2 are non-isomorphic, which is a contradiction. Hence $k \leq 4$.

Suppose $k = 4$. Then $\psi_1 = \{P \circ (v_1, v_2), (v_2, v_3), (v_3, v_4, v_1)\}$ and $\psi_2 = \{P \circ (v_1, v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ are two minimum simple acyclic graphoidal covers of G which determine two different partitions of q and hence ψ_1 and ψ_2 are non-isomorphic, which is a contradiction. Thus $k = 3$.

Thus by Claim 1, Claim 2 and Claim 3, G is a graph obtained by attaching a path to a vertex of a triangle.

The converse is obvious. □

The following is an interesting problem for further investigation.

Problem 2.8. *Characterize graphs having a unique minimum simple acyclic graphoidal cover.*

References

[1] B.D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.*, **18**(10)(1987), 882–890.

[2] S. Arumugam, B.D. Acharya and E. Sampathkumar, Graphoidal covers of a graph - A creative review, *Proceedings of the National workshop on Graph Theory and its Applications*, Manonmaniam Sundaranar University, Tirunelveli, Eds. S. Arumugam, B.D. Acharya and E. Sampathkumar, Tata McGraw Hill, (1996), 1–28.

[3] S. Arumugam and C. Pakkiam, Graphs with unique minimum graphoidal cover, *Indian J. pure appl. Math.*, **25**(11)(1994), 1147–1153.

[4] S. Arumugam and I. Sahul Hamid, Simple Graphoidal Covers in a Graph, *Journal Combin. Math. Combin. Comput.*, **64**(2008), 79–95.

[5] S. Arumugam and I. Sahul Hamid, Simple acyclic graphoidal covers in a graph, *Australasian Journal of Combinatorics*, **37**(2007), 243–255.

[6] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.*, **190**(1998), 66–77.

[7] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, 1972.

[8] C. Pakkiam and S. Arumugam, On the graphoidal covering number of a graph, *Indian J. pure appl. Math.*, **20**(4) (1989), 330–333.

[9] C. Pakkiam and S. Arumugam, The graphoidal covering number of unicyclic graphs, *Indian J. pure appl. Math.*, **23**(2)(1992), 141–143.