Degree Sets in Polygon Visibility Graphs

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Abstract

This paper presents some new results on permissible degree sets in polygon visibility graphs (PVGs). If the PVG has n vertices, we say it is an n-PVG. We also show some canonical construction techniques for PVGs with given degree sets.

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1 Introduction

A visibility graph is a triple $G = (V, O, E)$, where V is a set of vertices in the plane, O is a set of obstacles, and E is the set of all straight line edges connecting vertices in V but not intersecting any of the obstacles in O. In general, the obstacles in O can be any geometrical shapes, and the vertices in V can be any distinguished set of vertices on or off the obstacles. We will, however, restrict our discussion here to a special class of visibility graphs, whose vertices are the vertices of a simple polygon, P, and whose obstacles are the sides of P. We also restrict the set E to consist only of those edges that are interior to P or are sides of P. Such a graph is called a *polygon visibility graph* (PVG). If the polygon has n vertices, we call the corresponding polygon visibility graph an n-PVG.

Polygon visibility graphs have been studied for several decades now, and a good survey of the early results can be found in [7]. Please see [2, 3] for some recent results on visibility graphs. In this paper we obtain some results on the permissible degree sequences and degree sets for PVGs.

2 Degree Sequences and Degree Sets

We denote the degree sequence of a graph G as $< d_1^{n_1}, d_2^{n_2}, \dots, d_m^{n_m} >$, where the exponents indicate multiplicities, so that there are n_i vertices of degree d_i. The degrees are listed in ascending order. The *degree set* of G is then the set $\{d_1, d_2, \dots, d_m\}$ of distinct degrees of G. For example, the PVGs in Figure 1 have degree sequences $< 2^2, 3^2 >$ and $< 3^4 >$, respectively, while the PVG in Figure 2 has degree sequence $< 2, 3, 4^3, 5, 6 >$ and degree set $\{2, 3, 4, 5, 6\}$

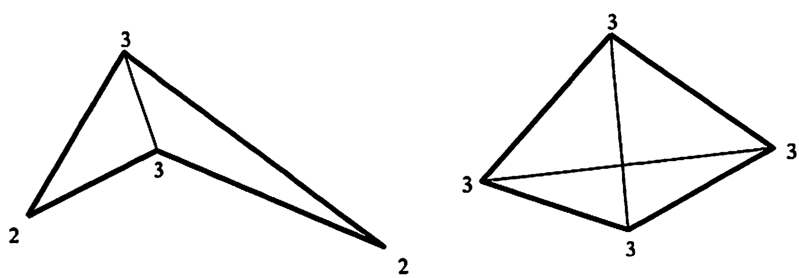

Figure 1

Hakimi [4] gave conditions for a sequence of integers to be the degree sequence of a graph. Kapoor et al [5] proved that every set of integers is realizable as the degree set of a graph, and also proved necessary and sufficient conditions for a given set S to be the degree set of a tree, a planar graph and an outerplanar graph. Ahuja and Tripathi [1] give some recent results on the possible size of a general graph that realizes a given degree set.

The question naturally arises: "Which degree sets are realizable as PVGs?"

The following two theorems give us a partial answer to this question and build some essential groundwork for a complete answer.

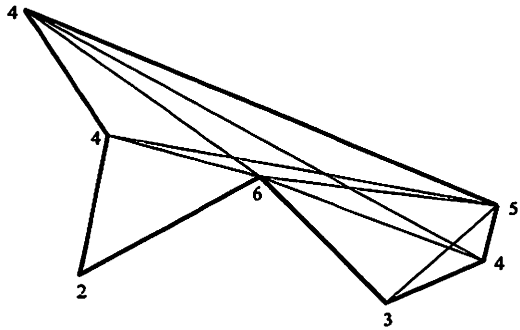

Figure 2

Theorem 2.1. *Every possible singleton degree set is realizable.*

Proof. We observe that if P is a convex polygon with $n \geq 3$ vertices, then the corresponding PVG is complete and has degree set $\{n-1\}$. $\square$

A few more definitions are in order. A *visibility clique* is a subset of the vertices of a PVG, which, together with the polygon edges and visibility edges connecting them, form a clique (maximal complete subgraph). In our constructions we often will use the notion of a spike. An *m-spike* is a set of m vertices which is inserted (glued) between two otherwise adjacent vertices of a polygon in such a way that the new vertices are not visible to any of the other vertices of that polygon. An *m-spur* is a set of m vertices inserted between two otherwise adjacent polygon vertices in such a way that at least some of the m vertices are visible to vertices of the original polygon other than the two between which they are inserted. In case the original polygon has only 2 vertices (is degenerate), any spike could also be considered a spur.

Theorem 2.2. *Given any set $S = \{i, j\}$, where $2 \leq i < j$, there is a PVG which has S as its degree set.*

Proof. We prove the theorem in two cases.

Case 1. $j < 2i$. Let $n_1 = j - i$ and $n_2 = 2i - j$. We construct a polygon as shown by the configuration in Figure 3.

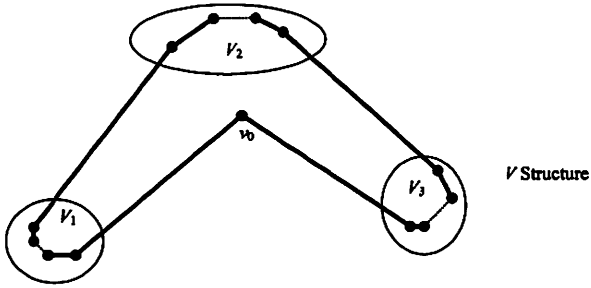

Figure 3

Here all of the vertices of the polygon except v_0 are divided into three convex chains V_1, V_2, and V_3, with n_1, n_2, and n_1 vertices, respectively. Notice that V_1 and V_3 are spurs of the configuration. Then each vertex in $V_1 \cup V_3$ has degree i, each vertex in V_2 has degree j, and vertex v_0 also has degree j.

Case 2. $j \geq 2i$. The degree set $\{i, j\}$ is realized by the configuration shown in Figure 4.

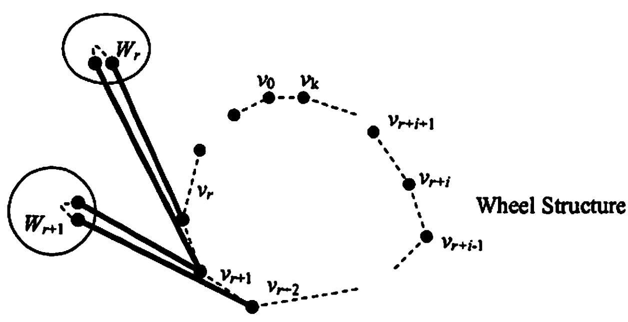

Figure 4

Here $v_0, \cdots v_s$ form a convex polygon with $s = j - 2i + 2$ and for each pair of vertices, $\{v_r, v_{r+1}\}$, we add an $(i-1)$-spike W_r. Thus, each vertex of W_r is of degree i, and each v_r is of degree $2(i-1)+s = 2i-2+j-2i+2 = j$. □

Notice that with a configuration similar to that of Configuration 1 shown in Figure 3 above, we can achieve degree sets $\{i, j, k\}$, with $i = n_1 + n_2, j = n_2 + n_3$, and $k = n_1 + n_2 + n_3$. The only constraints on the values of i, j, and k are that $i > 1$, $i < j < k$, and, since $n_2 > 0$, $k < i + j$.

The fundamental technique used in the constructions above is to begin with a convex set of vertices and add m-spikes between adjacent pairs of the original vertices. PVGs with degree sets of size greater than 2 can be constructed in many different ways, but in what follows we will develop a simple canonical way of constructing realizations of degree sets using as its basic building blocks the structures for degree sets of size 1 and 2 that we have already described and applying some simple bridging techniques. There are two types of bridges used in these constructions: the 1-2 bridge, which is formed from a simple 2-3 fan, and the convex k_1-k_2 bridge, with k_1 and k_2 each ≥ 2, both of which are illustrated in Figure 5.

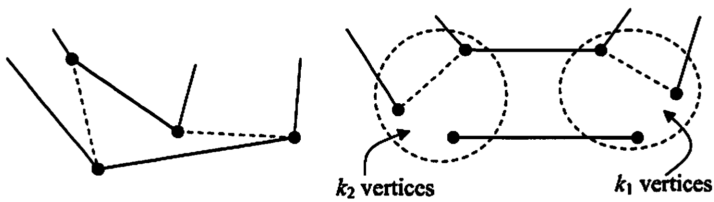

Figure 5

Lemma 2.1. *Any set* $S = \{i_1, i_2, \ldots, i_k\}$, *with* $1 < i_1$ *and* $i_r = i_1 + r - 1$, $r = 2, \ldots, k$, *can be realized as the degree set of a PVG.*

Proof. **Case** ($i_1 = 2$). When $k = 3$, we have the realization in Figure 6.

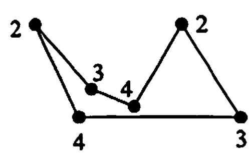

Figure 6

We can extend this easily to the case $k = 4$, by inserting a 3-3 bridge between one end of the 1-2 bridge and the adjacent end spike, as in Figure 7.

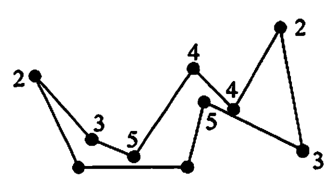

Figure 7

Notice that in this realization, there are four parts, the beginning and ending spike, both of degree 2, a middle spike of degree 3, and a transitional structure, or bridge, between them, in this case, what we will refer to as a 1-2-bridge since it adds 1 or 2 to the degrees of the vertices in the spikes on either end of the bridge.

Assume that for some $k \geq 4$, S is always realizable as above. We can extend this to $k+1$ by first constructing the realization of $S' = \{2, \ldots, k\}$, then inserting a $k-1$-spike as above between the $k-2$-spike and the 2 spike at the right end of the structure, joining it with 1-2-bridges on each side. The vertices of the $k-2$-spike will all have degrees $k-2, k-1$ and k, and the vertices of the adjacent spikes will remain as they were, and we have extended this to a realization of S.

Case $(i_1 > 2)$. This follows immediately from the previous case. Given an arbitrary set of integers $S = \{i_1, \ldots, i_k\}$, first extend it to a set S' such that $S' = \{2, \ldots, i_1 - 1, i_1, \ldots, i_k\}$. Now construct the realization of S' and trim off the part up to the i_1-spike plus the ending 2 spike and place an i_1 spike at the right end. Since k is at least 3, this can always be done. □

Lemma 2.2. *Any set* $S = \{i_1, i_2, \ldots, i_k\}$, *with* $4 \leq i_1, i_2 - i_1 > 1$ *and* $i_r = i_1 + \lfloor \frac{i_1}{2} \rfloor + 1$, $r = 2, \ldots, k$, *can be realized as the degree set of a PVG.*

Proof. The general schema for $k \geq 3$ is illustrated in Figure 8, where $i_2 = i_1 + k_1$, and $i_3 = i_1 + k_2$.

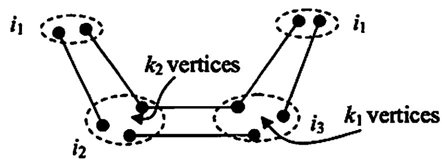

Figure 8

Note that, in this setting, since $k > 2$, i_1 must be greater than or equal to 4. For example, $\{4, 6, 7\}$ can be realized as in Figure 9.

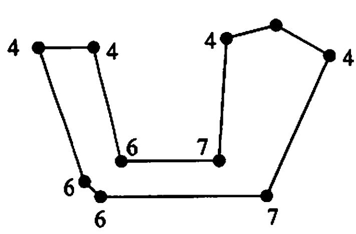

Figure 9

In general, we can chain copies of i_1 spikes with $k_r - k_{r+1}$ bridges connecting them, where k_r and k_{r+1} are each equal to $\lfloor \frac{i_1}{2} \rfloor + 1$, and $i_r = i_1 + k_{r-1}$ for all $r > 1$. Because of the requirement on the values of k_r, $k_r + k_{r+1} \leq i_1 + 1$, and this guarantees that the chain can be extended arbitrarily as illustrated in Figure 10. □

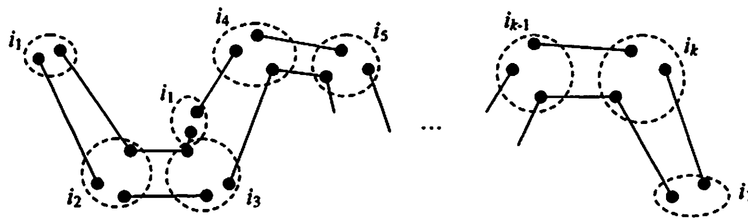

Figure 10

In the bridges used in the previous two lemmas, the vertices at the ends of the bridge overlap with some of the vertices of a spike. The proofs of the following two lemmas employ overlap along an entire spike of either a V structure or a wheel structure.

Lemma 2.3. *Any set* $S = \{i_1, i_2, \ldots, i_k\}$, *with* $i_1 + \lfloor \frac{i_1}{2} \rfloor + 1 < i_r < 2i_1$, *for* $r = 2, \ldots, k$, *can be realized as the degree set of a PVG.*

Proof. The cases where $i_1 = 2, 3$, or 4 degenerate to singleton sets, which are handled by Theorem 2.1. When $i_1 = 5$, or 6, we get only the 2 element sets $\{5, 9\}$ and $\{6, 11\}$, both of which are handled by Theorem 2.2. WLOG, then, we will assume $i_1 > 6$, and $k > 2$.

For $i_1 = 7$ and $k = 3, S = \{7, 12, 13\}$ and is realized as in Figure 11.

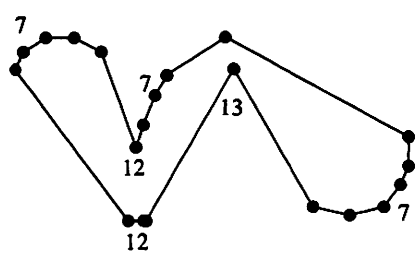

Figure 11

Before proceeding with an inductive argument, we observe that in any V structure realizing the set $S = \{i_1, i_r\}$ and satisfying the conditions of the lemma, the number of vertices at the end of each spike is at least half of i_1, and, consequently, the number of vertices in the shared portion of the V must be less than half of i_1, but greater than 1. Thus, any such V structure can be glued onto a smaller chain of V structures by overlapping one of its spikes with the right-most spike of the smaller chain.

Now, assume $S = \{i_1, i_2, \ldots, i_k\}$ satisfies the conditions of the lemma and that for all $j < k$, we know how to realize $S' = \{i_1, \ldots, i_j\}$ as a chain of V structures with an i_1 spike at both ends. Construct a V structure realizing $S'' = \{i_1, i_k\}$. Now glue S'' to the right end of S', overlapping the end spikes. This gives us a realization of S with an i_1 spike at both ends as in Figure 12, and, by induction on k, we are done. □

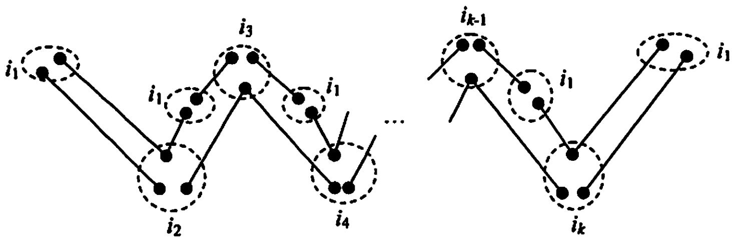

Figure 12

Lemma 2.4. *Any set $S = \{i_1, i_2, \ldots, i_k\}$, with $3 \leq i_1$, and $i_r \geq 2i_1$, for $r = 2, \ldots, k$, can be realized as the degree set of a PVG.*

Proof. By Theorem 2.2, we can construct a $\{i_1, i_r\}$ wheel for each $r = 2, \ldots, k$. Each of these wheels has at least 3 spokes (i_1-spikes), so, when $i_1 > 2$, we can glue them together in a chain by overlapping an i_1-spike of each $\{i_1, i_r\}$ wheel with an i_1 spike of the preceding $\{i_1, i_{r-1}\}$ wheel as in Figure 13. □

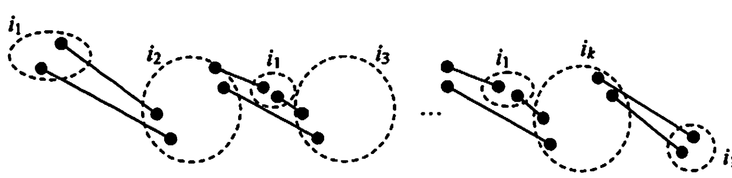

Figure 13

Given any set of integers $S = \{i_1, i_2, \ldots, i_k\}$, with $i_1 = 2$, we can clearly partition S into the disjoint subsets:

1. $S_b = \{i_r | i_r = i_1 + r - 1, r = 1, \ldots, s$, where s is maximal in $S\}$; we will refer to this as the *beginning sequence* of the elements of S;

2. $S_{h-} = \{i_r | r > s$, and $i_r = i_1 + \lfloor \frac{i_1}{2} \rfloor + 1\}$, the *lower middle sequence* of elements of S; if $|S_b| = 2$, we replace i_1 with i_2;

3. $S_{h+} = \{i_r | r > s$, and $i_1 + \lfloor \frac{i_1}{2} \rfloor + 1 < i_r < 2i_1\}$, the *upper middle sequence* of elements of S; if $|S_b| = 2$, we replace i_1 with i_2; and

4. $S_e, = \{i_r | r > s$, and $i_r = 2i_1\}$, the *ending sequence* of elements of S; if $|S_b| = 2$, we replace i_1 with i_2.

Any of these subsets may be empty, except S_b, which always contains at least i_1.

Lemmas 2.1 through 2.4 give us a canonical realization of each of the sets S_b, S_{h-}, S_{h+}, and S_e, respectively. For the set S_e, we required $i_1 > 2$. We observe that, for $i_1 > 2$, the realizations of all the possible combinations of these sets can be joined to form a larger PVG realizing the union of the sets in those combinations by simply gluing them together by overlapping the i_1 spikes (i_2 spikes, if $|S_b| = 2$) at the ends of each of the realizations.

Theorem 2.3. *Any set of integers $S = \{i_1, i_2, \ldots, i_k\}$ with $i_1 > 1$, and $i_1 < i_2 < \cdots < i_k$ can be realized as the degree set of a PVG which is the join of the realizations of the sets S_b, S_{h-}, S_{h+}, and S_e.*

Proof. The observation made in the previous paragraph takes care of the case when $i_1 = 3$.

Case $i_1 = 2$: If $i_2 = 3$, we can realize $S = \{2, 3, i_3, \ldots, i_k\}$ as shown in Figure 14.

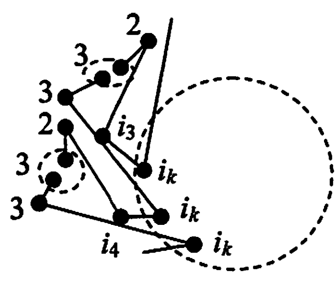

Figure 14

Some of the foot-like spikes in Figure 14 can be repeated as needed to bring the number of vertices in the core up to the required number to reach a vertex degree of i_k, that is, $i_k - 2$ vertices.

If $i_2 > 3$, we have proved the base cases of $k = 1$ and $k = 2$ in Theorems 2.1 and 2.2. Therefore assume as an induction hypothesis that $S = \{2, i_2, \ldots, i_j\}$ is realizable for all $j < k$, for some $k > 2$. To complete the proof, we need only extend this induction hypothesis.

Given any $S = \{2, i_2, i_3, \ldots, i_k\}$, consider $S' = \{i_2 - 2, i_3 - 2, \ldots, i_k - 2\}$, since $i_2 > 3$, we can realize S' by the induction hypothesis and the previous cases of the proof. Now, into this realization, insert a 2 spike in place of each polygon edge. This gives us the required realization of S. □

3 Degree Sets of Orthogonal Polygons

In this section we consider a special subclass of polygons. A polygon is *orthogonal* if its edges are alternatively horizontal and vertical. We first state some elementary properties of orthogonal PVGs.

Proposition 3.1. *If G is an orthogonal n-PVG, then*

1. *$\delta \geq 3$, and this bound is sharp.*
2. *There are an even number of even vertices.*
3. *If r is the number of reflex vertices, then $n = 2r + 4$.*
4. *Every reflex vertex has degree at least 5.*

Proof. The first is easily seen. For the second, note that an orthogonal polygon must have an even order. The proof of the third can be found in [7]. We include it here for the sake of completeness. The interior angle of a convex vertex is $\frac{\pi}{2}$ while that of a reflex vertex is $\frac{3\pi}{2}$. Since the sum of all interior angles is $(n-2)\pi$, we get $(n-r)\frac{\pi}{2} + r\frac{3\pi}{2} = (n-2)\pi$ and the result follows.

For the fourth, do a radial line sweep at the reflex vertex, starting along one of the two adjacent edges. As the line sweeps the first 90 degrees, it must encounter at least two vertices. As it sweeps the next 90 degrees, it must encounter at least one additional vertex, and as it sweeps the final 90 degrees, it will encounter at least two additional vertices. □

Proposition 3.2. *For an orthogonal PVG of order n,*

1. *If $n = 4$, the degree sequence is $< 3^4 >$*
2. *If $n \geq 4$, the degree sequence must have at least $\lceil \frac{n-4}{4} \rceil$ vertices of degree at least 5.*

Proof. Follows from above. □

We saw in Theorem 2.1 that convex polygons give us complete PVGs for each $n \geq 3$. Our next result shows that the CVGs of orthogonal polygons are never complete unless $n = 4$.

Proposition 3.3. *If P is an orthogonal polygon of order n, then the PVG of P has at most $\binom{n}{2} - 3r$ edges, where r is the number of reflex vertices of P. In particular, the only complete orthogonal PVG is K_4.*

Proof. Assume $n \geq 6$. Suppose the vertices of P (in counterclockwise order) are $v_1, v_2, \ldots, v_n$. If v_i is a reflex vertex then the line segments $v_{i-1}v_{i+1}$, $v_{i-1}v_{i+2}$ and $v_{i-2}v_{i+1}$ cannot be edges in the PVG (with the indices being considered mod n). □

4 Regularity

A regular graph of degree r has degree set $\{r\}$. For PVGs we have the following conjecture:

Conjecture 4.1. *A PVG is regular if and only if it is complete.*

We prove that this is true for small order ($n \leq 6$) PVGs. Note that the conjecture is trivially true for $n = 3$. We will need the following definition. An *ear* of a polygon P is a vertex v such that the line segment joining the two neighbors of v lies in the interior of P. In other words, the two neighbors of v on P are adjacent in the PVG of P. A theorem by Meisters [6] states that every polygon has at least two ears v and w such that vw is not a side of the polygon. Observe that if v is an ear on P of $n \geq 4$ vertices, and u and w are neighbors of v on P, then we get a new polygon P_1 of $n-1$ vertices by removing the vertex v and adding side uw. Also the PVG of P_1 is then an induced subgraph of the PVG of P.

Lemma 4.1. *For an n-PVG, G, with $n > 3$, if two vertices are adjacent in G, then at least one them has degree greater than 2.*

Proof. Let v_1 and v_2 be two adjacent vertices in G. If v_1 and v_2 are not neighbors on P, then each has degree at least 3. If v_1 and v_2 are adjacent on P, let v_3 be the other vertex adjacent on P to v_2. If v_1 can see v_3, then $deg(v_1) > 2$. If v_1 cannot see v_3, the triangle $v_1v_2v_3$ must contain another vertex v_{i_1} of G (an endpoint of a polygon edge that blocks the edge v_1v_3). Thus, there is a finite chain of nested triangles $v_2v_{i_k}v_3$. v_2 can see the vertex v_{i_k} of the innermost triangle of the chain, and thus $deg(v_2) > 3$. □

Lemma 4.1 leads to a special case of Conjecture 4.1

Lemma 4.2. *If an n-PVG is 3-regular, then $n = 4$ and the PVG is K_4.*

Proof. Clearly, $n \geq 4$. Let v be an ear of the polygon. Removal of v gives an induced $(n-1)$-PVG with two adjacent vertices of degree 2. By Lemma 4.1 it follows that $n - 1 = 3$. □

Theorem 4.1. *For $n \leq 6$, if a PVG is regular, then it is complete.*

Proof. As noted earlier, the case $n = 3$ is trivial. Assume $n \geq 4$. Suppose that G is r-regular. By Lemma 4.1, $r \geq 3$. If $n = 4$ or if $r = 3$, we are done by Lemma 4.2. Assume $n \geq 5$ and $r \geq 4$. If $n = 5$ and $r = 4$, or if $n = 6$ and $r = 5$, the PVG is complete. Finally suppose that $n = 6$ and $r = 4$. Removal of an ear gives a 5-PVG G with degree sequence $< 3^4, 4 >$. Suppose the vertices of the 5-polygon (say P_1) are v_1, v_2, v_3, v_4, v_5 (in order). Without loss of generality, suppose v_1 has degree 4. Clearly, in G, the removal of v_1 leaves a 4-cycle. Since v_2v_3, v_3v_4 and v_4v_5 are sides of P_1, it follows that v_2v_5 is in G. Hence v_1 must be an ear of P_1, and its removal must lead to a 4-PVG which cannot be a 4-cycle, a contradiction. □

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