The Binding Number of Trees and K(1,3)-free Graphs

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Abstract. The binding number of a graph G is defined to be the minimum of $|N(S)|/|S|$ taken over all nonempty $S \subseteq V(G)$ such that $N(S) \neq V(G)$. In this paper another look is taken at the basic properties of the binding number. Several bounds are established, including ones linking the binding number of a tree to the "distribution" of its end-vertices. Further, it is established that under some simple conditions, $K(1,3)$-free graphs have binding number equal to $(p(G)-1)/(p(G)-\delta(G))$ and applications of this are considered.

1. Introduction and Background

The concept of the binding number of a graph was introduced by Woodall [6] in 1973. It was an attempt to measure how "well-distributed" the edges of a graph are. Thus far, the results that have been obtained can largely be grouped into three broad families: firstly, basic or general results including some bounds and computational short-cuts; secondly, the binding numbers of some specific graphs or families thereof, and thirdly, conditions on the binding number which, together with other (simple) conditions, guarantee the presence of a required subgraph.

It is our aim to further explore the first two of these areas. Firstly, we consider the binding numbers of trees and forests; here we show that the binding number is directly related to how well-distributed the leaf edges are. Secondly, we examine those graphs whose binding number is $(p(G)-1)/(p(G)-\delta(G))$. For $K(1,3)$-free graphs, we give necessary and sufficient conditions for them to have this binding number. (A $K(1,3)$-free graph is one which does not contain $K(1,3)$ as an induced subgraph.) Thus, for example, ad hoc results on line graphs first proved in [5] follow readily.

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In this paper we consider only finite undirected graphs without loops or multiple edges. We shall use $p(G)$ to denote the number of vertices or *order* of a graph G, and the term *cut-set* to mean any set of vertices whose removal leaves a disconnected graph. Further, $N(S)$ denotes the neighborhood of a set S, while $N[S]$ denotes its closed neighborhood (i.e. $S \cup N(S)$). For other definitions not given here, see [2].

Define:

- ▷ For all graphs G, $\mathcal{F}(G) := \{ S : \emptyset \neq S \subseteq V(G) \,\&\, N(S) \neq V(G) \}$.
- ▷ The *binding number* of G, denoted by bind(G), is defined by

$$\text{bind}(G) := \min_{S \in \mathcal{F}(G)} \frac{|N(S)|}{|S|}.$$

Now, a *binding set* of G is any set $S \in \mathcal{F}(G)$ such that $\text{bind}(G) = |N(S)|/|S|$. Further, for all $S \subseteq V(G)$, the *excess* of S, denoted by exc(S), is given by $\text{exc}(S) = |N(S)| - |S|$.

We list some useful results.

Proposition 1. [3]
a) If G is a spanning subgraph of H then $\text{bind}(G) \leq \text{bind}(H)$.
b) $\text{bind}(\bigcup_{i=1}^{r} G_i) = \min\{\text{bind}(G_1), \ldots, \text{bind}(G_r), 1\}$.

Proposition 2. [6] *For all graphs G,* $\text{bind}(G) \leq (p(G)-1)/(p(G)-\delta(G))$.

Proposition 3. [6]
a) For $n \geq 3$, $\text{bind}(C_n) = 1$ *if n is even, and $(n-1)/(n-2)$ if n is odd.*
b) For $n \geq 1$, $\text{bind}(P_n) = 1$ *if n is even, and $(n-1)/(n+1)$ if n is odd.*

Proposition 4. *If G is a complete multipartite graph of order p and independence number β then* $\text{bind}(G) = (p-\beta)/\beta$.

Proposition 5. [4] *For all graphs G, if* $\text{bind}(G) < 1$ *then every binding set of G is independent.*

2. Trees and Forests

In this section, we prove some bounds on the binding number of a graph. We consider mainly the binding number of a tree or forest and show that this is directly related to the distribution of its end-vertices.

The first result relates binding number to the minimum and maximum degrees.

Theorem 1.
a) For all nonempty graphs G, $\text{bind}(G) \geq \delta(G)/\Delta(G)$.
b) If G has an r-factor for any $r \geq 1$ *then* $\text{bind}(G) \geq 1$.

Proof: a) If $\text{bind}(G) \geq 1$ then the statement is trivially true. Therefore, we may assume that $\text{bind}(G) < 1$. Let S be a binding set of G; then, by Proposition 5, S is independent. Let $M = \{uv \in E(G) : u \in S\}$. Then $|M| \leq \Delta(G) \cdot |N(S)|$ while since S is independent, $|M| \geq \delta(G) \cdot |S|$. Consequently, $\text{bind}(G) = |N(S)|/|S| \geq \delta(G)/\Delta(G)$.
b) This follows from part (a) and Proposition 1a. ∎

Equality is attained in the theorem for even cycles and (the relevant) complete bipartite graphs, inter alia.

Now, we define $\text{end}(G) = \{v \in V(G) : \deg v = 1\}$. Also, let $\text{ed}(v) = |N(v) \cap \text{end}(G)|$ and $\text{ed}(G) = \max\{\text{ed}(v) : v \in V(G)\}$. The following result is obviously true if $\text{ed}(G) = 0$, and, if $\text{ed}(G) \geq 1$, follows from taking $S = N(v) \cap \text{end}(G)$, where v is a vertex for which $\text{ed}(v) = \text{ed}(G)$.

Theorem 2. *For all graphs G,* $\text{ed}(G) \cdot \text{bind}(G) \leq 1$.

Theorem 1 shows that $\text{bind}(T) \geq 1/\Delta(T)$ for a tree T. The following provides a stronger (in general) lower bound.

Theorem 3. *For all non-trivial trees T,* $\text{bind}(T) > 1/(e+1)$ *where* $e = \text{ed}(T)$.

Proof: Let S be a binding set of T and let $x = |S - \text{end}(T)|$ and $y = |S \cap \text{end}(T)|$. Now, if $x = 0$ then $|N(S)|/|S| \geq 1/e$; hence we may assume that $x > 0$ so that $|N(S)| \geq x+1$. Further, if $v \in N(S)$ then $N(v) \cap \text{end}(T) \subseteq S$. Let r denote the number of vertices v in $N(S)$ such that $\text{ed}(v) = e$. Then $|N(S)| \geq r + (y - re)/(e-1)$ and $re \leq y$. Denoting

$$f_y(x) := \frac{\max\{x+1,\, r + (y-re)/(e-1)\}}{x+y},$$

we have that $|N(S)|/|S|$ is at least as large as the minimum of $f_y(x)$ taken over y and x. Now, $f_y(x)$ is minimised at $x + 1 = r + (y - re)/(e-1)$ and hence

$$\begin{aligned} \frac{|N(S)|}{|S|} &\geq \frac{r + (y-re)/(e-1)}{r + (y-re)/(e-1) - 1 + y} \\ &= \frac{y-r}{ey - r - e + 1} \end{aligned} \qquad (1)$$

$$\begin{aligned} &\geq \frac{y - y/e}{ey - (y/e) - e + 1} \\ &= \frac{y}{(e+1)y - e} \\ &> \frac{1}{e+1} \end{aligned}$$

and the result is proved. ∎

Now, a *caterpillar* is a tree T such that $T - \text{end}(T)$ is a path (called the spine). To see that the above theorem is sharp, let T be a caterpillar with $2a - 1$ vertices on its spine consecutively adjacent to $e, 0, e, 0, \ldots, e$ end-vertices ($a, e \geq 1$). Then T has binding number $a/(ae + a - 1)$ which tends to $1/(e+1)$ as $a \to \infty$. The above proof also yields as a corollary:

Theorem 4. *Let T be a non-trivial tree with a unique vertex v such that* $\text{ed}(v) = e = \text{ed}(T)$. *Then* $\text{bind}(T) = 1/e$.

Proof: That $1/e$ is an upper bound on the binding number follows from Theorem 2. That $1/e$ is a lower bound follows from (1) above, since $r \leq 1$ and so $|N(S)|/|S| \geq (y-1)/(ey-e)$. ∎

Examples of such trees are the comets first considered in [1]. Now, as a consequence of Theorems 2 and 3 and the fact that for a forest F, $\text{ed}(F)$ is the maximum value of ed over the components and $\text{bind}(F)$ is the minimum of the binding numbers of the components (by Proposition 1b), we have the following:

Theorem 5. *For all nonempty forests F,* $1/(e+1) < \text{bind}(F) \leq 1/e$ *where* $e = \text{ed}(F)$.

3. Graphs with $bind(G) = (p(G) - 1)/(p(G) - \delta(G))$

Here we consider those graphs G which have binding number equal to $\text{bind}(G) = (p(G) - 1)/(p(G) - \delta(G))$. (Recall that this value is an upper bound for the binding number by Proposition 2.) Many graphs, especially most of the specific product graphs that have been considered, have been shown to be in this class.

Indeed, we consider graphs with $\delta(G) \geq 1$ and define a hierarchy of properties:

E1: $\text{bind}(G) = (p(G) - 1)/(p(G) - \delta(G))$.

E2: For all $S \in \mathcal{F}(G)$ it holds that $\text{exc}(S) \geq \delta(G) - 1$.

E3: Property **E2** holds, with equality iff $S = \{v\}$ or $S = V(G) - N(v)$ where v is a vertex of degree $\delta(G)$.

The property **E2** was first considered by Saito and Tian [5]. Then the hierarchy is given by the following result.

Theorem 6.
a) If $\delta(G) \geq 1$ *then* **E3** $\Longrightarrow$ **E2** *and* **E2** $\Longrightarrow$ **E1**.
b) None of the converse implications holds.

Part (a) is obvious. Further, the noncomplete odd cycles satisfy **E2** but not **E3**. Saito and Tian [5] gave graphs which showed that **E1** does not imply **E2**. Another family of examples is given by $K_{b^2-2b} + 2K_b$ for $b \geq 3$ (where "+" denotes the join).

The complete graphs satisfy all the properties while noncomplete paths of even order and noncomplete cycles of odd order satisfy **E2** and **E1** only. However, those paths of odd order, complete multipartite graphs and even cycles which are not complete graphs satisfy none of these properties.

One may also consider the property:

F: Every binding set of G is of the form $V(G) - N(v)$ for some $v \in V(G)$.

Now, it is easily seen that **E2** implies **F** provided $\delta(G) \geq 2$. However, those complete multipartite graphs which are not complete graphs satisfy **F** but not **E1**. Further the graphs $K_{b^2-2b} + 2K_b$ for $b \geq 2$ satisfy **E1** but not **F**.

The next result gives some necessary conditions, in terms of the minimum degree and connectivity, for the above properties to hold.

Theorem 7.
a) [5] *If G satisfies* **E1** *and* $\delta(G) \geq 2$, *then* $\kappa(G) \geq \delta(G)/2$.
b) If G satisfies **E2** *then* $\kappa(G) \geq \delta(G) - 1$ *and* $p(G) \neq \delta(G) + 2$.
c) If G satisfies **E3** *then* $\kappa(G) = \delta(G)$ *and* $p(G) \neq \delta(G) + 2, \delta(G) + 3$.

Proof: We prove only parts (b) and (c). The result is obvious for complete graphs, so assume that G is not complete. In each case, the connectivity condition is necessary. To see this, take X to be a cut-set of G, and let $S = V(H)$ where H is a component of $G - X$; thus $\text{exc}(S) \leq (|X| + |S|) - |S|$. Hence in (b), $|X| \geq \delta(G) - 1$. Further in (c), if $|S| = 1$ then $|X| \geq \delta(G)$, while if $|S| \geq 2$ then S is not of the form $V(G) - N(v)$ where v is a vertex of minimum degree, so that again $|X| \geq \delta(G)$.

To prove the remainder, let T consist of two independent vertices. Then $\mathrm{exc}(T) \leq (p(G)-2)-2$. This yields the remaining necessary conditions in (b) and (c) noting that if $\mathrm{exc}(T)$ does equal $\delta(G)-1$, then T cannot be of the form $V(G)-N(v)$ for a vertex v of degree $\delta(G)$. ∎

The next theorem gives simple sufficient conditions for a $K(1,3)$-free graph to satisfy **E2** or **E3**.

Theorem 8. *Let G be a $K(1,3)$-free graph of order p and minimum degree $\delta \geq 3$.*
a) If $\kappa(G) \geq \delta - 1$ and $p \neq \delta + 2$ then G satisfies **E2**.
b) If $\kappa(G) = \delta$ and $p \neq \delta+2$, $p \neq \delta+3$, and $(\delta, p) \notin \{(3,8);(3,10);(4,9)\}$, then G satisfies **E3**.

Proof: We shall prove both cases simultaneously. Let $S \in \mathcal{F}(G)$. We wish to show that if the conditions of (a) hold then $\mathrm{exc}(S) \geq \delta - 1$, and that if the conditions of (b) hold then $\mathrm{exc}(S) \geq \delta$ unless S is of the form $\{v\}$ or of the form $V(G)-N(v)$ for some $v \in V(G)$ of degree δ.

Let S' be the set of isolated vertices of the induced graph $\langle S \rangle$. If $S' = \emptyset$ (so that $S \subseteq N(S)$), then it follows from $S \in \mathcal{F}(G)$ that $N(S)-S$ is a cut-set of G, and thus $\mathrm{exc}(S) = |N(S)-S| \geq \kappa(G)$. Therefore, we may assume that $S' \neq \emptyset$. Note that $\mathrm{exc}(S) \geq \mathrm{exc}(S')$.

Now, let $N^* = \{ v \in N(S') : |N(v) \cap S'| = 1 \}$. Since G is $K(1,3)$-free, it holds for all $v \in N(S')$ that $|N(v) \cap S'| \leq 2$ and for all $v \in N(S') - N^*$ that $N(v) \subseteq N[S']$. Hence, *either* N^* is a cut-set of G *or* $N[S']$ is the whole of $V(G)$.

Now, the number m of edges of G joining S' to $N(S')$ satisfies

$$\begin{aligned} \delta|S'| &\leq m = |N^*| + 2|N(S') - N^*| \\ &= 2\,\mathrm{exc}(S') + 2|S'| - |N^*|. \end{aligned} \tag{2}$$

Hence if $\mathrm{exc}(S') \leq \delta - 2$ then $|S'| \leq 2$, and, in fact, $|S'| = 2$, $N^* = \emptyset$ and $|N(S')| = \delta$. Since G is connected, N^* is not a cut-set of G, and so $N[S'] = V(G)$ and $p = \delta + 2$. This completes the proof of part (a).

So let us assume from now on that $\mathrm{exc}(S') = \mathrm{exc}(S) = \delta - 1$ and that $\kappa(G) = \delta \geq 3$. If $|S'| \geq 2$, then (2) gives $|N^*| \leq 2$, so that N^* is not a cut-set of G. Therefore, $N[S']$ is the whole of $V(G)$ and $p = 2|S'| + \mathrm{exc}(S') \geq \delta + 3$. Moreover, (2) gives

$$|S'| \leq 2(\delta-1)/(\delta-2),$$

so that $|S'| = 2$ and $p = \delta + 3$, unless $(\delta, p) \in \{(3,8);(3,10);(4,9)\}$.

It remains to consider the case when $|S'| = 1$, say $S' = \{v\}$, where $\deg v = \delta$ (since $\mathrm{exc}(S') = \delta - 1$). If $S' = S$ then $S = \{v\}$ which is allowed, so suppose $S' \neq S$. Then $N(S - S') - S \subseteq N(S')$ (since we are assuming that $\mathrm{exc}(S) = \mathrm{exc}(S')$), and indeed, as $N(S - S') - S$ is a cut-set of G, it holds that $|N(S - S') - S| \geq \delta$ and therefore $N(S - S') - S = N(S')$. Now *suppose* that $N[S] \neq V(G)$. Then, as G is connected, there exists an $x \in V(G) - N[S]$ adjacent to some $y \in N(S')$. But then y has three independent neighbours (v, x and one in $S - S'$) contradicting the requirement that G be $K(1,3)$-free. Hence $N[S] = V(G)$ and thus $S = V(G) - N(v)$, where v is a vertex of minimum degree. This concludes the proof of the theorem. ∎

We note that the last condition of (b) is not necessary; consider for example the graphs $K_\delta + (K_1 \cup K_{p-\delta-1})$. However, there do exist $K(1,3)$-free graphs G of the requisite order which do not satisfy **E3**. For $\delta = 3$ and $p = 8$, take $G = (K_3 \times K_3) - v$ (where "$\times$" denotes the cartesian product); for $\delta = 4$ and $p = 9$, take $G = K_3 \times K_3$; and for $\delta = 3$ and $p = 10$, take $G = L(K_5) - E(H)$ where H is a subgraph of $L(K_5)$ isomorphic to K_4. Nevertheless, from the above two theorems one may obtain the following result.

Theorem 9. *Let G be a $K(1,3)$-free graph of order p and minimum degree $\delta \geq 3$.*
a) G satisfies **E2** *iff $\kappa(G) \geq \delta - 1$ and $p \neq \delta + 2$.*
b) If $(\delta, p) \notin \{(3,8);(3,10);(4,9)\}$, then G satisfies **E3** *iff $\kappa(G) = \delta$ and $p \neq \delta + 2$, $\delta + 3$.*

As a direct application of the theorem follow some nice results. For example, the ad hoc results on specific line graphs and some of the results on total graphs in [5] are direct consequences of this theorem. While total graphs are not in general $K(1,3)$-free, we may add the following:

Theorem 10. *The total graphs $T(C_n)$ for $n \geq 4$ and $T(P_n)$ for $n \geq 1$ satisfy* **E2**.

Proof: The result for cycles follows immediately from the above theorem. Further, it holds that $T(P_n) \cong P^2_{2n-1}$. If $n > 1$ then this graph is hamiltonian and therefore a supergraph of C_{2n-1}. As $\delta(T(P_n)) = \delta(C_{2n-1})$ and C_{2n-1} satisfies **E2**, the result follows. ∎

Further, powers of cycles are $K(1,3)$-free. While the values of the binding numbers for such graphs were determined in [1], we may state the following result which follows directly from Theorem 9 and the observation that $\kappa(C^p_a) = 2a$ (provided $2a < p$).

Theorem 11.
a) If $2 \leq a \leq (p-4)/2$ *then* C_p^a *satisfies* **E3** *unless* $a = 2$ *and* $p = 9$.
b) If $2 \leq a \leq (p-3)/2$ *or if* $a = 1$ *and* p *is odd then* C_p^a *satisfies* **E2**.

This is itself useful; for example, the Harary graphs $H_{m,n}$ for odd m are formed from $C_n^{(m-1)/2}$ by the addition of edges. The following lemma shows that such graphs satisfy **E2**.

Lemma 12. *Let* G *be a graph which satisfies* **E3**. *If* G' *is a supergraph of* G *with* $V(G') = V(G)$ *and* $\delta(G') = \delta(G) + 1$ *then* G' *satisfies* **E2**.

Proof: The only candidates for sets S which have excess in G' less then $\delta(G') - 1$ and which lie in $\mathcal{F}(G')$, are of the form $\{v\}$ or $V(G) - N_G(v)$ where v is a vertex of G with $\deg_G v = \delta(G)$. However, the excess in G' of a singleton is always at least $\delta(G') - 1$, while the sets S of the latter case have $v \in N_{G'}(S)$ and thus $N_{G'}(S) = V(G')$. ∎

We were unable to determine the exact analogue of Theorem 9 for the property **E1**. By Theorem 7a, a necessary condition is that $\kappa(G) \geq \delta(G)/2$. However, the graphs $K_{b^2-2b-1} + 2K_b$ show that there is no constant $c < 1$ such that $\kappa \geq c \cdot \delta$ is sufficient.

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