Decompositions of Hypergraphs into Delta-systems and Constellations

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Abstract. A partition of the edge set of a hypergraph H into subsets inducing hypergraphs $H_1, \dots, H_r$ is said to be a *decomposition of H into* $H_1, \dots, H_r$. A uniform hypergraph $F = (\bigcup \mathcal{F}, \mathcal{F})$ is a Δ*-system* if there is a set $K \subseteq V(F)$, called the *kernel of* F, such that $A \cap B = K$ for every $A, B \in \mathcal{F}, A \neq B$. A disjoint union of Δ-systems whose kernels have the same cardinality is said to be a *constellation*. In the paper, we find sufficient conditions for existence of a decomposition of a hypergraph H into:

a) Δ-systems having almost equal sizes and kernels of the same cardinality,
b) isomorphic copies of constellations such that the sizes of their components are relatively prime.

In both cases, the sufficient conditions are satisfied by a wide class of hypergraphs H.

1. Introduction

In general, we follow the terminology of [3]. For a hypergraph H we denote by $V(H)$, $E(H)$ and $e(H)$ the set of vertices, the set of edges and the size of H, respectively. By the *degree* $\deg_H x$ of a vertex $x \in V(H)$ we mean the number of edges that contain x. Let $\Delta(H)$ and $\delta(H)$ stand for the maximum degree and the minimum degree of vertices in H, respectively. By $G \dot{\cup} H$ we mean the disjoint union of hypergraphs G and H and by nH the disjoint union of n copies of H. For every integer $k \geq 2$, a hypergraph with k-element edges only is called *k-uniform*. Finally, K_n^k and $K_{1,n}$ denote a complete k-uniform hypergraph of order n and a star of size n, respectively.

A *decomposition* of a hypergraph H into hypergraphs $H_1, \dots, H_r$ is a partition of the set $E(H)$ into nonempty subsets $E_1, \dots, E_r$ such that $(\bigcup E_i, E_i) = H_i$, for $i = 1, \dots, r$. Let $\mathcal{H}$ be a family of hypergraphs. A decomposition of H into $H_1, \dots, H_r$ is said to be an $\mathcal{H}$-decomposition if every hypergraph $H_i, i = 1, \dots, r$, is isomorphic to a hypergraph in $\mathcal{H}$. If $\mathcal{H} = \{F\}$ we write 'F-decomposition' instead of '$\{F\}$-decomposition'.

The decompositions of hypergraphs were mostly considered in the case of graphs (see Bermond and Sotteau [5] or Chung and Graham [11] for an exhaustive list of references). Several results are available for hypergraphs.

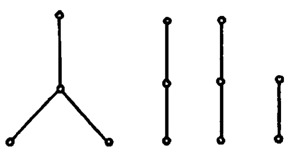
It seems that a special role in hypergraph decompositions is played by the so called Δ-systems.

A uniform hypergraph $F = (\bigcup \mathcal{F}, \mathcal{F})$ is called a Δ-system if there exists a set $K \subseteq V(F)$, called the *kernel* of F, such that $A \cap B = K$, for every $A, B \in \mathcal{F}$

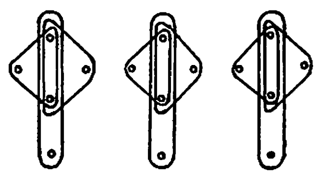
and $A \neq B$. Notice that for a Δ-system of size greater than 1, the kernel is unique. If the kernel is the empty set then the Δ-system is called a *matching*. In the case of graphs the only Δ-systems are matchings and stars.

A constellation is a somewhat more sophisticated variation of a Δ-system. Suppose that k and l are integers such that $0 < l < k$. Let $F_1, \ldots, F_t$ be disjoint k-uniform Δ-systems with l-element kernels and sizes $p_1, \ldots, p_t$, respectively. A *constellation* $\Delta(k, l, \mathbf{p})$, where $\mathbf{p} = (p_1, \ldots, p_t)$ is a hypergraph $F = F_1 \dot{\cup} \cdots \dot{\cup} F_t$. The hypergraphs F_i, for $i = 1, \ldots, t$, are called *components* of the constellation F. Clearly, every Δ-system is a constellation. Three examples of constellations are shown in Figure 1.

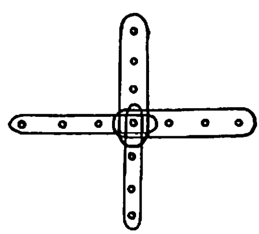
There is a number of papers concerning the decomposition of the complete k-uniform hypergraph K_n^k into Δ-systems (see [2], [4], [15], [17], [18], [20]-[23]). Lonc [15] proved that for a given Δ-system D and n sufficiently large, there is a D-decomposition of K_n^k if and only if the obvious divisibility condition $\binom{n}{k} \equiv 0 \pmod{e(D)}$ is satisfied.



$\Delta(2,1,(3,2,2,1))$



$\Delta(3,2,(3,3,3))$



$\Delta(4,1,(4))$

Figure 1.

The direction of research of this paper is a bit different. We try to find possibly general conditions under which a hypergraph can be decomposed into some 'Δ-system type' hypergraphs. We follow the direction of Lonc and Truszczyński [19] who found a minimal family $\mathcal{F}_{k,m}$ of k-uniform, m-edge hypergraphs having the following property: all, except for finitely many k-uniform hypergraphs H satisfying the obvious divisibility condition $e(H) \equiv 0 \pmod{m}$ have an $\mathcal{F}_{k,m}$-

decomposition. The family $\mathcal{F}_{k,m}$ turned out to be surprisingly small. It consists of Δ-systems and some hypergraphs of structure very close to them. For example, for $k = 2$, that is in the case of graphs, $\mathcal{F}_{2,m}$ consists of three graphs only, each being a constellation, namely: the star $K_{1,m}$, the matching mK_2 and the constellation $\Delta(2,1,(m-1,1)) = K_{1,m-1} \dot{\cup} K_2$. This result suggests that Δ-systems and related hypergraphs play the role of 'bricks' in hypergraph decompositions. Therefore, it seems to be of interest to examine decompositions of hypergraphs into Δ-systems and constellations.

The first of our main results is the following one. For any pair of Δ-systems Δ_1 and Δ_2 with kernels of the same cardinality and sizes p and $p+1$, respectively, we find a sufficient condition for a hypergraph to be $\{\Delta_1, \Delta_2\}$-decomposable. This sufficient condition is satisfied by a very large family of hypergraphs. In the case of graphs this family consists of all graphs G such that $\delta(G)$ is greater than a certain number that does not depend on the graph G. It has to be noted that this result has already been proved by Lonc [16] if Δ_1 and Δ_2 are stars. Our first result is related to results of Favaron *et al.* [13] and Favaron [12] who characterized the family of $\{K_{1,2}, K_{1,3}\}$-decomposable graphs and the family of $\{2K_2, 3K_2\}$-decomposable graphs, respectively.

The second of our results concerns C-decompositions of a hypergraph, where C is a constellation. It seems to be hopeless to determine the family of all C-decomposable hypergraphs for an arbitrary constellation C. It has been done for very special constellations like $2K_2$, $K_{1,2}$, $3K_2$ and $K_{1,2} \dot{\cup} K_2$ by Caro [9], Caro and Schönheim [10], Bialostocki and Roditty [6] and Favaron *et al.* [13], respectively. Alon [1] has proved that there is a constant $c = c(m)$ such that if $e(G) \geq c$ then mK_2-decomposition of a graph G exists if and only if $\Delta(G) \leq e(G)/m$.

In this paper we show that if $C = \Delta(k, l, \mathbf{p})$ is a constellation, where $\mathbf{p} = (p_1, \ldots, p_t)$, such that the greatest common divisor of the numbers $p_1, \ldots, p_t$ is equal to 1 then a certain large class of hypergraphs (to be specified later) consists of C-decomposable hypergraphs. In the case of graphs this class is the set of all graphs G satisfying the obvious divisibility condition $e(G) \equiv 0 \pmod{e(C)}$ and such that $\delta(G)$ is greater than a certain number that does not depend on the graph G.

2. Main Results, Examples and Problems

Let A be a subset of an edge in a k-uniform hypergraph $H = (V, \mathcal{E})$. By a *strong degree of A in H* (denoted by $d_H(A)$) we mean the size of a maximum-sized Δ-system with kernel A. This definition is an extension of a definition of the degree of a vertex introduced by Berge [3, p.429]. Note that, for a graph G without isolated vertices, the notions of degree of a vertex and strong degree of a 1-element set coincide, *i.e.* $d_G(x) = \deg_G x$, for every $x \in V(G)$.

Let $\mathcal{P}_i(V)$ stand for the set of all i-element subsets of the set V. Define

$$\delta_i(H) = \min_{A \in \nabla_i(\mathcal{E})} d_H(A),$$

for $i = 0, 1, \ldots, k-1$, where $\nabla_i(\mathcal{E}) = \{A \subseteq \mathcal{P}_i(V) : (\exists E \in \mathcal{E}) A \subseteq E\}$.

Throughout this section we assume that l and k are integers such that $l < k$. The following two theorems are the main results of this paper.

Theorem 1. *There is an integer $R = R(k, l, p)$ such that if a k-uniform hypergraph H satisfies the condition*

$$\delta_{k-1}(H) \geq R \tag{1}$$

then H can be decomposed into Δ-systems with l-element kernels and each of the size p or $p + 1$.

Theorem 2. *Let $\mathbf{p} = (p_1, \ldots, p_t)$ be a sequence of positive, relatively prime integers and $C = \Delta(k, l, \mathbf{p})$. There is an integer $P = P(k, l, \mathbf{p})$ such that every k-uniform hypergraph H satisfying the conditions*

$$e(H) \equiv 0 \pmod{e(C)} \quad \text{and} \tag{2}$$

$$\delta_{k-1}(H) \geq P \tag{3}$$

has a C-decomposition.

It is not difficult to check that each of the conditions (1) and (3) is satisfied by almost all k-uniform hypergraphs, *i.e.* the ratio of the number of k-uniform hypergraphs on n vertices satisfying (1) (respectively (3)) and the number of all k-uniform hypergraphs on n vertices goes to 1 as n goes to infinity.

The assumptions of Theorem 2 cannot, in general, be replaced by some weaker ones.

The first example shows that the assumption that the integers $p_1, \ldots, p_t$ are relatively prime cannot be eliminated.

Example 1: Let m, k, l, $p_1, \ldots, p_t$ be positive integers, $\mathbf{p} = (p_1, \ldots, p_t)$, $q = \sum_{j=1}^{t} p_j$ and $n = kqm$. Assume that $0 < l < k$ and that the greatest common divisor of $p_1, \ldots, p_t$ is equal to $d > 1$. Denote by G_m a hypergraph obtained from the complete k-uniform hypergraph on n vertices K_n^k, by deleting an edge. Let F_m be a hypergraph obtained from K_n^k by deleting a matching of size $q - 1$. Finally, let $H_m = G_m \dot{\cup} F_m$.

It is routine to check that $e(H_m) \equiv 0 \pmod{q}$ and $\delta_{k-1}(H_m) \geq (qm - 1)k$. Thus, for every $P = P(k, l, \mathbf{p})$ we can choose m such that $\delta_{k-1}(H_m) \geq P$. On the other hand, the hypergraph H_m does not have a $\Delta(k, l, \mathbf{p})$-decomposition for any m. Indeed, if it had such a decomposition then a decomposition of H_m

into Δ-systems of size d would exist. But this is not possible, since it is easy to show that $e(G_m) \equiv -1 \pmod{d}$. Consequently, H_m cannot be decomposed into Δ-systems of size d.

The next example shows that the condition (3) in Theorem 2 cannot be replaced by a condition of type $\delta_m(H) \geq P' = P'(m,k,l,\mathbf{p})$, for any $m = 0,1,\ldots,k-2$.

Example 2: Let k and P' be positive integers, V a set of cardinality $n+1 \geq 5kP'+1$, x a fixed element in V and m an integer such that $0 \leq m < k-1$. Let $X = V - \{x\}$. It follows from the inequality $n \geq m + (k-m)P'$ that for every $M \in \mathcal{P}_m(X)$, there is a family $\mathcal{E}_M \subseteq \mathcal{P}_k(X)$ such that $|\mathcal{E}_M| = P'$ and the hypergraph $(\bigcup \mathcal{E}_M, \mathcal{E}_M)$ is a Δ-system with kernel M. Let $\mathcal{E}' = \bigcup_{M \in \mathcal{P}_m(X)} \mathcal{E}_M$ and $\mathcal{F} = \{E \in \mathcal{P}_k(V) : x \in E\}$. It is not hard to check that $|\mathcal{P}_k(X) - \mathcal{E}'| \geq 5$. Let $|\mathcal{E}' \cup \mathcal{F}| \equiv r \pmod{5}, 0 \leq r < 5$. Choose arbitrarily $5-r$ distinct elements of $\mathcal{P}_k(X) - \mathcal{E}'$, adjoin them to $\mathcal{E}'$ and denote the resulting family by $\mathcal{E}$. Finally, define $H = (V, \mathcal{E} \cup \mathcal{F})$.

First, we show that $\delta_m(H) \geq P'$. To this end, consider $A \in \mathcal{P}_m(V)$. If $x \in A$ then $d_H(A) = \lfloor (n-m+1)/(k-m) \rfloor$. Otherwise, $d_H(A) \geq |\mathcal{E}_A| = P'$. In both cases, $d_H(A) \geq P'$, so $\delta_m(H) \geq P'$.

Clearly, $e(H) \equiv 0 \pmod{5}$.

We shall prove that H does not have a $\Delta(k,l,(3,2))$-decomposition, for $0 < l < k$. Suppose that such decomposition exists. At most 3 edges of each of the constellations forming this decomposition belong to $\mathcal{F}$. Since $|\mathcal{F}| = \binom{n}{k-1}$, there are at least $\frac{1}{3}\binom{n}{k-1}$ constellations $\Delta(k,l,(3,2))$ in the decomposition. Therefore, there are at least $\frac{5}{3}\binom{n}{k-1}$ edges in H. On the other hand, there are at most $\binom{n}{k-1} + \binom{n}{m}P' + 5$ edges in H. Thus, $\binom{n}{k-1} + \binom{n}{m}P' + 5 \geq \frac{5}{3}\binom{n}{k-1}$. This is a contradiction because the inequality does not hold under the assumptions $n \geq 5kP'$ and $0 \leq m < k-1$. Therefore, H does not have a $\Delta(k,l,(3,2))$-decomposition.

We are not able to find an example showing that the condition (1) in Theorem 1 cannot be replaced by a condition

$$\delta_m(H) \geq R' = R'(m,k,l,p),$$

for some $0 \leq l \leq m < k-1$. We suspect that it can.

Problem 1: For which integers l and m, $0 \leq l \leq m < k-1$, is the following statement true:

There is an integer $R' = R'(m,k,l,p)$ such that if a k-uniform hypergraph H satisfies the condition $\delta_m(H) \geq R'$ then H can be decomposed into Δ-systems with l-element kernels and each of size p or $p+1$?

Theorem 2 suggests another question. For which hypergraphs C (besides constellations) does Theorem 2 hold? This question is especially interesting in the case of graphs. Our Problem 2 suggests a possible answer.

Problem 2: Let $\mathcal{G}$ be a family of forests with components of relatively prime sizes. Prove or disprove:
For every $G \in \mathcal{G}$, there is an integer $P = P(G)$ (which does not depend on H) such that if $e(H) \equiv 0 \pmod{e(G)}$ and $\delta(H) \geq P$ then the graph H has a G-decomposition.

3. Proof of Theorem 1

We shall need three lemmas. The first of them is a well-known theorem of Hajnal and Szemerédi [14].

Lemma 3 (Hajnal, Szemerédi). *Let G be a graph. If $m \geq \Delta(G) + 1$ then there is a partition of the vertex set of G into m independent subsets of almost equal cardinalities.* ∎

(The sets $X_1, \ldots, X_n$ are said to be of *almost equal* cardinalities if $||X_i| - |X_j|| \leq 1$, for $i, j = 1, \ldots, n$.)

Lemma 4. *For every k-uniform hypergraph H and for every integer $m \geq k\delta(H)$, there is a decomposition of H into m matchings of almost equal sizes.*

Proof: Let G be the intersection graph for H, *i.e.* the graph whose vertices are the edges in H and two vertices are joined by an edge in it if the corresponding edges in H intersect. Clearly, $\Delta(G) \leq (\Delta(H) - 1)k$. Thus, by the assumptions, $m \geq k\Delta(H) \geq \Delta(G) + 1$. By Lemma 3, there is a partition of $V(G)$ into m independent sets of almost equal cardinalities. Since every partition of the vertex set of G into independent sets corresponds to decomposition of H into matchings, there is a decomposition of H into m matchings of almost equal sizes. ∎

Lemma 5. *If H is a k-uniform hypergraph then $e(H) \geq \Delta(H)\delta_{k-1}(H)/k$.*

Proof: Let x be vertex in H such that $\deg_H x = \Delta(H)$. For every edge E containing x, the set $E - \{x\}$ is the center of a Δ-system of size at least $\delta_{k-1}(H)$. Thus, the number of edges intersecting at least one of the edges containing x is at least $\delta_{k-1}(H)\Delta(H)/k$. Consequently, $e(H) \geq \Delta(H)\delta_{k-1}(H)/k$. ∎

Proof of Theorem 1: Let $R = k(k-l)^2\binom{k-1}{l}p$. We shall apply the Integer Ford-Fulkerson Theorem (cf. [8, p.51]):
Let $F = (X, C)$ be a digraph and let $f\colon C \to \mathbf{R}$ be a flow. There exists a flow $g : C \to \mathbf{Z}$ such that $g(c) = \lfloor f(c) \rfloor$ or $g(c) = \lceil f(c) \rceil$ for every arc $c \in C$. (The symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ stand for the integer part of x and the least integer not less than x, respectively.)

Let $H = (V, \mathcal{E})$ and define a digraph $\Gamma = (Y, A)$. Let $Y = \mathcal{E} \cup \nabla_{k-1}(\mathcal{E}) \cup$

$\{S,T\}$. Denote

$$A_1 = \{(E,B): E \in \mathcal{E}, \quad B \in \nabla_{k-1}(\mathcal{E}), \quad B \subseteq E\},$$
$$A_2 = \{(S,E): E \in \mathcal{E}\},$$
$$A_3 = \{(B,T): B \in \nabla_{k-1}(\mathcal{E})\} \qquad \text{and}$$
$$A_4 = \{(T,S)\}.$$

Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$ and finally, for $a \in A$, define

$$f(a) = \begin{cases} \frac{1}{k} & \text{for } a \in A_1 \\ 1 & \text{for } a \in A_2 \\ \frac{d_H(B)}{k} & \text{for } a = (B,T) \in A_3 \\ |\mathcal{E}| & \text{for } a \in A_4. \end{cases}$$

It is easy to verify that f is a flow. By the Integer Ford-Fulkerson Theorem, there is a flow g in Γ such that $g(a) = \lfloor f(a) \rfloor$ or $g(a) = \lceil f(a) \rceil$, for $a \in A$.

The flow g corresponds to a decomposition of H into Δ-systems with $(k-1)$-element kernels. In fact, assign to every set $B \in \nabla_{k-1}(\mathcal{E})$ the set of edges $\mathcal{E}_B = \{E \in \mathcal{E}: B \subseteq E \text{ and } g((E,B)) = 1\}$. The set $\mathcal{E}_B$ generates a Δ-system H_B of size at least $\lfloor d_H(B)/k \rfloor$ with B as its kernel. Moreover, every edge $E \in \mathcal{E}$ belongs to exactly one set $\mathcal{E}_B$. Consequently, the hypergraphs H_B, where $B \in \nabla_{k-1}(\mathcal{E})$, form a decomposition of H into Δ-systems of sizes greater than or equal to $\lfloor \delta_{k-1}(H)/k \rfloor \geq \lfloor R/k \rfloor = (k-l)^2 \binom{k-1}{l} p$.

Decompose every Δ-system H_B into $\binom{k-1}{l}$ Δ-systems H_B^D, $D \in \mathcal{P}_l(B)$, of almost equal sizes. Clearly, $e(H_B^D) \geq (k-l)^2 p$. Now, for every $D \in \nabla_l(\mathcal{E})$, denote by H^D the hypergraph generated by the set of edges $\mathcal{E}(H^D) = \bigcup_{B \supseteq D} \mathcal{E}(H_B^D)$. Obviously, the hypergraphs H^D, $D \in \nabla_l(\mathcal{E})$, form a decomposition of H. To prove the theorem, it suffices to show that H^D can be decomposed into Δ-systems with l-element kernels and each of size p or $p+1$, for every $D \in \nabla_l(\mathcal{E})$.

Remove the set D from every edge of H^D and denote by G^D the resulting $(k-l)$-uniform hypergraph. According to the construction of H^D, every $(k-1)$-element subset of an edge in H^D containing D is the kernel of a Δ-system of size at least $(k-l)^2 p$. Thus, every $(k-l-1)$-element subset of an edge in G^D is the kernel of a Δ-system of size at least $(k-l)^2 p$. Hence, $\delta_{k-l-1}(G^D) \geq (k-l)^2 p$. By Lemma 5,

$$e(G^D) \geq \Delta(G^D)\delta_{k-l-1}(G^D)/(k-l) \geq (k-l)p\Delta(G^D).$$

Let $m = \lfloor e(G^D)/p \rfloor$. Since $m \geq (k-l)\Delta(G^D)$, it follows from Lemma 4 that G^D can be decomposed into m matchings of almost equal sizes.

Let $e(G^D) = bp + r$, where $0 \leq r < p$. Since $e(G^D) \geq (k-l)p\Delta(G^D) \geq (k-l)p\delta_{k-l-1}(G^D) \geq (k-l)^3p^2 \geq p^2$, we get $b \geq p$ and $r/b < 1$. Therefore, the size of the smallest matching in the decomposition of G^D into m matchings is equal to $\lfloor e(G^D)/m \rfloor = \lfloor (bp+r)/\lfloor (bp+r)/p \rfloor \rfloor = \lfloor (bp+r)/b \rfloor = p$ and the size of the largest one is equal to $\lceil e(G^D)/m \rceil \leq p+1$. The decomposition of G^D into matchings of sizes p or $p+1$ corresponds to a decomposition of the hypergraph H^D into Δ-systems of sizes p or $p+1$ and with l-element kernels. This completes the proof. ∎

4. Proof of Theorem 2

To prove Theorem 2 we need several technical and rather complicated lemmas. Therefore, it seems useful to outline the steps of the proof first.

The crucial points of the reasoning are Theorem 1 and Lemmas 6, 7 and 10. The Lemmas 3, 4, 5, 8 and 9 play an auxiliary role. The hypergraph which is to be decomposed is usually denoted by H. In Theorem 1 and Lemmas 6, 7 and 10 we assume that the strong degree $\delta_{k-1}(H)$ is greater than a certain number independent of H. The number depends only on the parameters of the hypergraphs into which H is to be decomposed.

We use Theorem 1 to decompose H into Δ-systems, each of large (to be specified later) size p or $p+1$, with l-element kernels. Then (Lemma 6), we group the Δ-systems into constellations such that the number of the Δ-systems being components is suitably large in every constellation, and such that the sizes of the Δ-systems are still equal to p or $p+1$. We modify this decomposition (Lemma 7) to obtain a decomposition of H into constellations $C_1, \ldots, C_s$ of sizes being a multiplicity of the size of $C = \Delta(k, l, \mathbf{p})$ and such that both the number of components in every C_i and the sizes of the components are appropriately large. Finally, we apply Lemma 10 to decompose every constellation C_i into constellations isomorphic to C.

Lemma 6. *Let $l > 0$. There is an integer $T = T(k, l, p, q)$ such that every k-uniform hypergraph H satisfying the condition $\delta_{k-1}(H) \geq T$ can be decomposed into constellations $D_1, \ldots, D_s$ having, for $i = 1, \ldots, s$, the following properties:*

(6a) *every component of D_i has an l-element kernel,*

(6b) *$2q > q_i \geq q$, where q_i is the number of components in D_i and*

(6c) *the size of each of the components of D_i is p or $p+1$.*

Proof: Let $T = T(k, l, p, q) = \max\{R(k, l, p), k^2(p+1)^2 q\}$ (see Theorem 1 for the definition of $R(k, l, p)$). It follows from Theorem 1 that there is a decomposition Θ of H into delta-systems with l-element kernels and each of size p or $p+1$.

Let G be the graph whose vertices are the Δ-systems that form the decomposition Θ. Two vertices in G are joined by an edge if the vertex sets of the corresponding Δ-systems intersect.

Notice that

$$e(H) \leq |V(G)|(p+1) \tag{4}$$

because Θ is a decomposition of H into $|V(G)|$ Δ-systems of sizes at most $p+1$. Let D be a fixed Δ-system from the decomposition Θ.

The number of edges in H that intersect the set of vertices of D is not greater than

$$|V(D)|\Delta(H) \leq (l+(p+1)(k-l))\Delta(H).$$

On the other hand, the number is not less than the number of Δ-systems of Θ whose vertex sets intersect $V(D)$, *i.e.* it is not less than $\deg_G D$. The above two observations imply the inequality

$$\deg_G D \leq (l+(p+1)(k-l))\Delta(H)$$

for every Δ-system D from the decomposition Θ. Thus,

$$\Delta(G) \leq (l+(p+1)(k-l))\Delta(H). \tag{5}$$

Applying, in turn, (4), (5), the assumption $l > 0$, Lemma 5 and the definition of T we get

$$\frac{|V(G)|}{\Delta(G)+1} \geq \frac{e(H)/(p+1)}{(l+(p+1)(k-l))\Delta(H)+1}$$
$$\geq \frac{e(H)}{(p+1)^2 k\Delta(H)} \geq \frac{\delta_{k-1}(H)}{(p+1)^2 k^2} \geq q.$$

By virtue of Lemma 3, the vertex set of G can be partitioned into $\lfloor |V(G)|/q \rfloor$ independent sets of almost equal cardinalities. Since $q \leq \frac{|V(G)|}{\lfloor |V(G)|/q \rfloor} < 2q$, the cardinalities of the independent sets belong to the interval $[q, 2q)$. Clearly, this partition of $V(G)$ corresponds to a decomposition of H into constellations satisfying the conditions (6a), (6b) and (6c). ∎

Lemma 7. *Let n, p and l be positive integers such that $n < \frac{1}{2}p$ and $l > 0$. There is an integer $Q = Q(k, l, p, q)$ such that every k-uniform hypergraph H satisfying the conditions $\delta_{k-1}(H) \geq Q$ and $e(H) \equiv 0 \pmod{n}$ can be decomposed into constellations $C_1, \dots, C_s$ having, for $i = 1, \dots, s$, the following properties:*

(7a) *every component of C_i has an l-element kernel,*
(7b) *$q \leq q_i \leq 2q$, where q_i is the number of components in C_i,*
(7c) *the sizes of the components of C_i belong to the interval $(\frac{1}{2}p - n, p+1]$,*
(7d) $e(C_i) \equiv 0 \pmod{n}$.

Proof: Let $Q(k,l,p,q) = \max\{T(k,l,p,q), 8q^2(p+1)^2k^2\}$ (see Lemma 6 for the definition of $T(k,l,p,q)$). According to Lemma 6, there is a decomposition ψ of H into constellations $D_1, \dots, D_s$ satisfying the conditions (6a), (6b) and (6c). Let, for $i = 1, \dots, s$, $D_i^1, \dots, D_i^{r_i}$ be the components of D_i. By (6b), $q \leq r_i < 2q$ and by (6c), $p \leq e(D_i^j) \leq p+1$, for $j = 1, \dots, r_i$.

Let F be a graph with vertices $D_1, \dots, D_s$. Two vertices D_i and D_j form an edge in F if $V(D_i) \cap V(D_j) = \emptyset$.

We prove that $\delta(F) \geq \frac{1}{2}|V(F)|$. To this end, notice that, for $i = 1, \dots, s$,

$$|V(D_i)| < 2q(l + (k-l)(p+1)). \tag{6}$$

Moreover, it is easily seen that

$$e(H) < 2q(p+1)|V(F)|. \tag{7}$$

Applying, in turn, (6), Lemma 5, the definition of Q and (7), we get

$$\begin{aligned} \delta(F) &\geq |V(F)| - 2q(l + (k-l)(p+1))\Delta(H) \\ &\geq |V(F)| - 2qk(p+1)\frac{ke(H)}{\delta_{k-1}(H)} \\ &\geq |V(F)| - \frac{2qk^2(p+1)}{8q^2(p+1)^2k^2}e(H) \\ &\geq \frac{1}{2}|V(F)|. \end{aligned}$$

By the well-known Dirac Theorem (see Bollobás [7, p.132]), there exists a Hamiltonian path in F. Without loss of generality, we can assume that $D_1, \dots, D_s$ are the consecutive vertices of the path. Note that $D_i \dot\cup D_{i+1}$ is a constellation, for $i = 1, \dots, s-1$.

Now, we construct recursively the constellations $C_1, \dots, C_s$ that form a decomposition of H and satisfy the conditions (7a)–(7d).

Let $L_0 = D_1$. Suppose that we have already defined $L_0, L_1, \dots, L_{i-1}$, $C_1, \dots, C_{i-1}$. We define L_i and C_i for $0 < i < s$. Notice that there is an integer $m \in \left(\frac{1}{2}e(D_{i+1}^1) - n, \frac{1}{2}e(D_{i+1}^1)\right]$ such that $e(L_{i-1}) + m \equiv 0 \pmod{n}$. Decompose D_{i+1}^1 into two Δ-systems D'_{i+1} and D''_{i+1} of sizes m and $e(D_{i+1}^1) - m$, respectively. Let $C_i = L_{i-1} \dot\cup D'_{i+1}$ and $L_i = D''_{i+1} \dot\cup D_{i+1}^2 \dot\cup \dots \dot\cup D_{i+1}^{r_i}$. Finally, let $C_s = L_{s-1}$. The hypergraphs $C_1, \dots, C_s$ are constellations because $D_i \dot\cup D_{i+1}$ is a constellation, for $i = 1, \dots, s-1$. Moreover, according to the construction and the assumption $e(H) \equiv 0 \pmod{n}$, the hypergraphs $C_1, \dots, C_s$ form a decomposition of H satisfying the conditions (7a)–(7d). ∎

To prove the important Lemma 10, we need two auxiliary Lemmas 8 and 9.

Lemma 8. *Let $p_1, \ldots, p_t$ and $a_1, \ldots, a_q$ be sequences of integers and let $p_1 \geq \cdots \geq p_t > 0$. If*

$$a_i \geq p_1 t_q \sum_{j=1}^{t} p_j \qquad \text{for } i = 1, \ldots, q \tag{8}$$

$$a_i \equiv 0 \pmod{D_t} \qquad \text{for } i = 1, \ldots, q \tag{9}$$

(where D_t stands for the greatest common divisor of $p_1, \ldots, p_t$) and

$$\sum_{i=1}^{q} a_i \equiv 0 \pmod{\sum_{j=1}^{t} p_j} \tag{10}$$

then there are integers $\alpha_i^j \geq 0$, $j = 1, \ldots, t$, $i = 1, \ldots, q$ such that

$$\sum_{i=1}^{q} \alpha_i^1 = \cdots = \sum_{i=1}^{q} \alpha_i^t = \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j \quad \text{and}$$

$$\sum_{j=1}^{t} \alpha_i^j p_j = a_i, \quad \text{for } i = 1, \ldots, q.$$

Proof: We prove the lemma by induction on t. It holds for $t = 1$ because it suffices to put $\alpha_i^1 = \frac{a_i}{p_1}$, for $i = 1, \ldots, q$. Suppose that $t \geq 2$ and that the lemma is true for $t-1$. Denote by D_j the greatest common divisor of $p_1, \ldots, p_j$, for $j = 1, \ldots, t$. For $i = 1, \ldots, q-1$, there exists an integer

$$x_i \in \left(a_i / \sum_{j=1}^{t} p_j - D_{t-1}/D_t,\ a_i / \sum_{j=1}^{t} p_j \right] = I \tag{11}$$

such that

$$a_i/D_t - x_i p_t/D_t \equiv 0 \pmod{D_{t-1}/D_t}. \tag{12}$$

To see this, consider the remainders of the division of $a_i/D_t - x p_t/D_t$ by D_{t-1}/D_t for every $x \in I$. The remainders can not be equal for any x', $x" \in I$, $x' \neq x"$. Otherwise $(a_i/D_t - x' p_t/D_t) - (a_i/D_t - x'' p_t/D_t) = (x'' - x') p_t/D_t \equiv 0 \pmod{D_{t-1}/D_t}$. Since $|x' - x"| < D_{t-1}/D_t$ and p_t/D_t and D_{t-1}/D_t are relatively prime, $x' = x"$, a contradiction. Thus, for every $r = 0, 1, \ldots, D_{t-1}/D_t - 1$, there exists $x \in I$ such that $a_i/D_t - x p_t/D_t \equiv r \pmod{D_{t-1}/D_t}$. In particular, there exists x_i satisfying (11) and (12).

Let $\alpha_i^t = x_i$, for $i = 1, \ldots, q-1$ and let

$$\alpha_q^t = \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j - \sum_{i=1}^{q-1} \alpha_i^t.$$

Notice that by (11) and (8)

$$\alpha_i^t > a_i / \sum_{j=1}^{q} p_j - D_{t-1}/D_t \geq p_1 tq - p_1 \geq 0, \quad \text{for } i = 1, \ldots, q-1$$

and, by (11),

$$\begin{aligned}\alpha_q^t &\geq \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j - \sum_{i=1}^{q-1} a_i / \sum_{j=1}^{t} p_j \\ &= a_q / \sum_{j=1}^{t} p_j \geq 0 .\end{aligned}$$

Apply the induction hypothesis for the sequences $p_1, \ldots, p_{t-1}$ and $a'_1, \ldots, a'_q$, where $a'_i = a_i - \alpha_i^t p_t$, for $i = 1, \ldots, q$. It is therefore necessary to check that the assumptions (8), (9) and (10) are satisfied by these sequences.

1. Assumption (8).

For $i = 1, \ldots, q-1$,

$$\begin{aligned}a'_i &\geq a_i - p_t a_i / \sum_{j=1}^{t} p_j \\ &= a_i \sum_{j=1}^{t-1} p_j / \sum_{j=1}^{t} p_j \\ &\geq p_1 tq \sum_{j=1}^{t-1} p_j \\ &\geq p_1 (t-1) q \sum_{j=1}^{t-1} p_j .\end{aligned}$$

Moreover,

$$
\begin{aligned}
a'_q &= a_q - p_t\left(\sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j - \sum_{i=1}^{q-1} \alpha_i^t\right) \\
&> a_q - p_t \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j + p_t \sum_{i=1}^{q-1} a_i / \sum_{j=1}^{t} p_j - p_t(q-1) D_{t-1}/D_t \\
&= a_q \sum_{j=1}^{t-1} p_j / \sum_{j=1}^{t} p_j - p_t(q-1) D_{t-1}/D_t \\
&\geq p_1 t_q \sum_{j=1}^{t-1} p_j - q p_1^2 \\
&\geq p_1 (t-1) q \sum_{j=1}^{t-1} p_j .
\end{aligned}
$$

2. Assumption (9).

According to the definition of α_i^t, $a'_i = a_i - p_t\alpha_i^t \equiv 0 \pmod{D_{t-1}}$, for $i = 1, \ldots, q-1$. Moreover,

$$
\begin{aligned}
a'_q &= a_q - p_t \alpha_q^t \\
&= \sum_{i=1}^{q} a_i - \sum_{i=1}^{q-1} a_i - p_t \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j + p_t \sum_{i=1}^{q-1} \alpha_i^t \\
&= \left(\sum_{i=1}^{q} a_i\right)\left(\sum_{j=1}^{t-1} p_j\right) / \sum_{j=1}^{t} p_j - \sum_{i=1}^{q-1} (a_i - p_t \alpha_i^t) \\
&\equiv 0 \pmod{D_{t-1}}
\end{aligned}
$$

because

$$
\sum_{j=1}^{t-1} p_j \equiv 0 \pmod{D_{t-1}} .
$$

3. Assumption (10).

$$\sum_{i=1}^{q} a'_i = \sum_{i=1}^{q} a_i - p_t \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j$$
$$= \left(\sum_{i=1}^{q} a_i\right)\left(\sum_{j=1}^{t-1} p_j\right) / \sum_{j=1}^{t} p_j$$
$$\equiv 0 \pmod{\sum_{j=1}^{t-1} p_j}.$$

By the induction hypothesis, there exist integers $\alpha_i^j \geq 0$, $i = 1, \ldots, q$, $j = 1, \ldots, t-1$ such that

$$\sum_{i=1}^{q} \alpha_i^s = \sum_{i=1}^{q} a'_i / \sum_{j=1}^{t-1} p_j = \sum_{i=1}^{q} a_i / \sum_{j=1}^{t} p_j,$$

for $s = 1, \ldots, t-1$. Clearly,

$$\sum_{i=1}^{q} \alpha_i^t = \sum_{i=1}^{q} / \sum_{j=1}^{t} p_j.$$

Moreover,

$$\sum_{j=1}^{t-1} \alpha_i^j p_j = a'_i$$

so

$$\sum_{j=1}^{t} \alpha_i^j p_j = a_i, \quad \text{for } i = 1, \ldots, q.$$

This completes the proof of the lemma. ∎

Lemma 9. *Let B be a set of cardinality mt. Assume that the elements of B are colored with t colors $c_1, \ldots, c_t$ such that exactly m elements receive color c_i, for $i = 1, \ldots, t$. Moreover, let sets $B_1, \ldots, B_q$ form a partition of B. If*

$$|B_i| \leq m, \quad \text{for } i = 1, \ldots, q, \tag{13}$$

then there is a partition of B into m t-element subsets $F_1, \ldots, F_m$ such that elements of F_j, $j = 1, \ldots, m$, have distinct colors and $|F_j \cap B_i| \leq 1$, for $j = 1, \ldots, m$ and $i = 1, \ldots, q$.

Proof: Let $G = (X, Y; E)$ be a bipartite multigraph such that $X = \{B_1, \ldots, B_q\}$ and $Y = \{c_1, \ldots, c_t\}$ and multiplicity of an edge $B_i c_j$ is equal to the number of elements of B_i that are colored with c_j. Clearly, there is a one-to-one correspondence between the elements of B and the edges in G. According to (13), $\Delta(G) = m$. Since the chromatic index of a bipartite multigraph is equal to its maximum degree, G can be decomposed into m matchings of sizes $s_1 \geq \cdots \geq s_m$. Obviously, $s_1 \leq |Y| = t$. Thus, $mt = e(G) = s_1 + \cdots + s_m \leq ms_1 \leq mt$, so $t = s_1 = \cdots = s_m$. The decomposition of G into m matchings of size t corresponds to the required partition of B into the subsets $F_1, \ldots, F_m$. ∎

The next lemma is a corollary to Lemmas 8 and 9.

Lemma 10. *Assume that* $p_1, \ldots, p_t$ *and* $a_1, \ldots, a_q$ *are sequences of positive integers,* $p_1 \geq \cdots \geq p_t > 0$ *and* $l > 0$. *Let* $C = \Delta(k, l, (p_1, \ldots, p_t))$ *and* $K = \Delta(k, l, (a_1, \ldots, a_q))$. *If the conditions (8), (9) and (10) are satisfied and*

$$a_i \leq p_t \sum_{s=1}^{q} a_s \Big/ \sum_{j=1}^{t} p_j, \quad \textit{for } i = 1, \ldots, q, \tag{14}$$

then there is a C*-decomposition of* K.

Proof: Denote by $A_1, \ldots, A_q$ the components of the constellation K. We can assume, without loss of generality, that the size of A_i is equal to a_i, for $i = 1, \ldots, q$. By Lemma 8, there is a decomposition Θ of K into Δ-systems such that every Δ-system A_i is decomposed in Θ into α_i^j Δ-systems of size p_j, for $j = 1, \ldots, t$. Moreover, the number of Δ-systems of size p_j in Θ is equal to

$$m = \sum_{i=1}^{q} \alpha_i^j = \sum_{i=1}^{q} a_i \Big/ \sum_{j=1}^{t} p_j.$$

Let B_i be the set of Δ-systems that form the decomposition of A_i in Θ and let $B = \bigcup_{i=1}^{q} B_i$. Clearly, $|B| = mt$. Color every Δ-system of size p_j belonging to B with c_j, for $j = 1, \ldots, t$. Applying, in turn, the definition of α_i^j, Lemma 8, (14) and the definition of m we get

$$|B_i| = \sum_{j=1}^{t} \alpha_i^j \leq \frac{1}{p_t} \sum_{j=1}^{t} \alpha_i^j p_j = a_i / p_t \leq m, \quad \text{for } i = 1, \ldots, q.$$

By Lemma 9, the existence of a C-decomposition of K follows. ∎

Theorem 2 is now an easy consequence of Lemmas 7 and 10.

Proof of Theorem 2: Let $P(k, l, \mathrm{p}) = Q(k, l, p, q)$, where $q = \lceil 7 \sum_{j=1}^{t} p_j / p_t \rceil$ and $p = 6 p_1 t q \sum_{j=1}^{t} p_j$ (see Lemma 7 for the definition of $Q(k, l, p, q)$). According to Lemma 7, H can be decomposed into constellations $C_1, \ldots, C_s$ satisfying the conditions (7a)–(7d) with $n = \sum_{j=1}^{t} p_j = e(C)$. Let K be one of these constellations and suppose that $K = \Delta(k, l, (a_1, \ldots, a_{q'}))$. To prove the theorem, it suffices to show that K is C-decomposable.

Applying, in turn, (7c), the definition of p, and (7b) we obtain

$$a_i \geq \frac{1}{2}p - \sum_{j=1}^{t} p_j \geq 2qp_1 t \sum_{j=1}^{t} p_j \geq p_1 t q' \sum_{j=1}^{t} p_j, \qquad (15)$$

for $i = 1, \ldots, q'$. It follows by (7d) that

$$\sum_{i=1}^{q'} a_i = e(K) \equiv 0 \left(\mathrm{mod} \sum_{j=1}^{t} p_j \right).$$

Finally, by (7c), the definition of p, (15), (7b) and the definition of q we get

$$a_i \leq p + 1 \leq 7p_1 t q \sum_{j=1}^{t} p_j \leq \frac{7q}{q'^2} \sum_{i=1}^{q'} a_i \leq \frac{7}{q} \sum_{i=1}^{q'} a_i$$
$$\leq p_t \sum_{i=1}^{q'} a_i \Big/ \sum_{j=1}^{t} p_j, \quad \text{for } i = 1, \ldots, q'.$$

Since the integers $p_1, \ldots, p_t$ are relatively prime, all assumptions of Lemma 10 are satisfied. Consequently, K is C-decomposable. ∎

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