A Survey on Path Graphs Connectedness

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Abstract

The goal of this article is to provide an overview of all the results currently known regarding the connectedness of path graphs. The proofs we present are only those that illustrate the different techniques employed in obtaining the results.

This is an expository paper addressed to readers with a small degree of familiarity with the field of graph theory and its techniques.

1 Introduction

1.1 Definitions and notation

A *graph* is a pair $G = (V, E)$ where V is a finite set of *vertices*, and E is a subset of non-ordered pairs of vertices, called *edges*. If $e = uv \in E$ we say that u and v are *adjacent*, and also that u and v are the *endpoints* of the edge e.

For a vertex $v \in V$, the *neighborhood* of v is the set $N(v)$ of all vertices adjacent with v. That is,

$$N(v) = \{u : uv \in E\}.$$

The *degree* of a vertex v is $deg(v) = |N_G(v)|$. Sometimes it is useful to know that every vertex has a minimum number of neighbors, so the *minimum degree* of the graph G, denoted as $\delta(G)$, is the minimum degree over all vertices of G. In other occasions it is useful the concept of regularity regarding the degree. Therefore, a graph is *regular* if all of its vertices have the same degree. Particularly, a graph is said to be *δ-regular* if $deg(v) = \delta$, for every vertex v of G.

A *path* between two vertices u and v, or a *uv-path*, is an ordered sequence of vertices $u = x_0, x_1, x_2, \ldots x_{n-1}, x_n = v$ such that x_i is adjacent with x_{i+1},

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for all $i = 0, \ldots, n-1$ and $x_i \neq x_j$ if $i \neq j$, for all i, j such that $0 \leq i, j \leq n$. In other words, a path between u and v is a sequence of adjacent vertices that starts with u, ends with v, and does not repeat vertices. The *length* of a path is the number of edges it traverses. For example, the path $u = x_0, x_1, x_2, \ldots x_{n-1}, x_n = v$ has length n. Often we are interested in paths of minimal length between a given pair of vertices, so the *distance* between two vertices u and v is $d(u, v)$, defined as the length of a shortest uv-path. The maximum distance between all pairs of vertices in V is called the *diameter* of a graph G, denoted as $D = D(G)$. That is,

$$D(G) = \max\{d(u, v) : u, v \in V\}.$$

Notice that given a uv-path in G it might happen that $u = v$, and in that case we called the path $u = x_0, x_1, x_2, \ldots x_{n-1}, x_n = u$ is a *cycle* of *length* n. For some purposes it is useful to have a lower bound for the length of a cycle in a given graph. Thus, the *girth* of a graph G is $g(G)$, defined as the length of a shortest cycle in G.

A graph is called *connected* if there is a path between every pair of its vertices. A connected graph that does not have any cycle is a *tree*.

The reader is referred to [7, 12] for further terminology on graphs.

1.2 Connectedness

There are different measure of the connectedness of a graph. The *edge-connectivity* of a graph G is $\lambda(G)$ defined as the minimum number of edges whose removal disconnects G.

An important result for the study of the connectivity of a graph is Menger's Theorem [21], which states that the connectivity of a graph is related to the number of disjoint paths between pairs of distinct vertices of a graph. We will recall this result after giving a precise definition of disjoint paths.

Two uv-paths are said to be *edge-disjoint* if they have no edges in common.

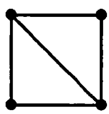
Theorem 1.1. [21] *The minimum number of vertices whose removal disconnects a pair of vertices u, v in G, equals the maximum number of vertex-disjoint uv-paths. Analogously, the minimum number of edges whose removal disconnects a pair of vertices u, v in G, equals the maximum number of edge-disjoint uv-paths.*

A main consequence of Menger's result is that it provides a constructive approach to measure the connectivity of a graph, which consists in finding disjoint paths between every pair of vertices.

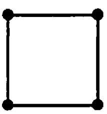
An *edge cut* in a graph G is a set of edges A such that $G - A$ is not connected. Therefore, it arises from the definition of edge-connectivity that for any graph G, $\lambda(G)$ is the cardinality of a minimal edge cut of G. Since $\lambda(G) \leq \delta(G)$, a graph G is said to be *maximally edge-connected* when $\lambda(G) = \delta(G)$.

A *connected component* of a non-connected graph is a maximal connected subgraph, i.e. a connected subgraph that is not a subgraph of any other connected subgraph of G, other than itself. If a minimum edge cut is removed from a graph G, the resulting graph necessarily has exactly two connected components, so and edge cut A can be identified with the pair $A = (C, \overline{C})$ where C and $\overline{C}$ represent the two components of the graph $G - A$.

Let us now turn our attention to the following example.



G_1



G_2

Fig. 1 Two maximally connected graphs, with $\lambda(G_1) = \lambda(G_2) = 2$.

In Figure 1 we are show two graphs, both with the same connectivity and both maximally connected. However, any set of two edges whose removal disconnects G_1 produces a connected component in G_A formed by a single vertex. This is not the case in G_2, where deleting two well-chosen edges might lead to two connected components, each of them consisting of an edge. To express this concept Boesch and Tindell introduced the notion of superconnectivity [5]. Next we present some terminology necessary for the definition of this new idea.

An edge cut is called *trivial* if $C = \{v\}$ or $\overline{C} = \{v\}$ for some vertex v with $deg(v) = \delta(G)$. A maximally edge-connected graph is called *super-λ* if every edge cut $(C, \overline{C})$ of cardinality $\delta(G)$ is trivial. The *superconnectivity* of a graph is denoted by $\lambda_1(G)$ and it is defined as $\lambda_1(G) = \min\{|(C, \overline{C})|, (C, \overline{C}) \text{ is a non trivial edge cut}\}$. Then, a graph G is super-λ if and only if $\lambda_1(G) > \delta(G)$.

1.3 Path graphs

Given a positive integer k, the k-path graph is an operator that associates a graph G with another graph denoted $P_k(G)$. The vertices of $P_k(G)$ represent the set of all paths of length k in G. Two vertices are adjacent in $P_k(G)$ whenever the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$

in G. Intuitively, this means that two vertices of $P_k(G)$ are adjacent if and only if they represent paths in G that can be obtained from each other by 'shifting'. Given a path $u_0, u_1, \ldots, u_k$ in G, the corresponding vertex in $P_k(G)$ will be denoted by $U = u_0u_1 \ldots u_k$.

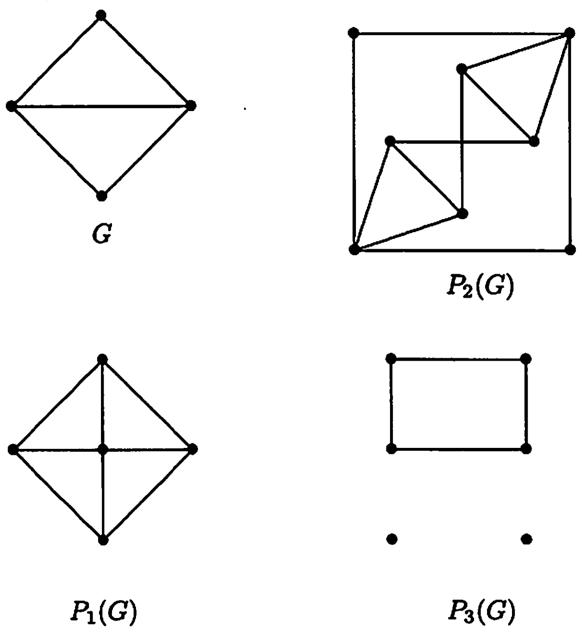


Fig. 2 The graph G and its corresponding 1, 2 and 3-path graphs.

Obviously, given a graph G, for any integer k such that $k > D(G)$, the diameter of G, the k-path graph is empty.

Path graphs were introduced by Broersma and Hoede in [6] as a natural generalization of line graphs. Indeed, for every graph G, the graph $P_1(G)$ coincides with the line graph of G denoted as $L(G)$. A characterization of P_2-path graphs was also given by Broersma and Hoede in [6], and a different characterization was later obtained by Li and Lin in [16]. Isomorphisms of 3-path graphs were studied jointly by Aldred, Ellingham, Hemminger and Jipsen in [1] and by Li in [18, 19]. Isomorphisms of 4-path graphs were studied by Li and Biao in [20]. Distance properties of path graphs were studied by Bela and Jurica [4] Knor and Niepel [14] and Chung, Ferrero, Taylor and Warshauer [8]. Crnković presented some insightful results on path graphs of incidence graphs [9].

Regarding the study of the connectedness of k-path graphs, it is important to distinguish the case $k = 1$, since 1-path graphs are line graphs and they have been widely studied. We mention here only those articles related to k-path graphs for the case $k \geq 2$. In this case, the first results were obtained by Li [17]. Later Knor [13, 15], Niepel [13, 15] and Mallah [15] continued the study, focusing mainly on finding conditions to guarantee that a path graph is connected. More recently, Balbuena [2, 3], Ferrero [2, 11, 10] and Garcia [3] had contributed to study measures of the connectivity and superconnectivity of path graphs.

Note that the path graph can be thought of as an operator on graphs, and therefore, we can study graphs arising from the iteration of the k-path graph operator. Indeed, the s-iterated k-path graph of G is the graph $P_k^s(G)$ defined as $P_k(G)$ if $s = 1$, and $P_k(P_k^{s-1}(G))$ if $s > 1$. Some important contributions in this direction were given by Prisner [22].

2 Connectedness of path graphs

2.1 Connected path graphs

In this subsection we survey conditions that guarantee that a path graph is connected.

Obviously if a graph G is not connected, $P_k(G)$ is not connected either, for any positive integer k. However, the connectivity of G is not enough to guarantee the connectivity of $P_k(G)$ for all positive integers k. Figure 3 gives an example of a graph G such that $P_1(G)$ is connected but $P_2(G)$ is not connected.

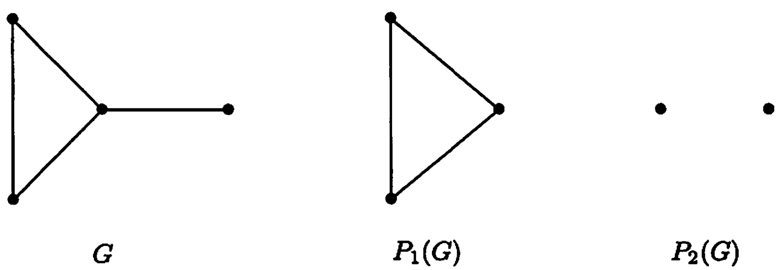


Fig. 3 The graph G and its corresponding 1 and 2-path graphs.

Therefore, before studying the connectivity of connected path graphs, we must address the problem of finding sufficient conditions for a path graph to be connected. Observe that as the previous example suggests, those conditions will involve not only the graph, but also the value k of the k-path graph under consideration.

The first and most general necessary and sufficient condition for a path graph to be connected was presented by Knor and Niepel [13]. In order to recall this result here we need to introduce some terminology.

Following the notation in [13], for a graph G and two integers k and t, $k \geq 2$ and $0 \leq t \leq k-2$, by $P^*_{k,t}$ we denote an induced tree in G with diameter $k+t$ and a diametric path $(x_t, x_{t-1}, \dots, x_1, v_0, v_1, \dots v_{k-t}, y_1, y_2, \dots, y_t)$ such that all the endvertices of $P^*_{k,t}$ are at distance no greater than t from v_0 or v_{k-t}, the degrees of $v_1, v_2 \dots v_{k-t-1}$ are 2 in $P^*_{k,t}$ and no vertex in $V(P^*_{k,t}) - \{v_1, v_2 \dots v_{k-t-1}\}$ is adjacent with a vertex in $V(G) - V(P^*_{k,t})$. The path $v_1, v_2 \dots v_{k-t-1}$ is the *base* of $P^*_{k,t}$, and for a path A of length k we say that $A \in P^*_{k,t}$ if and only if the base of $P^*_{k,t}$ is a subpath of A.

Theorem 2.1. [13] *Let G be a connected graph with girth at least $k+1$. Then, $P_k(G)$ is disconnected if and only if G contains a $P^*_{k,t}$, $0 \leq t \leq k-2$, and a path A of length k, such that $A \notin P^*_{k,t}$.*

Note that it is often the case that we can simplify the study of k-path graphs if we restrict k to small values. For example, it is sufficient - and easy to verify - that if G is connected also $P_1(G)$ is connected. Besides, if G is a connected graph with minimum degree $\delta \geq 2$, it is also possible to prove that $P_2(G)$ must be connected. For $k = 3$, Knor and Niepel [13] found a simplified version of their general result. We present their result next, after introducing some necessary notation.

Let P^0_3 denotes a subgraph of G induced by the vertices in a path of length 3, say v_0, v_1, v_2, v_3, such that neither v_0 nor v_3 has a neighbor in $V(G) - \{v_1, v_2\}$. A path A is in P^0_3 if and only if $A = v_0, v_1, v_2, v_3$. Analogously, P^0_4 denotes an induced subgraph of G with a path of length x, v_0, v_1, v_2, y in which every neighbor of v_0 and v_2 except v_0, v_1 and v_2 has degree 1, or it has degree 2 and in this case it is adjacent to v_1. Moreover, no vertex of $V(P^0_4) - \{v_1\}$ is adjacent to a vertex of $V(G) - V(P^0_4)$ in G. A path A of length 3 is in P^0_4 if v_0, v_1, v_2 is a subpath of A.

For a set of vertices S with no edges between its vertices, let K^*_4 denote a graph obtained from $K_4 \cup S$ by joining all vertices of S to one special vertex of K_4.

Let $K_{2,t}$ be a complete bipartite graph and let (X, Y) be a bipartition of $K_{2,t}$ where $X = \{v_1, v_2\}$. Join t sets of independent vertices by edges, each to one vertex of Y; further, glue a set of stars with at least 3 vertices by one endvertex, each to v_1 or to v_2; glue a set of triangles by one vertex, each either to v_1 or v_2; and finally, join v_1 to v_2 by an edge. The resulting graph is denoted by $K^*_{2,t}$.

Theorem 2.2. [13] *Let G be a connected graph such that $P_3(G)$ is not empty. Then, $P_3(G)$ is disconnected if and only if one of the following conditions holds*

1) G contains a P^0_t, $y \in \{3, 4\}$, and a path A of length 3 such that $A \notin P^0_t$

2) G is isomorphic to K^*_4

3) G is isomorphic to $K^*_{2,t}$, $t \geq 1$.

2.2 Connectivity of path graphs

Once established that a path graph is connected, we proceed to measure its connectivity.

A first result valid for any value of k was given by Ferrero in [10], and it establishes a lower bound for the connectivity of a path graph. The proof of that result is based on the construction of edge-disjoint paths between any two adjacent vertices, and since it illustrates some properties of path graphs, we will present it next.

Let k be a positive integer and let G be a connected graph with minimum degree $\delta > k$. Let $U = u_0 u_1 \dots u_k$ and $V = u_1 \dots u_k u_{k+1}$ be two adjacent vertices in $P_k(G)$. If $\delta > k$, there exist $b_i \in N(u_k) - \{u_1 \dots u_{k-1}, u_{k+1}\}$, $i = 1, \dots, \delta - k$ and $c_j \in N(u_1) - \{u_0, u_2 \dots u_k\}$, $j = 1, \dots, \delta - k$, and as a consequence, there exist vertices $u_1 \dots u_k b_i$ and $c_j u_1 \dots u_k$ in $P_k(G)$. Let γ be an isomorphism in the integer set $\{1, \dots, \delta - k\}$, then we can assign each b_i with one particular $c_{\gamma(i)}$. Therefore, there is a path $\mathcal{P}_i$: $u, u_1 \dots u_k b_i, c_{\gamma(i)} u_1 \dots u_k, v$ in $P_k(G)$, which obviously has length 3 and joins U and V. Indeed, those paths are proven to be disjoint, and the following result can be established.

Lemma 2.3. [10] *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > k$. Then, there exist $\delta - k$ edge disjoint paths of length* 3 *between any two adjacent vertices in $P_k(G)$.*

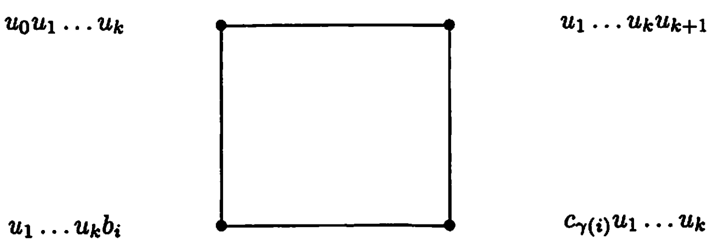


Fig. 4 Paths given in Lemma 2.3

Although it is not mentioned in [10], it is worth noticing that from the previous Lemma it can be derived the following result:

Theorem 2.4. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > k$. Then, $\lambda(P_k(G)) \geq \delta - k - 1$.*

However, the usefulness of this result is limited if one takes into account that $\delta(P_k(G)) \geq 2(\delta - 1)$. That is the reason why it is necessary to look at longer paths between any pair of adjacent vertices. For that purpose in [10] is also presented the following additional construction.

Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Let $U = u_0u_1 \dots u_k$ and $V = u_1 \dots u_k u_{k+1}$ be two adjacent vertices in $P_k(G)$. Since $\delta \geq k$, for each $i = 1, \dots, \delta - (k-1)$ there exists a choice of vertices $a_2^i, \dots, a_k^i$ and $b_2^i, \dots, b_{k+1}^i$ which determine the walks in G

$$P_i : a_2^i, \dots, a_k^i, u_0, u_1, b_2^i, \dots, b_{k+1}^i \text{ and } Q_i : b_{k+1}^i, \dots, b_2^i, u_1, \dots, u_{k+1}$$

Moreover, since $\delta \geq 2(k-1)$, for any integers i, j, $1 \leq i, j \leq \delta - (k-1))$ we can choose the vertices so that $a_k^i \neq a_k^j$ and $b_2^i \neq b_2^j$ if $i \neq j$. From each of those UV-walks in $P_k(G)$ it is obtained a different UV-path, and those paths are proven to be internally disjoint, so the following lemma can be obtained.

Lemma 2.5. [10] *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, there exist $\delta - (k-1)$ edge disjoint paths of length $3k-1$ between any two adjacent vertices in $P_k(G)$.*

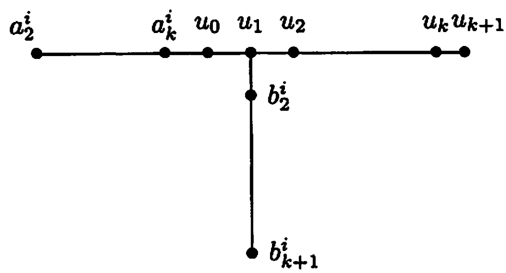


Fig. 5 Paths given in Lemma 2.5

It is shown in [10] that the paths obtained in the previous two lemmas, all together form a disjoint set of paths between any two vertices. As a consequence, we can now obtain a better lower bound for the connectivity of a path graph.

Theorem 2.6. [10] *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, $\lambda(P_k(G)) \geq 2(\delta - (k-1))$.*

Note that in general $\delta(P_k(G)) \geq 2(\delta - (k-1))$, but if G is δ-regular, then $\delta(P_k(G)) = 2(\delta - (k-1))$, so the following result can be derived.

Corollary 2.7. [10] *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then, $P_k(G)$ is maximally connected.*

Notice that the proofs for Lemmas 2.3 and 2.5 also work if the bound on the degree, $\delta > 2(k-1)$, is replaced by a more relaxed one, $\delta > k$, together with a lower bound on the girth g, which must be $g \geq k+1$. This was suggested in [10] and proven independently by Balbuena and Garcia in [3].

Theorem 2.8. [3] *Let k be a positive integer and let G be a connected graph with minimum degree $\delta \geq 3$ and girth at least $k+1$. Then, $\lambda(P_k(G)) \geq 2(\delta - 1)$.*

Corollary 2.9. [3] *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 3$ and girth at least $k+1$. Then, $P_k(G)$ is maximally connected.*

As we saw in the previous section, when restricting the study of k-path graphs to small values of k we can often obtain sharper results.

For instance, if $k = 2$ Theorem 2.6 says:

If G be a connected graph with minimum degree $\delta > 2$. Then, $\lambda(P_2(G)) \geq 2(\delta - 1)$.

However, $\lambda(P_2(G)) \geq 2(\delta - 1)$ still holds for some graphs G with minimum degree $\delta = 2$. This improvement was proved in [2] where the authors focused on 2-path graphs and studied the connectivity using a different technique. Instead of constructing disjoint paths, they study properties of edge cuts.

To understand the work in [2] it is necessary to establish a distinction between two different types of edges in 2-path graphs. Let G be a graph and let a, b, c and u, v, w be two paths in G that induce adjacent vertices in $P_2(G)$. Let us call the edge connecting abc and uvw in $P_2(G)$ an *ab-edge* or a *bc-edge*, depending on $v = a$ and $w = b$, or $u = b$ and $v = c$, respectively. For any given $ab \in E(G)$, let E^a_{ab} denote the set of vertices of $P_2(G)$ of the type xab, $x \in N_G(a) \setminus \{b\}$. Analogously, let E^b_{ab} denote the set of vertices of $P_2(G)$ of the type aby, $y \in N_G(b) \setminus \{a\}$. Then the following results hold.

Lemma 2.10. [2] *Let G be a connected graph with $\delta(G) \geq 2$. Let $A = (C, \overline{C})$ be an edge cut of $P_2(G)$, and let $ab \in E(G)$. If A contains ab-edges, then it contains at least* $\min\{deg(a) - 1, deg(b) - 1\}$ *ab-edges.*

Corollary 2.11. [2] *Let G be a graph with $\delta(G) \geq 2$ and $\lambda(G) \geq 2$. Let A be an edge cut of $P_2(G)$. Then there exist two different edges ab and cd in G, such that A contains both ab-edges and cd-edges.*

As a direct consequence of Lemma 2.10 and Corollary 2.11 it is obtained the following theorem.

Theorem 2.12. [2] *Let G be a connected graph with $\delta(G) \geq 2$.*

(a) $\lambda(P_2(G)) \geq \delta(G) - 1$;

(b) $\lambda(P_2(G)) \geq 2\delta(G) - 2$ *if* $\lambda(G) \geq 2$.

Corollary 2.13. [2] *Let G be a connected δ-regular graph with $\delta(G) \geq 2$. Then, $\lambda(P_2(G)) \geq 2\delta(G) - 2$ if $\lambda(G) \geq 2$.*

It is interesting to notice that the results on Theorem 2.12 are best possible. Indeed, if we consider the graph G formed by joining two triangles by a path of length 3. It is easy to see that $\delta(G) = 2$, $\lambda(G) = 1$ and $\lambda(P_2(G)) = 1$. Hence, Theorem 2.12 (a) is best possible for minimum degree equal to two. Moreover, as $\delta(P_2(G)) \geq 2\delta(G) - 2$, then Theorem 2.12 (b) is also best possible whenever the edge-connectivity of G is exactly $\lambda(G) = 2$.

The case $k = 3$ was also studied using particular properties of 3-path graphs that cannot be extended to larger values of k. Noticed that for $k = 3$ Theorem 2.6 says:

If G be a connected graph with minimum degree $\delta > 4$. Then,
$$\lambda(P_3(G)) \geq 2(\delta - 2).$$

Following a similar method to construct paths as the one that gave rise to Theorem 2.6, but adding some technical considerations due to the fact that $k = 3$, it was shown in [11] that:

Theorem 2.14. [11] *Let G be a connected graph with minimum degree $\delta \geq 4$, then $P_3(G)$ is maximally connected.*

Then, considering that $\delta(P_3(G)) \geq 2(\delta - 1)$, the following corollary comes immediately.

Corollary 2.15. [11] *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, $\lambda(P_3(G)) \geq 2(\delta - 1)$.*

Furthermore, in the case of 3-path graphs we can relax the conditions on the minimum degree of G in Theorem 2.14 if G is a triangle-free graphs.

Theorem 2.16. [11] *Let G be a connected triangle-free graph with minimum degree $\delta \geq 3$. Then $P_3(G)$ is maximally edge connected.*

Since G has no triangles, $\delta(P_3(G))$ is exactly $2(\delta - 1)$, and this allows us to estate the following corollary.

Corollary 2.17. [11] *Let G be a connected triangle-free graph with minimum degree $\delta \geq 3$. Then, $\lambda(P_3(G)) = 2(\delta - 1)$.*

2.3 Superconnectivity of path graphs

Once a graph is known to be maximally connected, it is interesting to know if the graph is or not superconnected, and if so, to study the values of its superconnectivity. For that reason, this section is restricted to those graphs that satisfy the conditions for maximal connectivity given in the previous section.

For the general case in which k is any positive integer, it was proven in [10] that if G is a connected δ-regular graph with $\delta > 2(k-1)$, then $P_k(G)$ is maximally connected. Furthermore, in this case the following result holds.

Theorem 2.18. [10] *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then, $P_k(G)$ is super-λ.*

Corollary 2.19. [10, 3] *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 3$ and girth at least $k+1$. Then, $P_k(G)$ is super-λ.*

The previous theorem gives rise to a trivial lower bound for the superconnectivity of path graphs, which is $\lambda_1(P_k(G)) \geq 2(\delta - k + 1) + 1$.

Besides, since there are more relaxed conditions to guarantee maximal connectivity in the cases $k = 2$ and $k = 3$, we must study the superconnectivity of 2 and 3-path graphs of the maximally connected graphs that satisfy them.

Theorem 2.20. [2] *Let G be a graph with minimum degree $\delta \geq 3$ and $\lambda(G) \geq 3$, such that $\delta(P_2(G)) = 2\delta - 2$. Then $P_2(G)$ is super-λ and $\lambda_1(P_2(G)) \geq 3(\delta - 1)$.*

For regular graphs the lower bound presented in the previous theorem can be improved:

Theorem 2.21. [2] *Let G be a δ-regular graph with $\delta \geq 4$ and $\lambda(G) \geq 4$. Then $P_2(G)$ is super-λ and $\lambda_1(P_2(G)) = 4\delta - 6$.*

We can obtain analogous results for $k = 3$ if restricted to graphs with $\delta \geq 4$, which is enough to guarantee maximal connectivity.

Theorem 2.22. [11] *Let G be a connected graph and with minimum degree $\delta \geq 5$. Then, $P_3(G)$ is super-λ.*

As a consequence of Theorem 2.16, to study the superconnectivity of 3-path graphs of graphs without triangles we only need the condition $\delta \geq 3$. Reasoning as in Theorem 2.22 it can be proved the following result.

Theorem 2.23. [11] *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 4$. Then, $P_3(G)$ is super-λ.*

3 Iterated path graphs

The conditions on a graph G to study the connectivity of $P_k(G)$ that we presented in the previous section can be classified into two different categories. On one side there are results that apply to graphs having a certain minimum degree, and on the other side, there are results that relax the lower bound on the degree and impose that the girth be large. The following result shows that the results based on large girth cannot be extended to iterated path graphs.

Lemma 3.1. [10] *Let k be an integer and let G be a connected graph with minimum degree $\delta > k$. Then, the girth of $P_k(G)$ is 3 if $k = 1$ or 2 and G has triangles, and 4 otherwise.*

In other words, the girth of a k-path graphs can never be greater than $k+1$ if $k \geq 3$.

However, the condition $\delta > 2(k-1)$ is preserved under the path graph iteration [13]. Therefore, by induction on the number of iterations of the path graph operator, the following corollary of Theorem 2.6 was given.

Corollary 3.2. [10] *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, for every $s \geq 1$, $\lambda(P_k^s(G)) \geq 2^s\delta - 2^s + 2$.*

Similarly, it can be proven the following corollary.

Corollary 3.3. *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then, for every positive integer s, $P_k^s(G)$ is maximally connected.*

Corollary 3.4. [10] *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, for every positive integer s, $P_k^s(G)$ is super-λ.*

Similarly, by induction on s the improvements obtained for 2 and 3-path graphs based on a minimum degree condition, can be generalized to $P_2^s(G)$ and $P_3^s(G)$.

Theorem 3.5. *Let G be a connected δ-regular graph with $\delta(G) \geq 2$ and $\lambda(G) \geq 2$. Then, $P_2^s(G)$ is maximally connected.* □

Theorem 3.6. [11] *Let G be a connected graph with minimum degree $\delta \geq 4$. Then $P_3^s(G)$ is maximally connected.* □

Corollary 3.7. [11] *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, $\lambda(P_3^s(G)) \geq 2^s\delta - 2^s + 2$.*

An exception in which the conditions on the girth $g \geq k+1$ are preserved under path graph iterations is the case of graphs without triangles and 3-path graphs. In that case, using induction on s, Theorem 2.16 and Corollary 2.17 we can obtain:

Theorem 3.8. *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then $P_3^s(G)$ is maximally edge connected.*

Corollary 3.9. *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, $\lambda(P_3^s(G)) = 2^s\delta - 2^s + 2$.*

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