On the Existence of Whim Domino Squares

Hung-Lin Fu and Chin-Lin Shue

Department of Applied Mathematics
National Chiao Tung University
Hsin-Chu, Taiwan
REPUBLIC OF CHINA

Abstract. The idea of a *domino square* was first introduced by J. A. Edwards *et al* in [1]. In the same paper, they posed some problems on this topic. One problem was to find a general construction for a whim domino square of side $n \equiv 3 \pmod 4$. In this paper, we solve this problem by using a direct construction. It follows that a *whim domino square* exists for each odd side [1].

1. Introduction.

Given an $n \times n$ chessboard (an $n \times n$ matrix of cells), can it be covered by a set of distinct dominoes (1×2 or 2×1 matrices) on the numbers $0, 1, \ldots, n$, so that the numbers appearing in each row and each column are all distinct? Clearly one must discard dominoes which have the same number at each end. Given a complete set of dominoes on the numbers $0, 1, 2, \ldots, n$, there are $\binom{n+1}{2} = \frac{1}{2}(n^2+n)$ dominoes from which one may choose. We call an $n \times n$ matrix covered in this way with dominoes based on the numbers $0, 1, 2, \ldots, n$ a domino latin square of side n (or, briefly, a domino square of side n). Since each domino covers two cells, and the number of cells in a domino square of side n is n^2, it is clear that domino squares of side n can only exist if n is even. It has been shown in [1] that there exists a domino square for each even side. If n is odd we adapt the definition slightly. This time we cover all cells of $n \times n$ matrix except for the central cell. We call such a square a domino latin square *with* a *hole in* the *middle*, or acronimically, a *whim* domino square, or a whimsy for short. In Figure 1.1, we present two known whimsy of side 3 and 5 respectively. A general construction for the whimsy of side $n \equiv 1 \pmod 4$ can be found in [1], but when $n \equiv 3 \pmod 4$, the problem remained open. In this paper, we solve this case, and thus complete the proof of the existence of whim domino squares of odd side.

0	1	2
1		3
3	2	1

0	1	3	2	4
1	2	0	3	5
4	5		0	2
2	3	5	4	0
3	4	2	5	1

Figure 1.1

2. The main results.

Before we construct whim domino squares, we need a lemma.

Lemma 2.1. *[2] If* $m \equiv 0,1 \pmod 4$, *then we can arrange the numbers* $1,2,\ldots,2m$ *into* m *pairs (A-system)* $(a_1,b_1),(a_2,b_2),\ldots,(a_m,b_m)$ *such that* $b_i - a_i = i$, $i = 1,2,\ldots,m$. *If* $m \equiv 2,3 \pmod 4$, *then the numbers* $1,2,\ldots,2m-1,2m+1$ *can be arranged into* m *pairs* $(c_1,d_1),\ldots,(c_m,d_m)$ *(B-system) such that* $d_i - c_i = i$, $i = 1,2,\ldots,m$.

Let A be the partial latin square given in Figure 2.1.

A:

0	1	2	$\ldots$	$n-1$
1	2	3	$\ldots$	n
2	3	4	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$n-2$	$n-1$	n		$n-4$
$n-1$	n	0	$\ldots$	$n-3$

Figure 2.1

Construction: $n = 4k+3$ and k is even.

1^0. We start with the latin rectangle $R = [r_{i,j}]$ obtained from A in Figure 2.1 by deleting the first column.

2^0. Permute the columns of R to obtain R' such that $r_{1,1} = a_1$, $r_{1,2} = b_1$, $r_{1,3} = a_2$, $r_{1,4} = b_2$, $\ldots$, $r_{1,n-2} = a_{2k+1}$, $r_{1,n-1} = b_{2k+1}$, where the pairs $(a_1,b_1),(a_2,b_2),\ldots,(a_{2k+1},b_{2k+1})$ form an A-system of the set $\{1,2,\ldots,4k+2\}$.

3^0. Let C be the column containing the elements $a_i - 1$ and $b_i - 1$, for $1 \leq i \leq 2k+1$, and $4k+2$ such that
  (a) $2k$ is in the $(2k+2)$-th position,
  (b) $4k+2$ is in the $(2k+3)$-th position, and,
  (c) for $1 \leq i \leq 2k+1$, elements $a_i - 1$ and $b_i - 1$ are in adjacent positions.

4^0. Construct an $n \times n$ array W such that the last column of W is C and (except for the last column) the ith row ($i = 1,2,\ldots,n$) of W is the $(j+1)$-th row of R' whenever the ith entry of C is j.

5^0. By pairing the entries (cells) of W, we obtain an $n \times n$ array which is covered by dominoes as in Figure 2.2.

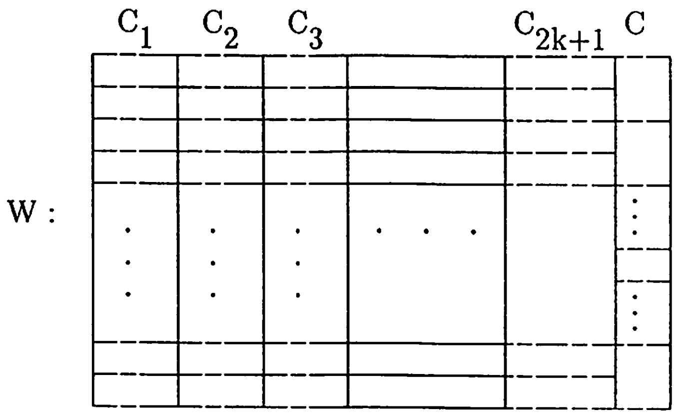

Figure 2.2

6^0. By permuting the columns of W, and possibly flipping all the dominoes in one column, we obtain an array W', where C is in the $(2k+3)$-th column and the column to the left of C, C', satisfies the following property: the $(2k+3)$-th cell of C' is filled with $2k$.

7^0. By changing the dominoes in the middle of W' (Figure 2.3) we obtain a whim domino square of side $4k+3$ with k an even number.

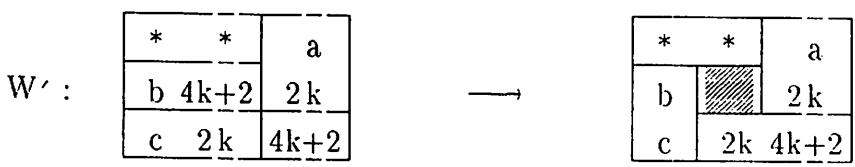

Figure 2.3

Lemma 2.2. *There exists a whim domino square of side n for each $n = 4k+3$ where k is an even number.*

Proof: (To follow the proof, it is advisable to look in the appendix at the example of the construction of a whim domino square of side 11). It is clear that no symbol is repeated in any row or column. None of the differences $b_i - a_i$ is congruent to $2k+2$ module $n+1 = 4k+4$. It follows that none of the dominoes

$$\boxed{0,\ 2k+2},\quad \boxed{1,\ 2k+3},\ldots,\ \boxed{2k+1,\ 4k+3}$$

occurs in the squares W or W'. In W' row $2k+3$ can be obtained from row $2k+2$ by adding $2k+2$ and reducing module $4k+4$. Therefore, the two dominoes which are introduced in the last stage, $\boxed{2k,\ 4k+2}$ and $\boxed{b, c}$, are both in the set of dominoes which do not occur in W or W'. It follows that we obtain finally a whim domino square of side n.

Theorem 2.3. *A whim domino square of side* n *exists for each* $n \equiv 3 \pmod 4$.

Proof: Let $n = 4k + 3$, k is an integer. By Lemma 2.2 we have the proof for the case k is an even integer. For the case k is an odd integer, we simply replace A-system by B-system in 2^0, $2k$ by $2k-1$ in 3^0, $4k+2$ by $4k+1$ in 3^0, the last column of A by $[n, 1, 2, \ldots, n-2]^T$, and the other steps are similar; we omit the details.

Acknowledgement.

The authors appreciate the helpful comments and modifications of Dr. A. J. W. Hilton and also the referee.

References

1. J.A. Edwards, G.M. Hamilton, A.J.W. Hilton, and Bill Jackson, *Domino squares,*, Annals of Discrete Mathematics **12** (1982), 95–111.
2. A.J.W. Hilton, *On Steiner and similar triple systems*, Math. Scand. **24** (1969), 208–216.

Appendix

(1°)

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	0
4	5	6	7	8	9	10	11	0	1
5	6	7	8	9	10	11	0	1	2
6	7	8	9	10	11	0	1	2	3
7	8	9	10	11	0	1	2	3	4
8	9	10	11	0	1	2	3	4	5
9	10	11	0	1	2	3	4	5	6
10	11	0	1	2	3	4	5	6	7
11	0	1	2	3	4	5	6	7	8

(2°)

8	9	1	3	4	7	2	6	5	10
9	10	2	4	5	8	3	7	6	11
10	11	3	5	6	9	4	8	7	0
11	0	4	6	7	10	5	9	8	1
0	1	5	7	8	11	6	10	9	2
1	2	6	8	9	0	7	11	10	3
2	3	7	9	10	1	8	0	11	4
3	4	8	10	11	2	9	1	0	5
4	5	9	11	0	3	10	2	1	6
5	6	10	0	1	4	11	3	2	7
6	7	11	1	2	5	0	4	3	8

(3°)

7
8
0
2
9
4
10
3
6
1
5

(4°, 5°)

3	4	8	10	11	2	9	1	0	5	7
4	5	9	11	0	3	10	2	1	6	8
8	9	1	3	4	7	2	6	5	10	0
10	11	3	5	6	9	4	8	7	0	2
5	6	10	0	1	4	11	3	2	7	9
0	1	5	7	8	11	6	10	9	2	4
6	7	11	1	2	5	0	4	3	8	10
11	0	4	6	7	10	5	9	8	1	3
2	3	7	9	10	1	8	0	11	4	6
9	10	2	4	5	8	3	7	6	11	1
1	2	6	8	9	0	7	11	10	3	5

(6°)

3	4	8	10	9	1	7	11	2	0	5
4	5	9	11	10	2	8	0	3	1	6
8	9	1	3	2	6	0	4	7	5	10
10	11	3	5	4	8	2	6	9	7	0
5	6	10	0	11	3	9	1	4	2	7
0	1	5	7	6	10	4	8	11	9	2
6	7	11	1	0	4	10	2	5	3	8
11	0	4	6	5	9	3	7	10	8	1
2	3	7	9	8	0	6	10	1	11	4
9	10	2	4	3	7	1	5	8	6	11
1	2	6	8	7	11	5	9	0	10	3

(7°)