Parameter Sets for a Class of Strongly Regular Graphs

Bert L. Hartnell
St. Mary's University
Halifax, Canada,

Neville Jeans and William Kocay
University of Manitoba
Winnipeg, Canada

Abstract. Strongly regular graphs are graphs in which every adjacent pair of vertices share λ common neighbours and every non-adjacent pair share μ common neighbours. We are interested in strongly regular graphs with $\lambda = \mu = k$ such that every such set of k vertices common to any pair always induces a subgraph with a constant number x of edges. The Friendship Theorem proves that there are no such graphs when $\lambda = \mu = 1$. We derive constraints which such graphs must satisfy in general, when $\lambda = \mu > 1$, and $x \geq 0$, and we find the set of all parameters satisfying the constraints. The result is an infinite, but sparse, collection of parameter sets. The smallest parameter set for which a graph may exist has 4896 vertices, with $k = 1870$.

1. Introduction.

We shall use the graph-theoretic notation of Bondy and Murty [1], so that a graph X has vertex set $V(X)$, edge set $E(X)$, and $\varepsilon(X)$ edges. If $V' \subseteq V(X)$, then $X[V']$ denotes the subgraph of X induced by V'.

The following problem arose in connection with *neighbourhood-connected* graphs. If $u \in V(X)$, then $N(u)$ denotes the *neighbourhood* of u, that is, $N(u) = \{v \mid uv \in E(X)\}$, and $N^*(u)$ denotes *closed* neighbourhood, $N^*(u) = N(u) \cup \{u\}$. A graph X is *neighbourhood-connected* (NC) if:

(1) X is connected; and
(2) $X - N^*(u)$ is connected and not complete, for all $u \in V(X)$.

Neighbourhood-connected graphs have been studied in [5] and [6]. A graph X is 2-neighbourhood-connected (2-NC) if it is NC and $X - N^*(u) - N^*(v)$ is connected but not complete, for all $u, v \in V(X)$. An interesting special case occurs when X is a regular graph and all $X - N^*(u) - N^*(v) \cong Y$, for a fixed graph Y. Let X be such a 2-NC graph.

It is usually more convenient to work with the complement of X, so let $\Gamma := \overline{X}$. Then $\Gamma[N_\Gamma(u) \cap N_\Gamma(v)] \cong \overline{Y}$, for all $u, v \in V(\Gamma)$, where $N_\Gamma(u)$ now denotes the neighbourhood in Γ. We relax this condition somewhat, and only require that the subgraphs $\Gamma[N_\Gamma(u) \cap N_\Gamma(v)]$ all have the same number of vertices and edges, for all pairs $u, v \in V(\Gamma)$. Therefore we take Γ to be an r-regular graph on n vertices such that $\Gamma[N_\Gamma(u) \cap N_\Gamma(v)]$ always contains k vertices and x edges.

A *strongly regular* graph (see [2], [3], [4], [7], [8]) is a regular graph in which every pair of adjacent vertices share λ common neighbours and every pair of non-adjacent vertices share μ common neighbours.

Lemma 1.1. Γ *is a strongly regular graph with* $\lambda = \mu = k$.

So Γ must satisfy the integrality conditions that strongly regular graphs do. In particular, the *Friendship Theorem* (see [3], [4]) states that no such Γ exists with $k = 1$. Since $k = 1$ implies that the common subgraph must always have a constant number $x = 0$ of edges, the results presented here can be considered a generalization of the Friendship Theorem. We prove that there are no such graphs with $k < 1870$, we determine constraints that the parameters must satisfy, and we find all parameter sets satisfying these constraints. The result is an infinite, but very sparse, set of parameters.

(Notice that the complete graph K_n always satisfies these conditions, with $r = n-1$ and $k = n-2$. Since this can be considered a trivial solution to the problem, we restrict our consideration to solutions with $n \neq r+1$.)

Lemma 1.2. *Let* Γ, r, k, *and* n *be as above. Then* $r - k = s^2$, *where* $s \mid k$ *and* $s \geq 2$, *and* $n = 1 + r(r-1)/k$.

Proof: Since every pair of vertices share k common neighbours, there are k paths of length 2 connecting any two vertices of Γ. Let A be the adjacency matrix of Γ. Then A^2 has off-diagonal elements all equal to k. Computing the eigenvalues and their multiplicities gives the stated conditions. If $s = 1$, then $r = k+1$ and $n = k+2$, so that Γ is a complete graph, which we have excluded, leaving $s \geq 2$.

Let $V = V(\Gamma)$, pick $v \in V$ and let $G = \Gamma[N_\Gamma(v)]$, the graph induced by the neighbourhood of v. Set $Q = \Gamma[V - N^*(v)]$. This is illustrated in Fig. 1. Clearly, G is a k-regular graph and Q is an s^2-regular graph. $[G, Q]$ stands for the edges of Γ with one end in G and one in Q.

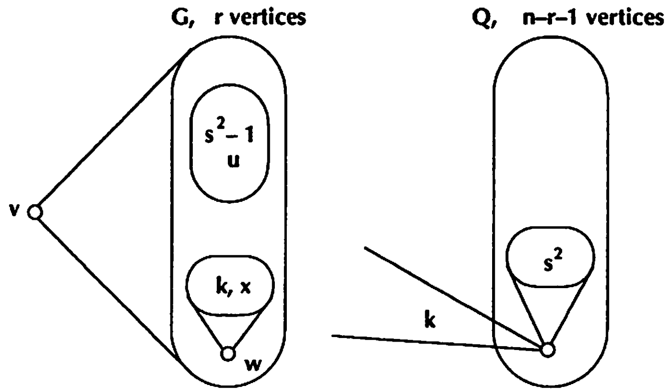

Fig. 1

If g and G are graphs, we write $n(g, G)$ for the number of *induced* subgraphs of G which are isomorphic to g. In particular, if K_i denotes a complete graph on i vertices, then $n(K_2, G)$ and $n(K_3, G)$ denote the number of edges and triangles of G, respectively. If the graph G is understood, we will sometimes shorten $n(g, G)$ to $n(g)$.

Lemma 1.3. $\varepsilon(G) = rk/2$ *and* $n(K_3, G) = rx/3$.

Proof: In G, the neighbourhood of every vertex contains k vertices and x edges, so that every point of G is on x triangles. Since G has r vertices, $n(K_3, G) = rx/3$.

Lemma 1.4. *Every edge of* Γ *is contained in exactly* k *triangles.*

Proof: Any pair of vertices of Γ share k common neighbours, so every edge is on k triangles.

Lemma 1.5. $x = \frac{k^2(k-1)}{2(r-1)}$.

Proof: We count the triangles of Γ which contain one or more edges of G. G has $kr/2$ edges, each of which is on k triangles, so $k^2r/2$ counts the edges of G according to the triangles containing them. Each of the $xr/3$ triangles of G are counted 3 times in this sum. Each edge of G is counted once for the triangle it forms with v, and each vertex of Q contributes x to the sum, since it shares k common neighbours with v. So $\frac{k^2r}{2} = xr + \frac{kr}{2} + x(n-r-1)$. Solving this for x gives the result, using Lemma 1.2.

In Section 3, we find all possible parameter sets s, k, x, r, n satisfying this formula.

Now let $\overline{K}_3$ denote the *empty* graph on 3 vertices (i.e., no edges), and $\overline{K}_2$ the empty graph on 2 vertices (a non-edge).

Lemma 1.6. *Every* $\overline{K}_2$ *of* Γ *is contained in exactly* $(n-2r+k-2)$ $\overline{K}_3$ *'s.*

Proof: There are $(n-2r+k-2)$ $\overline{K}_3$'s containing any non-edge, as illustrated in Fig. 2.

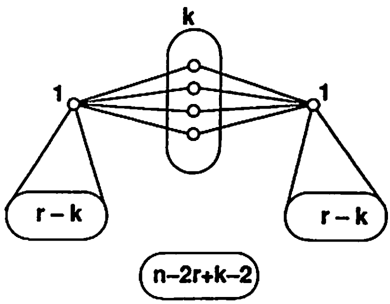

Fig. 2

We now count the number of $\overline{K}_3$'s of Γ containing one or more $\overline{K}_2$'s of either G or $[G, Q]$. There are several different kinds of such $\overline{K}_3$. This is illustrated in Fig. 3 where a name is given to the number of each type of $\overline{K}_3$ falling between G and Q. The $\overline{K}_3$'s are indicated by dashed lines. Then $n(W)$ denotes the number of $\overline{K}_3$'s of type W, etc.

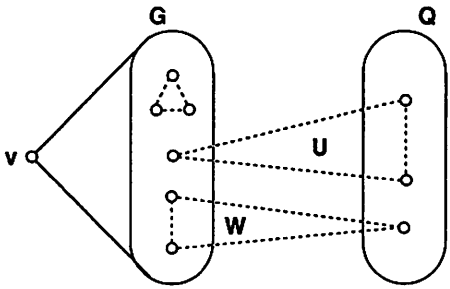

Fig. 3

Lemma 1.7. $(n - 2r + k - 2)n(\overline{K}_2, G) = 3n(\overline{K}_3, G) + n(W)$.

Proof: There are 2 kinds of $\overline{K}_3$ which use non-edges of G, as illustrated in Fig. 3. The number of $\overline{K}_3$'s totally contained in G is $n(\overline{K}_3, G)$ and there are $n(W)$ others. By Lemma 1.6 the above identity holds.

Lemma 1.8.

$$n(\overline{K}_2, G) = \binom{r}{2} - \frac{kr}{2} = \frac{r(s^2-1)}{2}, \quad \textit{and}$$

$$n(\overline{K}_3, G) = \frac{r}{3}\left\{\binom{s^2-1}{2} - \frac{k(s^2-1)}{2} + \binom{k}{2} - x\right\}.$$

Proof: Pick any vertex w in G (see Fig. 1). The k vertices adjacent to w induce a graph with x edges. Let the remaining $s^2 - 1$ vertices of G induce a subgraph with u edges. Then since G is k-regular, $k(k-1) - 2x = k(s^2-1) - 2u$. Solve this for u to get $u = \frac{k(s^2-1)}{2} - \binom{k}{2} + x$

So every w in G is contained in $\binom{s^2-1}{2} - u$ $\overline{K}_3$'s. Since G has r vertices, this gives the formula for $n(\overline{K}_3, G)$.

Lemma 1.9. $n(W) = \left\{\binom{s^2}{2} - \frac{ks^2}{2} + \frac{k^2}{2} - x\right\}\frac{r(s^2-1)}{k}$.

Proof: Each vertex w in Q is adjacent to k vertices of G which induce x edges, and non-adjacent to $r - k = s^2$ vertices of G which induce z edges (see Fig. 4). Since G is k-regular, $k^2 - 2x = ks^2 - 2z$, so that $z = (ks^2 - k^2)/2 + x$. So each vertex of Q is contained in $\binom{s^2}{2} - z$ $\overline{K}_3$'s contributing to $n(W)$. Since Q has $n - r - 1 = r(s^2-1)/k$ vertices, this gives the above formula.

Remark: Lemmas 1.7 and 1.9 give 2 ways of calculating $n(W)$. Equating the two formulas gives an equation which is always satisfied identically by the parameters.

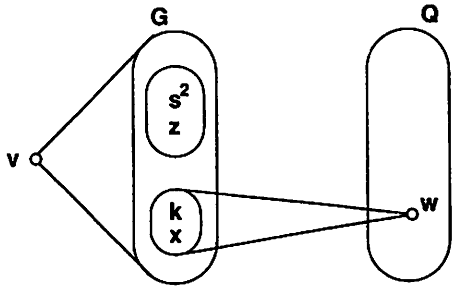

Fig. 4

Lemma 1.10. $2n(U) + 2n(W) = (n-2r+k-2).rs^2(s^2-1)/k$.

Proof: Counting the $\overline{K}_3$'s wich use non-edges of $[G,Q]$ according to the number of non-edges of $[G,Q]$ they contain gives $2n(U) + 2n(W)$. Since the number of non-edges of $[G,Q]$ is $(r-k)(n-r-1)$, we get the formula by applying Lemma 1.6.

Combining Lemmas 1.9 and 1.10 now gives a formula for $n(U)$. We state this as a theorem.

Theorem 1.11:

$$n(U) = \left\{\frac{s^2(s^2-1)}{k} - 2\right\}\frac{rs^2(s^2-1)}{2k} - \left\{\binom{s^2}{2} - \frac{ks^2}{2} + \frac{k^2}{2} - x\right\}\frac{r(s^2-1)}{k}.$$

2. An Inequality, and Counting Subgraphs.

Lemma 2.1. $2s^4(s^2-1)^2 + ks^2(s^2-1)^2 \geq 2ks^2(s^2-1) + 4k^2s^2 + k^3$.

Proof: Since $n(U)$ counts $\overline{K}_3$'s in Γ, the formula of Theorem 1.11 becomes the inequality $n(U) \geq 0$, which can then be reduced to the above expression (after some work).

We shall see later that $s^2 \leq k$. The substitution $\tau = s^2/k$ is useful; for then $0 < \tau \leq 1$ and we get a dimensionless formula. The inequality of Lemma 2.1 becomes

$$f(\tau) \equiv 2k^2\tau^4 + \tau^3(k^2-4k) - \tau^2(4k-2) - \tau(4k-3) - k \geq 0.$$

Lemma 2.2. *Let $k \geq 5$. If $\tau \leq k^{-1/3}$, then $f(\tau) < 0$.*

Proof: We first find the points of inflection of $f(\tau)$, by setting $f''(\tau) = 24k^2\tau^2 + 6\tau(k^2-4k) - 4(2k-1) = 0$. The roots are $\frac{1}{2k} - \frac{1}{8} \pm \frac{1}{8}\sqrt{1 + \frac{56}{k} - \frac{16}{k^2}}$. If the minus sign is chosen, the inflection point is clearly less than 0. Since $1+\frac{56}{k}-\frac{16}{k^2} \leq \left(1+\frac{28}{k}\right)^2$, the positive inflection point is $< 4/k$. In terms of s, this is $s^2 < 4$. It is easy to check that $f(4/k) = \frac{288}{k^2} + \frac{12}{k} - 16 - k$, which is less than 0 when $k \geq 5$. So $f(\tau)$ only begins to turn up after $\tau = 4/k$, so that there is only one real root, as depicted in Fig. 5.

We now evaluate $f(k^{-1/3}) = -2k^{2/3} - 4k^{1/3} - 4 + 3k^{-1/3} + 2k^{-2/3}$. Since this is less than 0 for all $k \geq 1$, we have $f(\tau) < 0$ for all $\tau \leq k^{-1/3}$ as required.

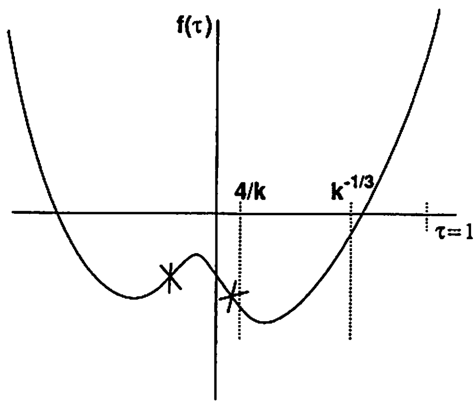

Fig. 5

So the *only* positive root of $f(\tau)$ is $> k^{-1/3}$. In terms of s, this is $s^2 > k^{2/3}$. (When k is large, the root is very near $k^{-1/3}$.) This will be useful in order to eliminate some of the possible sets of parameters for the graph Γ. It arose from counting $\overline{K}_3$ subgraphs, as depicted in Fig. 3. There are many other kinds of 3-point subgraph, according to both their position within G, and whether they have 0, 1, 2, or 3 edges. They can all be counted similarly to the methods used to find $n(U)$ and $n(W)$ in Section 1, but the inequalities arising from them are not of any help. However the following result is of some interest. Let B denote the 3-vertex induced subgraphs of Q containing exactly one edge (refer to Fig. 6).

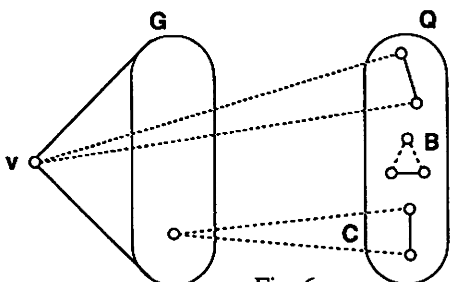

Fig. 6

Lemma 2.3. $n(B) = n(U)$.

Proof: By Fig. 2, every edge of Γ is on exactly $n - 2r + k$ B-type 3-vertex subgraphs. Since Q has $s^2(n - r - 1)/2$ edges, counting such subgraphs of Γ according to the number of edges of Q they contain gives a sum equal to $s^2(n - r - 1)(n - 2r + k)/2$. This must equal $n(B) + n(C) + \varepsilon(Q)$, where $n(C)$ counts the subgraphs illustrated in Fig. 6, since the point v is contained in $\varepsilon(Q)$ such subgraphs involving Q. Clearly $n(U) + n(C) = r\binom{n-2r+k}{2}$. This now gives $n(B) - n(U) = \frac{s^2(n-r-1)}{2}(n - 2r + k - 1) - r\binom{n-2r+k}{2}$, which reduces to $n(B) = n(U)$.

There are a number of other 3-vertex subgraph counts in Γ which turn out to be identical. They can all be proved similarly. It would be interesting to find a direct correspondence in the graph showing that the numbers must be equal.

By Lemma 1.2, we can write $A^2 = kJ + s^2 I$, where I is the identity matrix and J is the matrix of all 1's. Multiplying by A gives $A^3 = rkJ + s^2 A$. Entry $[A^3]_{vw}$ is the number of walks of length 3 connecting v to w. Since $N_\Gamma(v) \cap N_\Gamma(w)$ induces x edges, these x edges will contribute $2x$ to $[A^3]_{vw}$. Since the number of walks is determined by $rkJ + s^2 A$, one would expect this to give further information about the graph. There are evidently 2 cases. If v and w are adjacent, there are $rk + s^2$ walks; otherwise there are rk.

Let v and w be a given pair of adjacent vertices. Let X_1, X_2, X_3, and X_4 denote the 4 types of induced subgraph illustrated in Fig. 7, containing v and w as shown.

Lemma 2.4. $[A^3]_{vw} = (2r-1) + n(X_1) + n(X_2) + n(X_3) + n(X_4) = rk + s^2$, *if v and w are adjacent.*

Proof: Γ is r-regular, so there will be $2r - 1$ degenerate 3-walks connecting v to w, i.e., those in which an edge is retraced. All other walks will use 3 distinct edges, so that one of X_1, X_2, X_3, or X_4 must be induced.

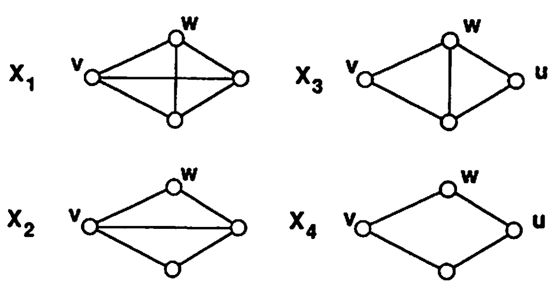

Fig. 7

We can now obtain X_1 and X_2 by counting. As noted above, $n(X_1) = 2x$. Comparing Figs. 1 and 7 shows that vertex u must be in the subgraph G, non-adjacent to w. Since $N_\Gamma(v) \cap N_\Gamma(w) \subseteq G$, and these k vertices induce x edges, there are $k(k-1)-2x$ edges in G between the vertices adjacent and non-adjacent to w. This gives $n(X_2) = k(k-1) - 2x$.

Corollary 2.5. $n(X_3) + n(X_4) = (k-1)(s^2-1)$.

Proof: This can be obtained by substituting the values for $n(X_1)$ and $n(X_2)$ into Lemma 2.4. However, the importance of this result is that it can also be obtained by direct counting. Referring to Fig. 7, we see that the vertex u must be in the subgraph Q of Fig. 1. Now $N_\Gamma(v) \cap N_\Gamma(w) \subseteq G$ has $k-1$ vertices other than w, either adjacent or non-adjacent to w. $n(X_3) + n(X_4)$ counts both these possibilities. Since w is joined to s^2-1 such vertices u, we have $n(X_3) + n(X_4) = (k-1)(s^2-1)$.

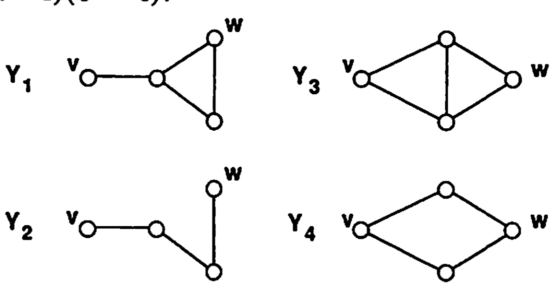

Fig. 8

If v and w are non-adjacent, then $[A^3]_{vw} = rk$. As in Lemma 2.4, we can write $[A^3]_{vw} = n(Y_1) + n(Y_2) + n(Y_3) + n(Y_4)$, where Y_1, Y_2, Y_3, and Y_4 are the types of subgraph shown in Fig. 8. A calculation similar to the above shows that $n(Y_1) = 2x, n(Y_2) = k^2 - 2x$, and that $n(Y_3) + n(Y_4) = ks^2$, which can be obtained either from $[A^3]_{vw}$ or directly, by counting.

So the use of $A^3 = rkJ + s^2A$ from the adjacency matrix algebra would not seem to give any new constraints on the parameters.

3. The Parameter Sets.

From Lemma 1.5 we have

$$2x(s^2+st-1) = s^2t^2(st-1), \text{ where } k = st, \quad s \geq 2, \text{ and } x \text{ is an integer.}$$

Lemma 3.1. $t^2 = m(s^2+st-1)$, *where* $m = 2x/s^2(st-1)$ *is an integer.*

Proof: We have $s^2 \mid 2x(s^2+st-1)$. Since $\gcd(s^2, s^2+st-1) = 1$, we have $s^2 \mid 2x$. Similarly, $\gcd(st-1, s^2+st-1) = 1$ so that $(st-1) \mid (2x/s^2)$ and the result follows.

Lemma 3.1 shows that s and t must be a solution of the diophantine equation $t^2 = m(s^2+st-1)$. In the following, we derive all solutions to this equation. We assume throughout that we are working only with integers (with a few evident exceptions).

Lemma 3.2. *s and t are a solution to the diophantine equation*

$$t^2 = m(s^2 + st - 1), \quad s \geq 2,$$

if and only if

$$t = (ms + a)/2, \text{ where } a^2 - s^2 d = -4m,\ d = m^2 + 4m, \text{ and } s \geq 2.$$

Proof: The equation $t^2 = m(s^2 + st - 1)$ can be rewritten as

$$t^2 - tms - m(s^2 - 1) = 0,$$

which is equivalent to $t = \frac{1}{2}\left(ms + \sqrt{m^2 s^2 + 4m(s^2 - 1)}\right)$, by the quadratic formula. (The minus sign of the quadratic formula can be ignored since we require $t > 0$.) Now since t is an integer, $m^2 s^2 + 4m(s^2 - 1) = a^2$, where a is a positive integer. Note that whenever the square root is integral we have $a \equiv ms \pmod 2$, so that if a is integral, so is t. The above equality can be rewritten as $a^2 - s^2 d = -4m$, where $d = (m^2 + 4m)$, and the result now follows.

The equation $a^2 - s^2 d = -4m$ is a Pellian equation which we will solve using elementary theory related to real quadratic fields. In the following paragraph we present a brief review of the relevant ideas. Readers interested in further background material should consult any standard algebraic number theory text.

Let $\delta = \sqrt{d}$ where $d > 0$ is not a perfect square, and denote the rationals by Q. Then

$$Q(\delta) \equiv \{x + y\delta \mid x, y \in Q\}$$

is a *real quadratic number field.* For each $\alpha = x + y\delta \in Q(\delta)$ the *norm* of α is $N(\alpha) \equiv x^2 - y^2 d$. It is easy to verify that $N(\alpha\beta) = N(\alpha)N(\beta)$. Let $Z \subseteq Q$ denote the set of integers. The *Z-module*

$$Z[1, \delta] \equiv \{x + y\delta \mid x, y \in Z\} \subseteq Q(\delta)$$

is an integral domain which is called an *order* of the field $Q(\delta)$. (Note that this order is not maximal for $d = m^2 + 4m$.) The elements of this order are referred to as *real quadratic integers.* Note that for each $\alpha \in Z[1, \delta], N(\alpha) \in Z$. Finally, a *unit* of an order O is an invertible element of O; that is $\alpha \in O$ is a unit of O if and only if $1/\alpha \in O$. It also follows that $\alpha \in O$ is a unit of O if and only if $N(a) = \pm 1$. Real quadratic orders have infinitely many units which can be generated as powers of a so-called *fundamental* unit.

Solving our Pellian equation is therefore equivalent to finding all $\alpha = a + s\delta \in Z[1, \delta]$ such that $N(\alpha) = -4m$. We will refer to such an α as a solution of the equation. Note that if α is a solution of the equation then so is $\eta\alpha$ where η is any unit of an order of $Q(\delta)$ such that $\eta\alpha \in Z[1, \delta]$ and $N(\eta) = 1$, since $N(\eta\alpha) = N(\eta)N(\alpha) = -4m$.

Theorem 3.3. *$N(a+s\delta) = -4m$ if and only if*

$$a + s\delta = \pm\eta^i(m+\delta), \text{ where } \eta = ((m+2)+\delta)/2 \text{ and } i \in Z.$$

Proof: If $a + s\delta$ is any solution of the equation $a^2 - s^2 d = -4m$ then

$$\alpha = (a+s\delta)\eta = ([a(m+2)+sd] + [a+s(m+2)]\delta)/2.$$

Now $d = m^2 + 4m$ and since $a^2 - s^2 d = -4m$ we have $a \equiv s \pmod 2$ if d is odd, and $a \equiv 0 \pmod 2$ if d is even. These results give

$$a(m+2) + sd \equiv 0 \pmod 2, \text{ and } a + s(m+2) \equiv 0 \pmod 2$$

and so $\alpha \in Z[1,\delta]$. It is easy to see that $N(\eta) = 1$, so that $N(\alpha) = N(a + s\delta)N(\eta) = -4m$, and so α is also a solution of the Pellian equation. A similar argument with $\eta^{-1} = ((m+2)-\delta)/2$ shows that $(a+s\delta)\eta^{-1}$ is also a solution. It therefore follows that $\pm\eta^i(m+\delta)$ is a solution for all $i \in Z$, since $m + \delta$ is obviously a solution.

It remains to show that every solution of the Pellian equation must be of this form. Let $a + s\delta$ be any solution of the equation with $|s| \geq 2$ (a and s may be negative). Notice that the 4 units $(\pm(m+2) \pm \delta)/2$ are just $\pm\eta^{\pm 1}$. Suppose first that a and s are positive. Let $\mu_1 = (-(m+2)+d)/2$ and let

$$a_1 + s_1\delta = \mu_1(a+s\delta) = ([-a(m+2)+sd] + [a - s(m+2)]\delta)/2.$$

Then

$$\begin{aligned} |s_1| &= |[a - s(m+2)]/2| = |[a - s\delta + s\delta - s(m+2)]/2| \\ &= |(-4m/[a+s\delta] - s[(m+2)-\delta])/2|, \text{ since } a^2 - s^2 d = -4m. \end{aligned}$$

Now $\delta^2 = m^2 + 4m$, so $m + 1 < \delta < m + 2$. Therefore $s[(m+2) - \delta]/2 < s/2$. Also, since $s \geq 2$, we have $4m/[a+s\delta] \leq 4m/2\delta < 2$. So $|s_1| < 1 + s/2 < s$. If, on the other hand, a or s (or both) had been negative, then choosing another of $\pm\eta^{\pm 1}$ as μ_1 to get $a_1 + s_1\delta = \mu_1(a+s\delta)$ makes $|s_1| < |s|$, since we can always choose μ_1 to make $s_1 = [|a| - |s|(m+2)]/2$, as above. We can now choose μ_2 so that $a_2 + s_2\delta = \mu_2(a_1 + s_1\delta)$ has $|s_2| < |s_1|$, etc.

It follows that we can find $\mu = \mu_1\mu_2\ldots\mu_i = \pm\eta^j$, such that $a_i + s_i\delta = \mu(a+s\delta)$ is a solution of the Pellian equation with $|s_i| \leq 1$. Hence $a_i + s_i\delta$ is one of $\pm m \pm \delta$. Since $(m+\delta)\eta^{-1} = -m + \delta$, multiplying by one of ± 1 or $\pm\eta^{-1}$ will bring $a_i + s_i\delta$ into the form $m + \delta$. So every solution of the Pellian equation can be obtained from the solution $m + \delta$ by multiplying by some power of η, and ± 1.

Remark: Note that $\eta = ((m+2)+\delta)/2$ is a unit of the order $Z[1,(m+\delta)/2]$ but not of $Z[1,\delta]$; however $\eta^2 \in Z[1,\delta]$ if m is even, and $\eta^3 \in Z[1,\delta]$ if m is odd.

Corollary 3.4. *The positive solutions of the diophantine equation* $t^2 = m(s^2 + st - 1)$, $s \geq 2$ *are given by*

$t = (ms + a)/2$, *where* $a + s\delta = \eta^i(m + d)$, $\eta = [(m+2) + \delta]/2$, *and* $i > 0$.

Proof: We must select from the result in Theorem 3.3 the positive solutions with $s \geq 2$. Note that this requires that $a + s\delta \geq 2\delta > m + \delta$. Since $\eta = [(m+2) + \delta]/2 > 1$, it is easy to see that we must choose the plus sign in $\pm\eta^i(m + \delta)$ and take $i > 0$.

We can now write down all the parameters s, t, k, x, etc. of the strongly regular graphs Γ in terms of m, δ, and η. Let $m \geq 1$ be arbitrary but fixed.

Theorem 3.5. *Let* $m \geq 1$, $\delta^2 = m^2 + 4m$, *and* $\eta = [(m+2) + \delta]/2$. *Define*

$$a_i + s_i\delta = \eta^i(m + d), \quad \text{and} \quad t_i = (ms_i + a_i)/2, \quad \text{for } i \geq 0.$$

Then $a_i = m(\eta + 1)(\eta^i - \eta^{-i-1})/\delta$, $s_i = (\eta - 1)(\eta^i + \eta^{-i-1})/\delta$, *and* $t_i = m(\eta^{i+1} - \eta^{-i-1})/\delta$.

Proof: This is most easily proved with generating functions. Introduce an indeterminate z, and sum the above formula for $a_i + s_i\delta$ to get

$$\sum_{i\geq 0}(a_i + s_i\delta)z^i = (m+\delta)\sum_{i\geq 0}(\eta z)^i = \frac{(m+\delta)}{1 - \eta z}$$

Now re-arrange the right hand side to separate the terms involving δ.

$$\frac{(m+\delta)}{1-\eta z} = \frac{(m+\delta)(1-\eta^{-1}z)}{(1-\eta z)(1-\eta^{-1}z)}$$

Comparing this with the above summation shows that

$$\sum_{i\geq 0} a_i z^i = \frac{m(1+z)}{1-(m+2)z+z^2} \text{ and } \sum_{i\geq 0} s_i z^i = \frac{(1-z)}{1-(m+2)z+z^2}$$

To extract the terms a_i and s_i use partial fractions :

$$\frac{1}{1-(m+2)z+z^2} = \frac{1}{(1-\eta z)(1-\eta^{-1}z)} = \frac{1}{\delta\eta^{-1}(1-\eta z)} - \frac{1}{\delta\eta(1-\eta^{-1}z)}$$

The coefficient of z^i in this power series is evidently $[\eta^{i+1} - \eta^{-i-1}]/\delta$. We can now use the summations for a_i and s_i to write down

$$a_i = m(\eta+1)(\eta^i - \eta^{-i-1})/\delta \text{ and } s_i = (\eta - 1)(\eta^i + \eta^{-i-1})/\delta.$$

The formula for t_i then gives

$$t_i = m(\eta^{i+1} - \eta^{-i-1})/\delta.$$

Notice that δ is very nearly $m+2$; in fact $(m+2) - 2/(m+2) < \delta < m+2$, so that $\eta = [(m+2) + \delta]/2$ satisfies $(m+2) - 1/(m+2) < \eta < m+2$. Since $m \geq 1$, we can ignore the terms η^{-i} when i is large to give asymptotic values for a_i, s_i, t_i and $k_i = s_i t_i$:

$$\begin{aligned}
a_i &\approx m(\eta + 1)\eta^i/\delta \approx m(m+3)(m+2)^{i-1}, \\
s_i &\approx (\eta - 1)\eta^i/\delta \approx (m+1)(m+2)^{i-1}, \\
t_i &\approx m\eta^{i+1}/\delta \approx m(m+2)^i, \\
k_i &\approx m(m+1)(m+2)^{2i-1}.
\end{aligned}$$

We now consider for which of these solutions a strongly regular graph Γ may exist. The parameters r, n, and x of Γ can now be calculated. By Lemma 3.1, $x = ms^2(st-1)/2$. That this must be an integer is not necessarily guaranteed by the solution in terms of m. Lemma 3.6 determines when this is so.

Lemma 3.6. *If m is odd and $i \equiv 2 \pmod 3$ then $x = ms_i^2(s_i t_i - 1)/2$ is not an integer; in all other cases x is an integer.*

Proof: Since $ms_i^2(s_i t_i - 1)$ is odd if and only if m and s_i are odd, and t_i is even, we see that x will be an integer if and only if we do not have m and s_i odd or t_i even. We therefore must show that this is equivalent to $i \equiv 2 \pmod 3$.

Assume that m is odd. It is easy to check that $\eta^3 = (m^3 + 6m^2 + 9m + 2)/2 + (m^2 + 4m + 3)\delta/2$, both terms of which are integers when m is odd. So $\eta^{3i} = b + c\delta$ where b and c are integers. Furthermore since $b^2 - c^2 d = 1$ and $d = m^2 + 4m \equiv 5 \pmod 8$, we see that $b \equiv 1 \pmod 2$ and $c \equiv 0 \pmod 4$. Now

$$a_{3i} + s_{3i}\delta = (m+\delta)(b + c\delta) = (mb + cd) + (mc + b)\delta.$$

Thus $t_{3i} = (ms_{3i} + a_{3i})/2 = mb + c(m^2 + 2m)$ is odd since m and b are odd, and c is even. We also have

$$a_{3i+1} + s_{3i+1}\delta = (a_{3i} + s_{3i}\delta)\eta = [(mb + cd) + (mc + b)\delta][(m+2) + \delta]/2$$

which gives $s_{3i+1} = (m+1)b + c(m^2 + 3m)$ which is clearly even, so x will be an integer. Finally

$$a_{3i-1} + s_{3i-1}\delta = (a_{3i} + s_{3i}\delta)\eta^{-1} = [(mb + cd) + (mc + b)\delta][(m+2) - \delta]/2$$

from which we get

$$a_{3i-1} = -bm + cd, \quad s_{3i-1} = b - cm, \quad t_{3i-1} = (ms_{3i-1} + a_{3i-1})/2 = 2cm.$$

This makes m and s odd, but t even, so that x is not an integer when $i \equiv 2 \pmod 3$, but is integral in all other cases.

We now have 3 constraints on the parameters :

(1) they form a solution to the Pellian equation;

(2) $x = ms^2(st-1)/2$ must be an integer, i.e., $i \not\equiv 2 \pmod 3$ when m is odd;

(3) the inequality of Lemma 2.1, $f(\tau) \geq 0$, must be satisfied.

Since $\tau = s^2/k = s/t$, write $\tau_i = s_i/t_i$. The first solution (for any given m) is $a_1 + s_1\delta = \eta(m+\delta) = (m^2+3m) + (m+1)\delta$. Then $t_1 = m^2 + 2m$, $k_1 = m(m+1)(m+2)$ and $\tau_1 = (m+1)/m(m+2)$.

Strongly regular graphs must also satisfy the *Krein* conditions (see [3]). In terms of the present notation these are the following 2 inequalities, since Γ has eigenvalues $\pm s$:

I. $(s+1)(k+s-s^2) \leq (s^2+k+s)(s-1)^2$

II. $(s-1)(s^2+s-k) \leq (s^2+k-s)(s+1)^2$

Lemma 3.7. *The Krein conditions are always satisfied by the solutions to the Pellian equation.*

Proof: Substituting $k = st$ into I and multiplying it out reduces it to $3t \leq s^2 + st - 1$. By Lemma 3.2, this gives $3m \leq t$. Since $t_1 = m^2 + 2m$ and $m \geq 1$, we see that I is always satisfied, since t_i increases with i. Substituting $k = st$ into II reduces it to $s^2 + st + 3t \geq 1$. Since $s \geq 2$ this is also always satisfied.

If we now substitute the values of k_1 and t_1 into the inequality $f(\tau) \geq 0$ we have :

Lemma 3.8. $f(\tau_1) = -2(m+1)(m+2) = -2k_1/m$.

Proof: The calculation is tedious but straightforward.

So s_1, t_1, k_1 is never an acceptable set of parameters. When m is odd, neither is s_2, t_2, k_2, by Lemma 3.6. Because of the exponential form of the solution, this means that the acceptable solutions will be "large".

Lemma 3.9. *Let $m \geq 1$ be given, and let $i \geq 1$. Then $\frac{1}{m+1} \leq \tau_i \leq \frac{1}{m+\frac{1}{2}}$, and the asymptotic value of τ ($i \to \infty$) is $(\eta-1)/m\eta$.*

Proof: By Theorem 3.5,

$$\tau_i = \frac{(\eta-1)[1+\eta^{-2i-1}]}{m\eta[1-\eta^{-2i-2}]}$$

which gives the asymptotic value, since $\eta \geq 1$.

By Lemma 3.2, $a_i^2 = m^2s_i^2 + 4m(s_i^2-1) = s_i^2(m^2+4m-4m/s_i^2)$. Since $s_i \geq 2$ and $m \geq 1$, this gives $s_i^2(m+1)^2 \leq a_i^2 < s_i^2(m+2)^2$. From $t_i = (ms_i + a_i)/2$, we then get $(m+1/2)s_i \leq t_i < (m+1)s_i$; in other words, $\frac{1}{m+1} < \tau_i \leq \frac{1}{m+\frac{1}{2}}$.

Corollary 3.10. $f(\tau_i) > 0$ *for all* $i \geq 2$ *and all* $m \geq 1$.

Proof: Discarding the terms $2\tau^2 + 3t$ from $f(\tau)$ (see Lemma 2.1) leaves

$$f(\tau)/k > 2k\tau^4 + \tau^3(k-4) - 4\tau^2 - 4\tau - 1.$$

By Corollary 3.4, $s_2 = m^2 + 3m + 1, t_2 = m(m+1)(m+3)$, and $k_2 = s_2 t_2$. Since $k_i \geq k_2$, for $i \geq 2$, then using the bounds of Lemma 3.8 for τ_i gives

$$\begin{aligned} f(\tau_i)/k_i > & \frac{2m(m+3)(m^2+3m+1)}{(m+1)^3} + \frac{m(m+3)(m^2+3m+1)}{(m+1)^2} \\ & -4\left\{\frac{1}{(m+1)^3} + \frac{1}{(m+\frac{1}{2})^2} + \frac{1}{(m+\frac{1}{2})}\right\} - 1 \\ > & \, 2m + (m^2+3m+1) - 4\{1/8 + 4/9 + 2/3\} - 1 > 0, \end{aligned}$$

since $m \geq 1$.

Hence, all solutions s_i, t_i of the Pellian equation also satisfy the inequality when $i \geq 2$. As we shall see, these tend to be large solutions.

The following table shows, in order of increasing k, the first 5 acceptable solutions; that is, those not eliminated by the inequality $f(\tau) \geq 0$, or by the divisibility condition $2\,|ms^2(st-1)$.

m	s	t	k	r	x	n	remark
1	13	21	273	442	22984	715	not possible
2	11	30	330	451	39809	616	not possible
1	34	55	1870	3026	1080282	4896	?
4	29	140	4060	4901	6827238	5916	?
2	41	112	4592	6273	big	8569	not possible

This comprises all solutions with $k \leq 20000$ not eliminated by these two conditions. The first 2 parameter sets, $k = 273$ and $k = 330$, give a non-integral value for $n(\overline{K}_3, G)$ from Lemma 1.9, so that they are not possible. The solution with $k = 4592$ requires that the subgraph Q of Γ be 613-regular, with 2295 vertices, which rules it out. The two parameter sets marked by "?" are still unsettled. So if there are any strongly regular graphs Γ satisfying these properties, they must be large. The smallest might have 4896 vertices with $k = 1870$, or 5916 vertices with $k = 4060$. After this we need $k > 20000$. There are an infinite number of parameter sets arising as solutions of the Pellian equation, which satisfy the inequality $f(\tau) \geq 0$, but they tend to become sparser and sparser as k increases. We would like to prove that, like the friendship theorem, there are no graphs at all which satisfy these properties. This would seem to require the discovery of a new constraint other than the counting and algebraic constraints that we have used.

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