On the number of multiplicative partitions

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Abstract. Let $f(n)$ denote the number of essentially different factorizations of n. In this paper, we prove that for every odd number > 1, we have $f(n) \leq c\frac{n}{\log n}$, where c is a positive constant.

Consider the set $T(n) = \{(m_1, m_2, \ldots, m_s) \mid n = m_1 m_2 \cdots m_s,\ m_i > 1,\ 1 \leq i \leq s\}$, where n and m_i, $1 \leq i \leq s$, are all natural numbers and identify those partitions which differ only by the order of the factors. We define $f(n) = |T(n)|$, $n > 1$, and $f(1) = 1$.

In 1983, John F. Hughes and J. O. Shallit [2] have proved $f(n) \leq 2n^{\sqrt{2}}$.

In 1987, Chen Xiao-Xia [1] has proved $f(n) \leq n$.

We easily prove that $f(n) = 0(n^\alpha)$, $\alpha < 1$, does not hold. In fact, let $B(n)$ denote the nth Bell-number and $a_n = P_1 P_2 \cdots P_n$, P_i being the ith prime. We have $\log f(a_n) = \log B(n) \sim n \log n$ and $\log a_n = \sum_{i=1}^{n} \log P_i = \sum_{P \leq P_n} \log P \sim P_n \sim n \log n$. It follows that $\lim_{n\to\infty} \frac{\log f(a_n)}{\log a_n} = 1$. If $f(n) \leq An^\alpha$, then $\overline{\lim_{n\to\infty}} \frac{\log f(n)}{\log n} \leq \alpha < 1$. So we get a contradiction.

In this paper, we shall prove the following:

Theorem. *For every odd number > 1, we have*

$$f(n) \leq c\frac{n}{\log n},$$

where c is a positive constant.

Throughout this paper, let $P(n)$ be the largest prime factor of n and $P_1(n)$ the smallest.

To prove Theorem, we need the following:

Lemma. *If $n > 1$, then $f(n) \leq \sum_{d \mid \frac{n}{P(n)}} f(d)$.*

Proof: If $n = \prod_{j=1}^{r-1} P_j^{\alpha_j} \cdot P_r$, $P_1 < P_2 < \cdots < P_r$, consider the sets: $T_{j_1 j_2 \cdots j_{r-1}}(n) = \{(P_1^{\alpha_1 - j_1} P_2^{\alpha_2 - j_2} \cdots P_{r-1}^{\alpha_{r-1} - j_{r-1}} P_r,\ m_2, \cdots, m_s) \mid n = P_1^{\alpha_1 - j_1} P_2^{\alpha_2 - j_2} \cdots P_{r-1}^{\alpha_{r-1} - j_{r-1}} P_r m_2 \cdots m_s,\ m_i > 1, 2 \leq i \leq s\}$, $0 \leq j_i \leq \alpha_i$, $1 \leq i \leq r-1$, where also identify those partitions which differ by the order of the factors.

We easily see $|T_{j_1 j_2 \cdots j_{r-1}}(n)| = f(P_1^{j_1} P_2^{j_2} \cdots P_{r-1}^{j_{r-1}})$, $T(n) = \bigcup_{j_1=1}^{\alpha_1} \bigcup_{j_2=1}^{\alpha_2} \cdots \bigcup_{j_{r-1}=1}^{\alpha_{r-1}} T_{j_1 j_2 \cdots j_{r-1}}(n)$ and only when $j_k = i_k$, $1 \le k \le r-1$, $T_{j_1 j_2 \cdots j_{r-1}}(n) \cap T_{i_1 i_2 \cdots i_{r-1}}(n) \ne 0$. Hence we obtain

$$\begin{aligned} f(n) = |T(n)| &= \sum_{j_1=0}^{\alpha_1} \sum_{j_2=0}^{\alpha_2} \cdots \sum_{j_{r-1}=0}^{\alpha_{r-1}} |T_{j_1 j_2 \cdots j_{r-1}}(n)| \\ &= \sum_{j_1=0}^{\alpha_1} \sum_{j_2=0}^{\alpha_2} \cdots \sum_{j_{r-1}=0}^{\alpha_{r-1}} f(P_1^{j_1} P_2^{j_2} \cdots P_{r-1}^{j_{r-1}}) \\ &= \sum_{d \mid \frac{n}{P(n)}} f(d). \end{aligned} \tag{1}$$

If $n = \prod_{j=1}^{r} P_j^{\alpha_j}$, $P_1 < P_2 < \cdots < P_r < P_{r+1}$, $\alpha_r \ge 2$, let $n_1 = \frac{nP_{r+1}}{P_r}$. Consider a mapping from $T(n)$ into $T(n_1)$:

$$(m_1, m_2, \cdots, m_s) \to (m_1, \cdots, \frac{m_q P_{r+1}}{P_r}, \cdots, m_s),$$

where $n = m_1 m_2 \cdots m_s$, $m_i > 1$, $1 \le i \le s$ and $m_q = \max_{P_r / m_i}\{m_i\}$. We easily see that it is a 1-1 mapping from $T(n)$ onto a subset of $T(n_1)$. So, by (1), we get

$$f(n) \le f(n_1) = \sum_{d \mid \frac{n_1}{P(n_1)}} f(d) = \sum_{d \mid \frac{n}{P_r}} f(d) = \sum_{d \mid \frac{n}{P(n)}} f(d). \qquad \blacksquare$$

Proof of Theorem: It is well-known that $d(n) = o(n^\epsilon)$ and $\log n = o(n^\epsilon)$ for every positive ϵ, where $d(n)$ is the number of divisors of n. Hence we have

$$d(n) \le c_0 n^{\frac{1}{3}} \tag{2}$$

$$\log n \le c_1 n^{\frac{1}{3}}, \tag{3}$$

where c_0, c_1 are constants and $c_0 c_1 > 1$.

It is easy to prove that

$$\sum_{d \mid \frac{n}{P(n)}} d \le \frac{n}{P_1(n) - 1}. \tag{4}$$

In fact, let $n = \prod_{i=1}^{r} P_i^{\alpha_i}$, $P_1 < P_2 < \cdots < P_r$. We have

$$\sum_{d \mid \frac{n}{P(n)}} d = \frac{P_1^{\alpha_1} - 1}{P_1 - 1} < \frac{n}{P_1(n) - 1}, \quad (r = 1)$$

or ,

$$\sum_{d \mid \frac{n}{P(n)}} d = \frac{P_r^{\alpha_r}-1}{P_r-1}\prod_{i=1}^{r-1}\frac{P_i^{\alpha_i+1}-1}{P_i-1} = \frac{P_r^{\alpha_r}-1}{P_1-1}\prod_{i=1}^{r-1}\frac{P_i^{\alpha_i+1}-1}{P_{i+1}-1}$$

$$\leq \frac{P_r^{\alpha_r}}{P_1-1}\prod_{i=1}^{r-1}\frac{P_i^{\alpha_i+1}}{P_i} \leq \frac{n}{P_1(n)-1}. \quad (r \geq 2)$$

Let $c = 6c_0c_1$, we shall show that $f(n) \leq c\frac{n}{\log n}$ for all odd numbers > 1 by means of induction.

When $n = 3$, we have $f(3) = 1 < c\frac{3}{\log 3}$.

Suppose $f(d) \leq c\frac{d}{\log d}$ for all odd numbers which > 1 and $\leq n-2$, where $n-2$ is an odd number and ≥ 3. We shall prove that $f(n) \leq c\frac{n}{\log n}$.

By Lemma, we have

$$f(n) \leq \sum_{d \mid \frac{n}{P(n)}} f(d) = \sum_{\substack{d \mid \frac{n}{P(n)} \\ d \leq n^{\frac{3}{5}}}} f(d) + \sum_{\substack{d \mid \frac{n}{P(n)} \\ d > n^{\frac{3}{5}}}} f(d) = S_1 + S_2 \tag{5}$$

By (2), (3) and $f(d) \leq d$, we get

$$S_1 \leq n^{\frac{3}{5}} \sum_{d \mid \frac{n}{P(n)}} 1 \leq n^{\frac{3}{5}} d(n) \leq c_0 n^{\frac{4}{5}} = \frac{c_0 n}{n^{\frac{1}{5}}} \leq \frac{c_0 n}{\frac{1}{c_1}\log n} = c_0c_1\frac{n}{\log n}. \tag{6}$$

By (4), $P_1(n) > 2$ and the supposition of induction, we get

$$S_2 \leq 6c_0c_1 \sum_{\substack{d \mid \frac{n}{P(n)} \\ d > n^{\frac{3}{5}}}} \frac{d}{\log d} \leq 6c_0c_1\frac{5}{3\log n}\sum_{d \mid \frac{n}{P(n)}} d \leq 6c_0c_1\frac{5}{3\log n}\frac{n}{P_1(n)-1}$$

$$\leq 5c_0c_1\frac{n}{\log n}. \tag{7}$$

By (5), (6) and (7), we get

$$f(n) \leq c_0c_1\frac{n}{\log n} + 5c_0c_1\frac{n}{\log n} = c\frac{n}{\log n}.$$

Our theorem is now proved by induction. ∎

Editor's Note.

After this volume was in print, we learned that the conjecture of Hughes and Shallit, namely, that $f(n) \leq n/(\log n)$ for $n \neq 144$, has been proved by F.W. Dodd and L.E. Mattics, *Rocky Mountain Journal of Mathematics* **17** (1987), pp.797–813.

References

1. Chen Xiao-Xia, *On multiplicative partitions of natural number*, Acta Mathematica Sinica **30** (1987), 268–271.
2. John F. Hughes and J. O. Shallit, *On the number of multiplicative partitions*, Amer. Math. Monthly **90** (1983), 468–471.