The graphs determined by an adjacency property

Katherine Heinrich
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B. C. V5A 1S6
Canada

Abstract. We determine all graphs G of order at least $k+1$, $k \geq 3$, with the property that for any k-subset S of $V(G)$ there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbours in S. Such graphs have exactly $k+1$ vertices and consist of a family of vertex-disjoint cycles. When $k = 2$ it is clear that graphs with this property are the so-called friendship graphs.

Let G be a graph with the property that for any two vertices in the graph there is a unique vertex adjacent to both of them. The Friendship Theorem states that in such a graph there must be a vertex which is adjacent to all other vertices. From this it follows easily that all such graphs (the friendship graphs) consist of a set of edge-disjoint 3-cycles which share a common vertex. This was first proved by Erdös, Rényi and Sós [4], and later alternate proofs were given by Wilf [7] and Longyear and Parsons [6].

Many variations of the Friendship theorem have been suggested. They are typically concerned with specifying either the number of paths between any two vertices or the size of the neighbourhood of any k-subset of vertices. Descriptions of these generalizations can be found in Bondy [1] and Delorme and Hahn [3].

Caccetta, Erdös and Vijayan [2] have studied graphs G with the property that for any two subsets A and B of $V(G)$ with $|A \cap B| = 0$ and $|A \cup B| = k$, there exists a vertex u, $u \notin A \cup B$, so that u is adjacent to all vertices of A and none of B. Prior to this Exoo [5] had considered graphs in which the sizes of A and B are fixed, and for each such A and B there are at least t vertices, $u_1, u_2, \cdots, u_t$, adjacent to each of A and to none of B. If we suppose that $|A| = 2, |B| = 0, t = 1$, and moreover that u_1 is unique, then the graphs with this property are precisely the friendship graphs. (Without uniqueness we obtain all graphs with diameter at most 2 in which each edge lies in a 3-cycle.)

Discussions with Alspach and Caccetta on graphs with such properties led to the following problem which might be viewed as a generalization of the Friendship Theorem: Determine all graphs G of order at least $k+1$ with the property that for any k-subset S of $V(G)$ there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbours in S. The case $k = 2$ is, of course, the Friendship Theorem so we will only consider $k \geq 3$.

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Main Theorem. *If G is a graph of order at least $k+1$, $k \geq 3$, with the property that for any k-subset S of $V(G)$ there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbours in S, then G has exactly $k+1$ vertices and is regular of degree 2. (So G is a vertex-disjoint union of cycles.)*

We prove the main theorem via the following series of steps. In Theorem 1 we will show that if $k = 3$, then G must be a cycle of length four. In Theorem 2 we will show that if $|V(G)| = k+2$, $k \geq 2$, then G is regular of degree 2 and hence is a vertex-disjoint union of cycles. Finally the proof of the main theorem is completed in Theorem 4 when we show that there are no graphs with more than $k+1$ vertices, $k > 3$, which have the desired property.

For any graph G we will say that a k-subset A of $V(G)$ is good if there exists a vertex x, $x \in V(G) - S$, so that x has exactly two neighbours in A. If there is no such x we say that A is not good.

Theorem 1. *If G is a graph or order at least four with the property that for any 3-subset S of $V(G)$ there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbours in S, then $G \cong C_4$ (the cycle of length 4).*

Proof: Let G be a graph as described in the theorem. We will first prove that if G has a path of length four on the vertex-set W, then the subgraph of G induced on W is either K_5 or $K_5 - e$ (the graph K_5 with the edge e deleted).

Let $W = \{1,2,3,4,5\}$ and let the path of length four have edges 12, 23, 34 and 45. Since 2 and 4 cannot both be adjacent to exactly two of $\{1,3,5\}$, we must have one of the edges 14 or 25. Without loss of generality we can assume that we have the edge 25. Next, so that 3 and 5 do not both have exactly two neighbours in $\{1,2,4\}$, we must have one of the edges 13 and 15. Again without loss of generality we add the edge 13. Currently each of 1 and 4 has exactly two neighbours in $\{2,3,5\}$. So we must have one of the edges 15 and 24. If we have 15, then since 1 and 2 now both have two neighbours in $\{3,4,5\}$ we must add either 24 or 14. On either addition we get isomorphic graphs so we can assume that the edge added is 14. Now consider vertices 4 and 5 and the set $\{1,2,3\}$. Either 24 or 35 must be added and in either case we get $K_5 - e$. So assume that instead of adding 15 we added 24. Vertices 1 and 5 and the 3-set $\{2,3,4\}$ require that one of the edges 14 and 35 be added. Either choice yields an isomorphic graph so we can suppose that we add 14. Now consider vertices 1 and 3 and the 3-set $\{2,4,5\}$. From this we must add one of the edges 15 or 35. Both yield $K_5 - e$ and we are done.

It now follows that if G has a path of length four, then $|V(G)| \geq 5$ and G is either a complete graph or a complete graph with an edge deleted. (To see this argue as follows. Let $V(K_5) = V(K_5 - e) = \{1,2,3,4,5\}$ and suppose that $e = 12$. Consider the 3-subset $\{3,4,5\}$. There is a vertex x, $x \notin \{1,2,3,4,5\}$, with exactly two neighbours in $\{3,4,5\}$. Without loss of generality we may assume $3x \in E(G)$. But we now have a path of length four on the vertex set

$\{1,2,3,4,x\}$, and hence the subgraph induced on these vertices is either K_5 or $K_5 - e$. Now consider the set $\{1,2,3,5,x\}$ which also contains a path of length four and hence induces either K_5 or $K_5 - e$. So we have K_6 or $K_6 - e$. Continue in this manner.) Clearly each graph contains a 3-subset which is not good.

So G has no path of length four. Let $x \in V(G)$ so that $\deg(x) = \Delta(G) \geq 2$ (as is easily seen). Then, to avoid a path of length four, $V(G) - \{x\} \cup \Gamma(x) \cup \Gamma^2(x)$ (where $\Gamma(x)$ is the set of neighbours of x, and $\Gamma^2(x)$ the set of neighbours of $\Gamma(x)$ but excluding x.)

Suppose $\Gamma^2(x) = \emptyset$. If $|\Gamma(x)| \geq 4$, there is at most one edge in $G[\Gamma(x)]$ and hence no 3-set in $\Gamma(x)$ is good. So $|\Gamma(x)| = 3$ (as $|V(G)| \geq 4$) and the 3-set $\Gamma(x)$ is not good.

Therefore $\Gamma^2(x) \neq \emptyset$. Clearly $G[\Gamma^2(x)]$ can contain no edges. If $y \in \Gamma(x)$ then either y is adjacent to all or none of $\Gamma^2(x)$. If both y and $x \in \Gamma(x)$ have non-empty neighbourhoods in $\Gamma^2(x)$, then $|\Gamma^2(x)| = 1$. But this implies that $|\Gamma(x)| \leq 2$. Hence $|\Gamma(x)| = 2$ and G is C_4. Thus exactly one vertex of $\Gamma(x)$ is adjacent to all vertices of $\Gamma^2(x)$. If $|\Gamma(x)| \geq 3$, there can be no edges in $G[\Gamma(x)]$ and hence no 3-set on $\Gamma(x)$ can be good. So $|\Gamma(x)| = 2$, but now $\{x\} \cup \Gamma(x)$ is not good. ∎

Theorem 2. *If G is a graph with $|V(G)| = k+1$ so that for any k-subset S of $V(G)$ there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbours in S, then G is regular of degree two.*

Proof: Suppose $|V(G)| = k+1$. For each y, $y \in V(G)$, let $S = V(G) - \{y\}$. Then y has exactly two neighbours in S and hence G is regular of degree 2. ∎

Before proving the last theorem we need the following lemma.

Lemma 3. *Let G be a graph of order at least $k+2$, $k > 3$, with the property that for any k-subset S of $V(G)$ there is a unique vertex $x, x \in V(G) - S$, with exactly two neighbours in S. Then G has a vertex of degree at least k.*

Proof: We first prove that for any vertex y in the graph G (as described in the statement of the lemma), either $\deg(y) = 2$ or $\deg(y) \geq k$.

Let $y \in V(G)$ and suppose that $\deg(y) \neq 2$ and $\deg(y) \leq k-1$. Let S_1 be a k-subset of $V(G)$ so that $\{y\} \cup \Gamma(y) \subseteq S_1$. By assumption there is a unique vertex $y_1 \in V(G) - S_1$ so that $|\Gamma(y_1) \cap S_1| = 2$. Let $S_2 = S_1 \cup \{y_1\} - \{y\}$ and note that $yy_1 \notin E(G)$. Then there is a unique vertex $y_2 \in V(G) - S_2$ so that $|\Gamma(y_2) \cap S_2| = 2$. Clearly $y_2 \neq y$. Observe that $y_1 y_2 \in E(G)$ (or we contradict the uniqueness of y_1). Now let $S_3 = S_2 \cup \{y_2\} - \{y_1\}$. There is a unique vertex $y_3 \in V(G) - S_3$ so that $|\Gamma(y_3) \cap S_3| = 2$. Clearly y_3 is distinct from y, y_1 and y_2, and $y_2 y_3 \in E(G)$. But now as both y_1 and y_2 cannot have exactly two neighbours in $S = S_3 \cup \{y_3\} - \{y_2\}$, we must have $y_1 y_3 \in E(G)$. But now both y_2 and y_3 have exactly two neighbours in S_2. We have reached a contradiction.

We next show that all vertices in G can not have degree two and so there must be a vertex of degree at least k. If G is regular of degree two, then G has non-adjacent vertices x and y and $|\Gamma(x) \cup \Gamma(y)| \leq 4$. There is a k-subset S in $V(G) - \{x, y\}$ so that $\Gamma(x) \cup \Gamma(y) \subseteq S$, and it is not good. ∎

Theorem 4. *There is no graph $G, |V(G)| > k+1$ and $k > 3$, with the property that for any k-subset S of $V(G)$ there is a unique vertex x, $x \in V(G) - S$, which has exactly two neighbours in S.*

Proof: The proof is quite straightforward but unfortunately many cases must be considered. Let x be a vertex of G with $\deg(x) \geq k$ (such a vertex exists by Lemma 3). We will write $V(G) = \{x\} \cup \Gamma(x) \cup \Gamma^2(x) \cup T$ but our argument will use only the vertices of $\{x\} \cup \Gamma(x) \cup \Gamma^2(x)$.

Let A be a $(k-1)$-subset of $\Gamma(x)$. For the set $A \cup \{x\}$ there is a unique vertex y_1, $y_1 \in (\Gamma(x) \cup \Gamma^2(x)) - (A \cup \{x\})$, so that $|\Gamma(y_1) \cap (A \cup \{x\})| = 2$.

Case 1 $y_1 \in \Gamma(x)$: For the set $A \cup \{y_1\}$ there is a unique vertex y_2, $y_2 \in (\Gamma(x) \cup \Gamma^2(x)) - (A \cup \{y_1\})$, $(y_2 \neq x)$ so that $|\Gamma(y_2) \cap (A \cup \{y_1\})| = 2$.

1.1 $y_2 \in \Gamma(x)$: If $y_1 y_2 \in E(G)$, then both y_1 and y_2 have exactly two neighbours in $A \cup \{x\}$. So $y_1 y_2 \notin E(G)$. Consider the k-set $A \cup \{y_2\}$. There is a unique vertex y_3, $y_3 \in (\Gamma(x) \cup \Gamma^2(x)) - (A \cup \{y_2\})$, and $y_3 \neq x, y_1$, so that $|\Gamma(y_3) \cap (A \cup \{y_2\})| = 2$.

1.1.1 $y_3 \in \Gamma(x)$: Clearly $y_2 y_3 \notin E(G)$ (or both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$), and $y_3 y_1 \in E(G)$ (else both y_2 and y_3 have exactly two neighbours in $A \cup \{y_1\}$). But now both y_1 and y_2 have exactly two neighbours in $A \cup \{y_3\}$. A contradiction.

1.1.2 $y_3 \in \Gamma^2(x)$: On considering $A \cup \{x\}$ we see that $y_3 y_2 \in E(G)$. Since $|A| \geq 3$, there exists a vertex $y \in A$ so that both $yy_1 \notin E(G)$ and $yy_3 \notin E(G)$. Let $B = A \cup \{x\} \cup \{y_2\} - \{y\}$. But B is a k-subset and $|\Gamma(y_3) \cap B| = |\Gamma(y_1) \cap B| = 2$ and we obtain a contradiction.

This completes the subcase 1.1.

1.2 $y_2 \in \Gamma^2(x)$: Our first observation is that $y_1 y_2 \in E(G)$ (else y_2 also has exactly two neighbours in $A \cup \{x\}$). Since $|A| \geq 3$, we can choose a vertex $y \in A$ so that both $yy_1 \notin E(G)$ and $yy_2 \notin E(G)$. Let $A' = A - \{y\}$. Then there is a vertex y_3, $y_3 \in (\Gamma(x) \cup \Gamma^2(x)) - (A' \cup \{x, y_2\})$, which has exactly two neighbours in $A' \cup \{x, y_2\}$. Clearly $y_3 \neq x, y_1, y_2$. Also $y_3 \neq y$ (else both y and y_2 have exactly two neighbours in $A' \cup \{x, y_1\}$).

1.2.1 $y_3 \in \Gamma(x)$ and $y_2 y_3 \in E(G)$: Note that $yy_3 \notin E(G)$ or else both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$. Let $y^* \in A'$ so that $y^* y_1 \in E(G)$. But now each of y_1 and y_3 have exactly two neighbours in $A \cup \{x, y_2\} - \{y^*\}$ and we have a contradiction.

1.2.2 $y_3 \in \Gamma(x)$ and $y_2 y_3 \notin E(G)$: It immediately follows that $yy_3 \in E(G)$ as otherwise both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$. But now both y_1 and y_3 have exactly two neighbours in $A \cup \{y_2\}$. A contradiction.

1.2.3 $y_3 \in \Gamma^2(x)$ and $y_2 y_3 \in E(G)$: Now $yy_3 \in E(G)$ or both y_1 and y_3 have exactly two neighbours in $A \cup \{y_2\}$. But now both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$.

1.2.4 $y_3 \in \Gamma^2(x)$ and $y_2 y_3 \notin E(G)$: To avoid both y_1 and y_3 having exactly two neighbours in $A' \cup \{x, y_1\}$ we must have $y_1 y_3 \in E(G)$, and to avoid both y_1 and y_3 having exactly two neighbours in $A \cup \{x\}$ we must have $yy_3 \in E(G)$. We now consider the unique vertex y_4, $y_4 \in (\Gamma(x) \cup \Gamma^2(x)) - (A' \cup \{x, y_3\})$, satisfying $|\Gamma(y_4) \cap (A' \cup \{x, y_3\})| = 2$. Clearly $y_4 \neq x, y_1, y_2, y_3$. Suppose that $y_4 = y$. Then $|\Gamma(y) \cap A'| = 0$. Now choose $y^* \in A'$ so that $y^* y_2 \notin E(G)$. Then both y and y_2 have exactly two neighbours in $A' \cup \{x, y_1, y_3\} - \{y^*\}$. A contradiction so $y_4 \neq y$.

1.2.4.1 $y_4 \in \Gamma(x)$ and $y_3 y_4 \in E(G)$: Let $y^* \in A'$ so that $y^* y_1 \in E(G)$. To avoid both y_1 and y_4 having exactly two neighbours in $A \cup \{x, y_3\} - \{y^*\}$ we must have $yy_4 \in E(G)$. But now both y_1 and y_4 have exactly two neighbours in $A \cup \{x\}$. A contradiction.

1.2.4.2 $y_4 \in \Gamma(x)$ and $y_3 y_4 \notin E(G)$: It follows immediately that $yy_4 \in E(G)$, or else both y_1 and y_4 have exactly two neighbours in $A \cup \{x\}$. But now both y_1 and y_4 have exactly two neighbours in $A \cup \{y_3\}$.

1.2.4.3 $y_4 \in \Gamma^2(x)$ and $y_3 y_4 \in E(G)$: First, $y_1 y_4 \in E(G)$ or both y_2 and y_4 have exactly two neighbours in $A' \cup \{y_1, y_3\}$, and second $yy_4 \in E(G)$ or both y_2 and y_4 have exactly two neighbours in $A \cup \{y_1\}$. But now both y_1 and y_4 have exactly two neighbours in $A \cup \{x\}$.

1.2.4.4 $y_4 \in \Gamma^2(x)$ and $y_3 y_4 \notin E(G)$: As in the previous case we find that $y_1 y_4 \in E(G)$ or else both y_2 and y_4 have exactly two neighbours in $A' \cup \{y_1, y_3\}$. Now choose $y^* \in A'$ so that both $y^* y_4 \in E(G)$ and $y^* y_2 \notin E(G)$ (since y_4 has two neighbours in A' and y_2 only one this is possible). But now both y_2 and y_4 have exactly two neighbours in $A' \cup \{x, y_1, y_3\} - \{y^*\}$ and we have a contradiction.

This completes subcase 1.2.4, so completing case 1.

Case 2 $y_1 \in \Gamma^2(x)$: For the set $A \cup \{y_1\}$ there exists a unique vertex y_2, $y_2 \in (\Gamma(x) \cup \Gamma^2(x)) - (A \cup \{y_1\})$, $(y_2 \neq x, y_1)$ so that $|\Gamma(y_2) \cap (A \cup \{y_1\})| = 2$.

2.1 $y_2 \in \Gamma^2(x)$: We observe immediately that $y_1 y_2 \in E(G)$. For the set $A \cup \{y_2\}$ there is a unique vertex y_3, $y_3 \in (\Gamma(x) \cup \Gamma^2(x)) - (A \cup \{y_2\})$, $(y_3 \neq x, y_1)$ so that $|\Gamma(y_3) \cap (A \cup \{y_2\})| = 2$.

2.1.1 $y_3 \in \Gamma^2(x)$: We must have $y_2 y_3 \in E(G)$ or else both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$. Now choose $y^* \in A$ so that both $y^* y_1 \in E(G)$ and $y^* y_3 \notin E(G)$. We now find that both y_1 and y_3 have exactly two neighbours in $A \cup \{x, y_2\} - y^*$. A contradiction.

2.1.2 $y_3 \in \Gamma(x)$: First, $y_2 y_3 \notin E(G)$ as otherwise both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$, and second $y_1 y_3 \in E(G)$ or both y_2 and y_3 have exactly two neighbours in $A \cup \{y_1\}$. Now we know there is a unique vertex y_4, $y_4 \in (\Gamma(x) \cup \Gamma^2(x)) - (A \cup \{y_3\})$, so that $|\Gamma(y_4) \cap (A \cup \{y_3\})| = 2$. Clearly $y_4 \neq x, y_1, y_2, y_3$.

2.1.2.1 $y_4 \in \Gamma^2(x)$: We begin by observing that $y_3 y_4 \in E(G)$ as otherwise both y_1 and y_4 have exactly two neighbours in $A \cup \{x\}$. Choose $y^* \in A$ so that both $y^* y_1 \in E(G)$ and $y^* y_4 \notin E(G)$. Then both y_1 and y_4 have exactly two neighbours in $A \cup \{x, y_3\} - y^*$. A contradiction.

2.1.2.2 $y_4 \in \Gamma(x)$: First, $y_3 y_4 \notin E(G)$ or else both y_1 and y_4 have exactly two neighbours in $A \cup \{x\}$. This then implies that $y_2 y_4 \in E(G)$ or both y_3 and y_4 have exactly two neighbours in $A \cup \{y_2\}$. Next choose $y^* \in A$ so that both $y^* y_3 \in E(G)$ and $y^* y_2 \notin E(G)$. But now each of y_2 and y_3 has exactly two neighbours in $A \cup \{x, y_4\} - y^*$. A contradiction.

This completes the proof of subcase 2.1.

2.2 $y_2 \in \Gamma(x)$: This is the last possibility so we can assume that for any $(k-1)$-subset $B \subseteq \Gamma(x)$, the unique vertex with exactly two neighbours in $B \cup \{x\}$ lies in $\Gamma^2(x)$. To begin observe that $y_1 y_2 \notin E(G)$ as otherwise both y_1 and y_2 have exactly two neighbours in $A \cup \{x\}$. Choose $y \in A$ so that $y y_1 \in E(G)$ and consider the set $B = A \cup \{y_2\} - \{y\}$. Now, there is a unique vertex y_3, $y_3 \in \Gamma^2(x) - (B \cup \{x\})$ so that $|\Gamma(y_3) \cap (B \cup \{x\})| = 2$.

2.2.1 $y_2 y_3 \in E(G)$: Now, $y_3 y \notin E(G)$ or else both y_1 and y_3 have exactly two neighbours in $A \cup \{x\}$. But this implies that both y_1 and y_3 have exactly two neighbours in $A \cup \{y_2\}$.

2.2.2 $y_2 y_3 \notin E(G)$: Then $y_3 y \in E(G)$ as otherwise both y_1 and y_3 have exactly two neighbours in $A \cup \{y_2\}$. Also $y_1 y_3 \in E(G)$ else y_1 and y_2 each have exactly two neighbours in $A \cup \{y_3\}$. Choose $y^* \in A$ so that both $y^* y_3 \in E(G)$ and $y^* y_1 \notin E(G)$. We now see that both y_1 and y_3 have exactly two neighbours in $A \cup \{x, y_2\} - \{y^*\}$. A contradiction.

This completes the proof of case 2 and hence the proof of the theorem. ∎

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