MIXED TELEPHONE PROBLEMS

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Abstract. We consider a generalization of the well-known gossip problem: Let the information of each point of a set X be conveyed to each point of a set Y by k-party conference calls. These calls are organized step-wisely, such that each point takes part in at most one call per step. During a call all the k participants exchange all the information they already know.

We investigate the mutual dependence of the number L of calls and the number T of steps of such an information exchange. At first a general lower bound for $L \cdot k^T$ is proved. For the case that X and Y equal the set of all participants, we give lower bounds for L or T, if T resp. L is as small as possible. Using these results the existence of information exchanges with minimum L and T is investigated. For $k = 2$ we prove that for even n, there is one of this kind iff $n \leq 8$.

1. Introduction.

We consider the following version of the well-known telephone or "gossip" problem: Let a set V of n persons (points) be given, and suppose that each person of a subset X knows an item of information which is not known to any of the others. We arrange a step-wise exchange of information such that at its end each person of a subset Y knows all the units of information originally known to one of the persons of X. Each step consists of some parallel k-party conference calls ($k \geq 2$), every person takes part in at most one call per step. During such a call the k participants exchange all the information, which they already have. Any sequence of this kind is called a (X, Y)-complete information flow (IF), and we are interested in its length L, that is, the number of calls, and its time T, that is, the number of steps. Already a lot of results have been proved. The reader is referred to the introduction of [6] and to [10] for a survey on well-known results and a more extensive list of references. For motivating the present paper we explain the situation in the "classical" case $X = Y = V$, $k = 2$, $n \geq 4$. For the sake of brevity we use "complete" instead of "(V, V)-complete". Then in [1, 3, 10] it has been proved that any complete IF contains at least $2n - 4$ calls. Even there are complete IF of this length, but the construction used in the above-mentioned papers yields one with no less than $2\lceil \log_2 n \rceil - 2$ steps. This is much, because in [5] it has been shown that the smallest possible time of any complete IF equals $\lceil \log_2 n \rceil$ (if n even) or $\lceil \log_2 n \rceil + 1$ (if n odd). But also the complete IF of this time are "bad" in some sense, since, for example, for even n they need $\frac{n}{2} \lceil \log_2 n \rceil$ calls, that is, much more than the smallest possible number $2n - 4$.

Let us call a complete IF ideal, if it has the smallest possible number of calls and of steps. For which n we can find an ideal IF? We will answer this question completely for even n at the end of our paper. The other sections contain generalizations of this problem.

In Section 2 we start with the most general case of arbitrarily chosen k, X, Y. Here an inequality is shown, which expresses the mutual dependence of L and T. More precisely we estimate the length of a complete IF with smallest possible time in Section 3, and the time of a complete IF with smallest possible length in Section 4. Using these results the existence of ideal IF is investigated in Section 5.

At first we should prove a technical proposition.

Proposition 1. *For any positive integers* k, r, s $(r \leq s)$ *and any real* a_i $(i = r, \ldots, s)$,

$$k^s \cdot \sum_{i=r}^{s} k^{-i} \cdot a_i = (k-1) \cdot \sum_{j=0}^{s-r-1} k^j \cdot \left(\sum_{i=r}^{s-j-1} a_i \right) + \sum_{i=r}^{s} a_i .$$

Proof: We start from

$$k^s \cdot \sum_{i=r}^{s} k^{-i} a_i = \sum_{i=r}^{s} k^{s-i} a_i = \sum_{i=r}^{s-1} (k^{s-i} - 1) a_i + \sum_{i=r}^{s} a_i ,$$

represent the coefficient in the first sum as a geometric progression and change the sums:

$$\sum_{i=r}^{s-1} (k^{s-i} - 1) a_i = (k-1) \sum_{i=r}^{s-1} \left(\sum_{j=0}^{s-i-1} k^j \right) a_i = (k-1) \sum_{j=0}^{s-r-1} \sum_{i=r}^{s-j-1} k^j a_i .$$

∎

2. Distributing subgraphs.

We use the basic and more general model introduced in detail in [6] and should briefly recall the notation. Throughout the paper let n and k be natural numbers with $n \geq k \geq 2$, and $V := \{1, 2, \ldots, n\}$. Let $H := (V, E)$ denote any connected k-uniform hypergraph on V with edge-set E, that is, a collection of k-element subsets of V. To the edges we assign finite sets of natural numbers, ϕ is possible. The arising numeration φ of H is called information flow (IF) iff for any pair of different adjacent $e, e' \in E$, $\varphi(e) \cap \varphi(e') = \phi$ holds. Furthermore we call the elements of $\varphi(e)$ numbers of e. A call of φ on H is a pair $c = (e; r)$ with $e \in E$, $r \in \varphi(e)$ and the set of all calls with fixed number r is the rth step S_r. We will also write in this situation that c takes place on e during the rth step and that every point $v \in e$ participates in c or "$v \in c$" for the sake of brevity.

For any $v, w \in V$, a path $e_1, e_2, \ldots, e_m$ in H is called φ-monotonic (v, w)-path iff $v \in e_1$, $w \in e_m$ and there are numbers $r_i \in \varphi(e_i)$ with $r_i < r_2 < \ldots < r_m$. Then information is conveyed from v to w during the consecutive calls $(e_1; r_1)$, $(e_2; r_2), \ldots, (e_m; r_m)$, and we write that w knows v's information. To include the case $v = w$ formally, any path of length 0 should be called φ-monotonic, too. Let us fix now two nonempty subsets $X, Y \subseteq V$. Iff for any points $x \in X$, $y \in Y$ there is a φ-monotonic (x, y)-path, then φ is called (X, Y)-complete, or complete if $X = Y = V$.

We measure the quality of an IF with two parameters,

the length $L(H, \varphi) := \sum_{e \in E} |\varphi(e)|$, and

the time $T(H, \varphi) := |\bigcup_{e \in E} \varphi(e)|$.

If there is no danger of ambiguity, then L or T are used instead of $L(H, \varphi)$ or $T(H, \varphi)$, respectively.

In this section we will investigate the mutual dependence of L and T. Because we are interested in "best possible" IF between the n points, a call can be carried out between any group of k points. So throughout the paper we restrict the above-introduced model to the case that $H = H_n^k := (V, \binom{V}{k})$ is the complete k-uniform hypergraph on n points. (Note that not every edge must be used during the IF!) Of course, results to the appropriate problems for restricted hypergraphs would be very interesting. Let any (X, Y)-complete IF φ on H_n^k be fixed arbitrarily. At first, for an arbitrarily chosen point $x \in X$ and for $t = 1, 2, \ldots, T$, let $p_t(x)$ denote the number of calls that pass along x's information during the tth step. Collecting all these calls for $t = 1, 2, \ldots, T$ we get a numbered subgraph of H_n^k which describes the transmission of the unit of information originally known to x only to all the other points. The investigation of all those so-called distributing subgraphs and their connection to the IF will lead to the first results. We start with a lemma on one distributing subgraph.

Lemma 1. *For any $x \in X$ and $t = 1, 2, \ldots, T$,*

$$\sum_{i=1}^{t} p_i(x) \geq \lceil (k^{t-T}|Y| - 1)/(k-1) \rceil .$$

Proof: We prove

$$k^{t-T}\left[1 + (k-1)\sum_{i=1}^{t} p_i(x)\right] \geq |Y| \quad (t = 1, 2, \ldots, T)$$

by showing that this inequality holds for $t = T$ and that the left-hand side is monotone decreasing in t. In each call that conveys x's information we can find at least one participant who knows this item already before that call, that is, at most $k-1$ additional points can be informed about it. Consequently, after t steps, at most $1+(k-1)\sum_{i=1}^{t} p_i(x)$ points know x's information, where the summand 1 represents x itself. This implies:

$$1+(k-1)\sum_{i=1}^{T} p_i(x) \geq |Y|,$$

because after the step $t = T$ all the points of Y must know that unit of information, and

$$p_{t+1}(x) \leq 1+(k-1)\sum_{i=1}^{t} p_i(x) \quad (t = 1, 2, \ldots, T-1),$$

because any two calls of the same step do not have common participants, but each of the $p_{t+1}(x)$ calls has at least one participant who learned x's information during the first t steps. Adding $\sum_{i=1}^{t} p_i(x)$ to our last inequality, we get

$$\sum_{i=1}^{t+1} p_i(x) \leq 1+k\sum_{i=1}^{t} p_i(x),$$

and finally

$$k^{T-(t+1)}\left[1+(k-1)\sum_{i=1}^{t+1} p_i(x)\right]$$
$$\leq k^{T-(t+1)}\left[k+(k-1)k\sum_{i=1}^{t} p_i(x)\right] = k^{T-t}\left[1+(k-1)\sum_{i=1}^{t} p_i(x)\right],$$

that is, the asserted monotonicity holds. ∎

Now we will get a lower bound for the number L of calls in terms of our parameters $p_t(x)$ by considering all the distributing subgraphs.

Lemma 2.

$$L \geq \sum_{t=1}^{T}\left\lceil k^{-t}\sum_{x\in X} p_t(x)\right\rceil.$$

Proof: First we prove by induction over t that any point x knows at most k^{t-1} items of information before the tth step (resp. after S_{t-1}). The case $t = 1$ is trivial because of our basic model. Suppose we proved the statement for t. Obviously, it remains true if x does not take part in a call of S_t. If x participates, then it can learn additionally all the items known to the $k-1$ other participants of the same call.

By induction hypothesis, each of the k points knows at most k^{t-1} items, hence x (and all the others also) knows no more than k^t items after S_t resp. before S_{t+1}.

Consequently, each call of S_t transmits at most k^t items of information, or in other words, it belongs to at most k^t distributing subgraphs. Therefore, in $\sum_{x\in X} p_t(x)$ every call of S_t is counted no more than k^t times, that is, $\sum_{x\in X} p_t(x) \leq k^t|S_t|$. Since obviously $L = \sum_{t=1}^{T} |S_t|$, this implies the assertion. ▮

A lower bound for L depending on T is given by the minimum of the right-hand side in the inequality of Lemma 2, where the numbers $p_t(x)$ have to fulfill the restrictions given in Lemma 1. This can be formulated as a problem of integer programming, but we were not able to solve it exactly in the general case. Thus we should use certain estimates. Note that logarithms are taken base k throughout the paper.

Theorem 1. *For any nonempty $X, Y \subseteq V$ and any (X,Y)-complete IF φ on H_n^k,*

$$L(H_n^k,\varphi)\cdot k^{T(H_n^k,\varphi)} \geq \frac{1}{k}|X|\,|Y|\lceil \log|Y|\rceil - \frac{1}{k-1}\left(\frac{1}{m}-\frac{1}{k}\right)|X|\,|Y|,$$

where $m := |Y|/k^{\lceil \log|Y|\rceil-1}$.

Proof: Starting from Lemma 2 we get

$$L \geq \sum_{t=1}^{T}\left\lceil k^{-t}\sum_{x\in X} p_t(x)\right\rceil \geq \sum_{t=1}^{T}\left(k^{-t}\sum_{x\in X} p_t(x)\right) = \sum_{x\in X}\sum_{t=1}^{T}(k^{-t}p_t(x)), \quad (1)$$

and by Proposition 1,

$$L \geq \sum_{x\in X} k^{-T}\left\{(k-1)\sum_{j=0}^{T-2} k^j\left(\sum_{i=1}^{T-j-1} p_i(x)\right) + \sum_{i=1}^{T} p_i(x)\right\}.$$

Since all the coefficients in the last expression are non-negative, we may continue the inequality using Lemma 1 for all $x \in X$:

$$L \geq |X|k^{-T}\left\{(k-1)\sum_{j=0}^{T-2} k^j\lceil(k^{-j-1}|Y|-1)/(k-1)\rceil + \lceil(|Y|-1)/(k-1)\rceil\right\}. \quad (2)$$

It is well-known and easy to see that the number of points knowing a fixed unit of information can increase at most by the factor k per step (see, for example, [8]). Thus $T \geq q := \lceil \log|Y|\rceil$, because all the points of Y have to know that unit of information at the end. Consequently, the summation in (2) runs up to $q-2$ at least. But for $j \geq q-1$, $\lceil (k^{-j-1}|Y|-1)/(k-1) \rceil = 0$ in that sum. Therefore (2) becomes

$$L \geq |X| k^{-T} \left\{ (k-1) \sum_{j=0}^{q-2} k^j \lceil (k^{-j-1}|Y|-1)/(k-1) \rceil + \lceil (|Y|-1)/(k-1) \rceil \right\}.$$

From this we get our final result by using $\lceil z \rceil \geq z$ for each summand:

$$L \cdot k^T \geq |X| \left\{ \sum_{j=0}^{q-2} (k^{-1}|Y| - k^j) + (|Y|-1)/(k-1) \right\}$$

$$= |X| \left\{ \frac{1}{k}|Y|(q-1) - (k^{q-1}-1)/(k-1) + |Y|/(k-1) - 1/(k-1) \right\}$$

$$= \frac{1}{k}|X|\,|Y|q - \frac{1}{k-1}|X|\,|Y| \left(\frac{1}{m} - \frac{1}{k} \right), \text{ since } k^{q-1} = |Y|/m. \quad ∎$$

It might be useful to have another representation of this bound.

Corollary 1.

$$L \cdot k^T \geq \frac{1}{k}|X|\,|Y| \log |Y| + K \cdot |X|\,|Y|,$$

where $K := \frac{1}{k-1}\left(1 - \frac{1}{m}\right) - \frac{1}{k} \log m$ *and* $0 < K < \frac{1}{k}$.

Proof: Setting $q = \log |Y| + (1 - \log m)$ in Theorem 1, we get the first inequality. It remains to prove the bounds for the coefficient K. Since $1 = |Y|/k^{\log |Y|} < m \leq |Y|/k^{\log |Y| - 1} = k$, we have $K < \frac{1}{k-1}\left(1 - \frac{1}{k}\right) - \frac{1}{k} \log 1 = \frac{1}{k}$. On the other hand, it is a well-known fact that the term $\left(1 + \frac{1}{x}\right)^{x+1}$ for $x > 0$ monotone decreases. Substituting $x = \frac{1}{y-1}$ with $y > 1$ we get that $y^{1+(y-1)^{-1}}$ increases in y. Therefore $m^{m/(m-1)} \leq k^{k/(k-1)}$ and by elementary calculations, $K < 0$. ∎

Remarks:

1. In the course of the proof we found (2) by replacing each $\sum_{i=1}^{t} p_i(x)$ by its lower bound given in Lemma 1. Considering the problem as one of integer programming as mentioned above, we should point out that indeed equality holds in all the auxiliary conditions (Lemma 1) at the same time, if for any $x \in X$ and $t = 1, 2, \ldots, T$, we set

$$p_t(x) = \lceil (k^{t-T}|Y|-1)/(k-1) \rceil - \lceil (k^{t-1-T}|Y|-1)/(k-1) \rceil \geq 0.$$

2. Reversing the IF φ, that is, replacing each number t by $T+1-t$, we get a (Y, X)-complete IF φ^- on H_n^k, the so-called inverse IF, with the same number of calls and steps as φ has. Therefore Theorem 1 also holds after

changing X and Y, what would yield a better bound if $|X| > |Y|$. So we could explain the non-symmetry of the lower bound given in Theorem 1 by considering this under the assumption $|X| \leq |Y|$ without loss of generality.

Indeed Theorem 1 expresses the mutual dependence of L and T. It is easy to see that both parameters cannot become "small" at the same time. So one has to pay with more calls or steps for less steps or calls, respectively. This very general statement should be made more concrete in the remaining part of our paper.

The most interesting case is to estimate one of our parameters L or T if the value of the other is supposed to be as small as possible. In the following sections we should present some results concerning these situations, where we restrict ourselves to complete IF, that is, $X = Y = V$. Let us start with

3. Fast complete information flows.

From [8] we know for any complete IF φ on H_n^k, $T \geq \lceil \log n \rceil$ and this bound can be achieved if k/n. So throughout this section we consider the case k/n. Let a complete IF φ on H_n^k be chosen such that $T = \lceil \log n \rceil$, that is, with as few steps as possible.

Lemma 3. *a) For* $t = 1, 2, \ldots, T$, $|S_t| \leq n/k$.
b) For any $v \in V$, $p_1(v) = 1$.
c) $|S_1| = |S_T| = n/k$.

Proof: a) immediately follows from the fact that by definition every point can take part in at most one call per step.

b) For any $v \in V$, we get from Lemma 1 with $t = 1$, $T = \lceil \log n \rceil$:

$$p_1(v) \geq \lceil (nk^{1-\lceil \log n \rceil} - 1)/(k-1) \rceil .$$

Since $\lceil \log n \rceil - 1 < \log n$, also $n > k^{\lceil \log n \rceil - 1}$ and finally $p_1(v) \geq 1$ follows. On the other hand, $p_1(v) \leq 1$ because in the first step, v's information can be transmitted in that single call only which v itself takes part in.

c) Consequently, every point of V indeed must take part in one call of the first step, that is, $|S_1| = n/k$. Finally, we consider the inverse IF φ^- defined in Remark 2 at the end of Section 2. It is complete and contains $T = \lceil \log n \rceil$ steps. Hence, as we just proved, there are n/k calls in its first step. But the calls of this step correspond to the calls of the last step of φ, that is, we have $|S_t| = n/k$, too. ∎

This enables us to improve Theorem 1 here.

Theorem 2. *If k/n, then for any complete φ on H_n^k with $\lceil \log n \rceil$ steps,*

$$\frac{m}{k^2} n \lceil \log n \rceil + \frac{2k-3}{k^2(k-1)}(k-m)n \le L(H_n^k, \varphi) \le \frac{1}{k} n \lceil \log n \rceil,$$

where $m := n/k^{\lceil \log n \rceil - 1}$.

Proof: By Lemma 3a, $L = \sum_{t=1}^{T} |S_t| \le T \cdot \frac{n}{k} = \frac{1}{k} n \lceil \log n \rceil$. To prove the lower bound we proceed as in Section 2, but let us introduce some more notation for the purposes of Section 5. Using Lemma 3b in Lemma 1 and Lemma 3c in Lemma 2 we get a "new" optimization problem (OP):

for $v = 1, \dots, n$ and $t = 2, \dots, T-1$, $p_t(v) \ge 0$, integer;

$$\sum_{i=2}^{t} p_i(v) \ge \left\lceil \frac{1}{k-1}(k^{t-T} n - k) \right\rceil; \quad 2\frac{n}{k} + \sum_{t=2}^{T-1} \left\lceil k^{-t} \sum_{v=1}^{n} p_t(v) \right\rceil \to \text{MIN}.$$

Let us denote the minimum of this function under the auxiliary conditions by $l_1(n)$. We estimate $l_1(n)$ by determining the minimal value $l_2(n)$ of the smaller expression

$$2\frac{n}{k} + \sum_{t=2}^{T-1} \sum_{v=1}^{n} k^{-t} p_t(v)$$

as in the proof of Theorem 1. We get with Proposition 1:

$$l_2(n) = 2\frac{n}{k} + nk^{-T+1} \left\{ (k-1) \sum_{j=0}^{T-4} k^j \left\lceil \frac{1}{k-1}(k^{-j-2} n - k) \right\rceil + \left\lceil \frac{1}{k-1} \left(\frac{n}{k} - k \right) \right\rceil \right\},$$

and as in the proof of Theorem 1:

$$\begin{aligned} l_2(n) &\ge 2\frac{n}{k} + n^2 k^{-T-1}(T-3) - \frac{n}{k(k-1)} + \frac{n^2}{k-1} k^{-T} \\ &= \frac{m}{k^2} n \lceil \log n \rceil + \frac{2k-3}{k^2(k-1)}(k-m)n =: l_3(n), \end{aligned}$$

because $T = \lceil \log n \rceil$ and $k^T = \frac{nk}{m}$. ∎

Remark 3: Obviously,

$$0 \le l_1(n) - l_2(n) < T - 2 \quad \text{and}$$

$$0 \le l_2(n) - l_3(n) < nk^{-T+1} \left((k-1) \sum_{j=0}^{T-4} k^j + 1 \right) = nk^{-2},$$

that is,

$$l_3(n) \le l_1(n) \le l_3(n) + n/k^2 + \lceil \log n \rceil - 2 .$$

Thus, even the exact solution of (OP) would improve the lower order terms of the lower bound in Theorem 2 only.

By several authors ([7, 4]) it has been shown that any complete IF on H_n^k has no less than $2\lceil (n-k)/(k-1) \rceil$ calls. Because this is a linear term in n, our Theorem already shows that IF with $\lceil \log n \rceil$ steps demand essentially more calls, the order of magnitude is $n\lceil \log n \rceil$ in this case.

If n itself is a power of k, then $m = k$ and from the above theorem it immediately follows $L = \frac{n}{k}\lceil \log n \rceil$. So, in some sense, our inequalities are even strong.

The exact determination of the number of calls is also possible in some more cases.

Theorem 3. *If k/n and $n > k^{\lceil \log n \rceil} - (k-1) \cdot k^{\lceil \lceil \log n \rceil /2 \rceil}$, then any complete IF φ on H_n^k with $\lceil \log n \rceil$ steps contains exactly $\frac{n}{k}\lceil \log n \rceil$ calls.*

Proof: We prove $|S_t| = n/k$ for $t = 1, \dots, \lfloor T/2 \rfloor$. Then by considering the inverse IF φ^- the same also holds for the remaining steps, and altogether $L = \sum_{t=1}^{T} |S_t| = \frac{n}{k} T$ as asserted.

Let us assume that in the step t, where $t \le \lfloor T/2 \rfloor$, the point $v \in V$ does not take part in any call. We already mentioned that at most k^{t-1} points know v's item of information before S_t. During S_t at most $(k-1)(k^{t-1}-1)$ additional points can learn it because v itself does not work. Hence, after S_t at most $k^t - (k-1)$ points know that item. In the rest of the IF this number can increase at most by the factor k per step, that is, at the end at most

$$k^{T-t}(k^t - (k-1)) = k^T - (k-1)k^{T-t} \le k^T - (k-1)k^{\lceil T/2 \rceil} < n$$

points will know v's initial item of information. Since this contradicts the completeness of φ, each point must take part in a call of S_t, that is, $|S_t| = n/k$. ∎

If n lies at the beginning of the interval $\left(k^{\lceil \log n \rceil - 1}; k^{\lceil \log n \rceil}\right]$, then in Theorem 2 a great gap remains. So the exact determination of the minimal value for the number of calls is still an open problem for these n.

4. Short complete information flows.

Let us consider now a complete IF φ on H_n^k with

$$L = \begin{cases} \lceil \frac{n-k}{k-1} \rceil + \lceil \frac{n}{k} \rceil & \text{if } n \le k^2 \\ 2\lceil \frac{n-k}{k-1} \rceil & \text{if } n \ge k^2 . \end{cases}$$

From [7] and [4] we know that this is the smallest possible length for any complete IF. One can easily check that the inequality of Theorem 1 is fulfilled by $L = \lceil (n-k)/(k-1) \rceil + \lceil n/k \rceil$ and $T = \lceil \log n \rceil$ if $n \le k^2$. Hence we cannot get a better estimate than the general lower bound for the number of steps in this case.

Theorem 4. *If $n \geq k^2$, then for any complete IF φ on H_n^k with $2\lceil \frac{n-k}{k-1} \rceil$ calls, $T(H,\varphi) > \log n + \log\log n - \log \frac{2k}{k-1}$.*

Proof: For $|X| = |Y| = n$ we get from Theorem 1:

$$Lk^T \geq \frac{1}{k} n^2 \lceil \log n \rceil \geq \frac{1}{k} n^2 \log n$$

and $T \geq 2 \log n + \log\log n - 1 - \log L$. But $L < 2\left(\frac{n-k}{k-1} + 1\right) = 2\frac{n-k}{k-1} < 2\frac{n}{k-1}$, that is, $\log L < \log n + \log \frac{2}{k-1}$. ∎

However, we cannot give any construction achieving this bound. So we should present a short complete IF with $2\lceil \log n \rceil$ steps on the complete k-uniform hypergraph on n points.

Lemma 4. *If $n = 1 + r(k-1)$ for some non-negative integer r, then there is an IF χ_n with r calls and $\lceil \log n \rceil$ steps, such that at its end each point knows the information of 1.*

Proof: To each point v we join the uniquely determined vector $\underline{v} = (v_1, v_2, \ldots, v_T)$ with $v_i \in \{0, 1, \ldots, k-1\}$ and $v - 1 = \sum_{i=1}^{T} v_i k^{i-1}$. Obviously $\underline{1} = (0, 0, \ldots, 0)$. Now we define χ_n with the help of this vector-representation: The first step consists of one call between the points $(j, 0, 0, \ldots, 0)$ for $j = 0, 1, \ldots, k-1$. After this all these points know 1's information.

For $t = 2, \ldots, T-1$, during the tth step it is carried out a call on every edge of the type

$$\{(i_1, i_2, \ldots, i_{t-1}, j, 0, \ldots, 0) : \ j = 0, \ldots, k-1\},$$

where $(i_1, i_2, \ldots, i_{t-1})$ is any $(t-1)$-tuple with elements of $\{0, 1, \ldots, k-1\}$. Since $k^t \leq k^{T-1} < n$ indeed all of these vectors correspond to points of V. So such a step contains k^{t-1} calls, and after them each of the k^t points $w \in V$ with $w_{t+1} = w_{t+2} = \ldots = w_T = 0$ knows the information originally known to 1.

We have $n - k^{T-1} = r(k-1) - (k^{T-1} - 1) = (k-1)\,(r - (1 + k + \ldots + k^{T-2}))$ $=: (k-1)r'$. Hence the remaining $n - k^{T-1}$ points can be shared into r' disjoint groups $V_1, V_2, \ldots, V_{r'}$ of exactly $k-1$ elements. Then for $i = 1, 2, \ldots, r'$, in the last step S_T of χ_n a call takes place on the edge $V_i \cup \{i\}$. Since $0 < n - k^{T-1} \leq k^T - k^{T-1} = (k-1)k^{T-1}$, we have $r' \leq k^{T-1}$, that is, the point $i \leq r' \leq k^{T-1}$ already after the first $T-1$ steps knew 1's information. Therefore after S_T all n points know the information of 1, and $L = \sum_{t=1}^{T-1} k^{t-1} + r' = r$ by the definition of r'. ∎

Figure 1 shows the example $k = 3, n = 15 = 1 + 7 \cdot 2$. We denoted each point $v \in V$ by $\underline{v}$ and the calls of the 1. or 2. or 3. step by - - - - - or - - - or ———, respectively.

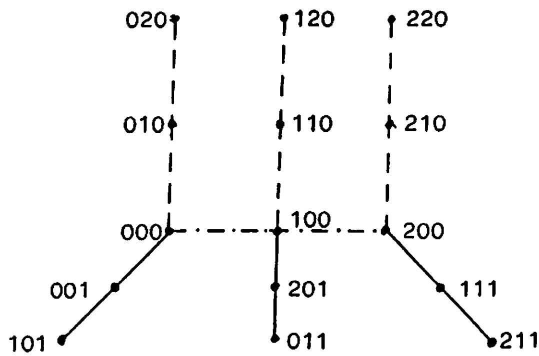


Figure 1

In the following proof we also use the inverse IF χ_n^-, which ensures the transmission of the information of all points of V to 1.

Theorem 5. *If $n = k^2 + r(k-1)$ for some non-negative integer r, then there is a complete IF with $2\lceil\frac{n-k}{k-1}\rceil$ calls and no more than $2\lceil\log n\rceil$ steps.*

Proof: Let be $r \equiv s \bmod k^2$ $(0 \le s < k^2)$, $r_i := \lfloor r/k^2 \rfloor$ for $i = 1,\ldots,k^2 - s$ and $r_i := \lceil r/k^2 \rceil$ for $i = k^2 - s + 1,\ldots,k^2$. Then $\sum_{i=1}^{k^2} r_i = r$, and we may share the $n - k^2 = r(k-1)$ points $k^2 + 1,\ldots,n \in V$ in exactly k^2 disjoint subsets $V_1,\ldots,V_{k^2}$ with $|V_i| = r_i(k-1)$. Now on each subset $V_i \cup \{i\}$ we carry out $\chi^-_{|V_i|+1}$, such that i learns the information of all points of V_i. By Lemma 9 this requires $\sum_{i=1}^{k^2} r_i = r$ calls and $\lceil\log(1+|V_i|)\rceil \le \lceil\log(1+\lceil r/k^2\rceil(k-1))\rceil$ steps.

After this there take place parallel calls on the edges $\{(i-1)\cdot k + j\colon j = 1,2,\ldots,k\}$ for $i = 1,2,\ldots,k$ in one step, and in the next step parallel calls on the edges $\{(i-1)\cdot k + j\colon i = 1,2,\ldots,k\}$ for $j = 1,2,\ldots,k$. Altogether these are $2k$ calls in 2 steps, and after them each point $1,2,\ldots,k^2$ knows all information.

Finally we carry out $\chi_{|V_i|+1}$ on $V_i \cup \{i\}$, such that all information is transmitted from i to the points of $V_i\cdot$ $(i = 1,\ldots,k^2)$. This IF has $2k+2r$ calls and $2\lceil\log(1+\lceil r/k^2\rceil(k-1))\rceil + 2$ steps. But

$$2\left\lceil\frac{n-k}{k-1}\right\rceil = 2\left\lceil\frac{k^2-k+r(k-1)}{k-1}\right\rceil = 2\lceil k+r\rceil = 2k+2r \quad \text{and}$$

$$T \le 2\left\lceil\log\left(1+\frac{r}{k^2}(k-1)+(k-1)\right)\right\rceil + 2$$

$$\le 2\left\lceil\log\left(k+\frac{r}{k}(k-1)\right)\right\rceil + 2 = 2\left\lceil\log\frac{n}{k}\right\rceil + 2 = 2\lceil\log n\rceil. \quad \blacksquare$$

In the case $k = 2$ one can use a complete IF on 8 points with 3 steps and 12 calls (see Section 5, Figure 2) instead of the above-used mid-part of the IF. Then even $T \leq 2\lceil \log_2 n \rceil - 3$ follows. It remains a gap to the lower bound proved in Theorem 4. We conjecture that the exact minimum number of steps is nearer to this constructed value.

5. Ideal information flows.

Let us turn now to our original question. A complete IF φ on the complete k-uniform hypergraph H_n^k on n points is called ideal, if there is no complete IF φ' on H_n^k with $L(H_n^k, \varphi') < L(H_n^k, \varphi)$ and no complete IF φ'' with $T(H_n^k, \varphi'') < T(H_n^k, \varphi)$, that is, an ideal IF has minimum length and minimum time among all complete IF. Again we restrict ourselves to the case k/n, where we know that any ideal IF on H_n^k has exactly $\lceil \log n \rceil$ steps and $\lceil \frac{n-k}{k-1} \rceil + \lceil \frac{n}{k} \rceil$ (if $n \leq k^2$) or $2\lceil \frac{n-k}{k-1} \rceil$ (if $n \geq k^2$) calls.

Theorem 6.

a) *If* $n \leq k^2$ *and* k/n, *then there is an ideal IF on* H_n^k.
b) *If* $k = 2$, *then there is an ideal IF on the complete graph* K_n *for every even* $n \leq 2^3 = 8$.

Proof:

a) The statement is obvious for $n = k$, because a complete exchange of information can be realized with one call. Let be $k < n = pk$. We consider the IF defined in [8]. It consists of the steps

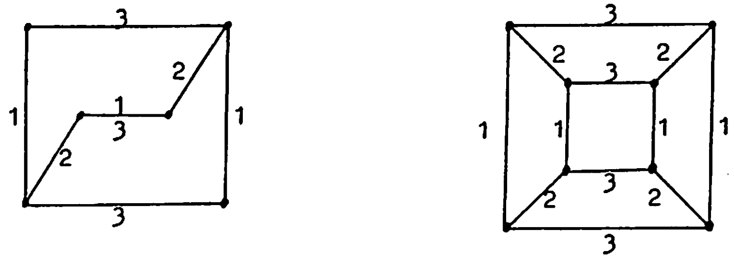
$$S_t = \left\{ \left(\left\{ s + j\frac{k^t - 1}{k-1} + 1 : \; j = 0, 1, \ldots, k-1 \right\} ; t \right) : \; s = 0, k, 2k, \ldots, (p-1)k \right\}$$

for $t = 1, 2, \ldots, \lceil \log n \rceil$, where the numbers of the points have to be computed modulo n. In the mentioned paper the completeness of this IF has been proved. Furthermore we have $T = \lceil \log n \rceil = 2$, and $L = 2p$, because in every step exactly p calls take place. But since $p \leq k$ we have $p - 1 < (p-1)\frac{k}{k-1} = (1 + \frac{1}{k-1})(p-1) = (p-1) + \frac{p-1}{k-1} \leq p$, that is, $\lceil (p-1)\frac{k}{k-1} \rceil = p$ and $L = 2p = \lceil (p-1)\frac{k}{k-1} \rceil + p = \lceil \frac{n-k}{k-1} \rceil + \lceil \frac{n}{k} \rceil$. Hence the above defined IF indeed turns out to be ideal.

b) For the remaining cases $n = 6$ and $n = 8$ we give an example for an ideal IF in Figure 2. ∎

An immediate consequence of Theorem 1 is that ideal IF cannot exist for arbitrarily large n. More precisely we get the following results.

Lemma 5. *Suppose* $n > k^2$ *and* k/n. *If there is some ideal IF on* H_n^k, *then* $n < B \cdot k^{\lceil \log n \rceil - 1}$ *where* $B = \frac{3}{(\lceil \log n \rceil - 3)(1 - \frac{1}{k}) + 1}$.



$n = 6$
$L = 2n - 4 = 8$
$T = \lceil \log_2 n \rceil = 3$

$n = 8$
$L = 2n - 4 = 12$
$T = \lceil \log_2 n \rceil = 3$

Figure 2

Proof: On the one hand we have

$$L = 2\left\lceil \frac{n-k}{k-1} \right\rceil \leq 2\left(\frac{n-k}{k-1} + \frac{k-2}{k-1}\right) = \frac{2n-4}{k-1}.$$

On the other hand we may apply Theorem 2. Using $T = \lceil \log n \rceil$ and $m = n/k^{T-1}$ this yields:

$$L \geq Tk^{-T-1}n^2 + \frac{2k-3}{k(k-1)}n - k^{-T-1} \cdot \frac{2k-3}{k-1}n^2.$$

Altogether this means

$$k^{-T-1} \cdot \frac{T(k-1) - (2k-3)}{(k-1)}n^2 + \frac{2k-3}{k(k-1)}n \leq \frac{2}{k-1}n - \frac{4}{k-1},$$

and

$$\frac{(T-3)(k-1)+k}{k^T}n^2 - 3n + 4k \leq 0, \tag{3}$$

and with $(T-3)(k-1) + k = k \cdot \frac{3}{B}$: $\frac{3}{Bk^{T-1}}n^2 - 3n < 0$. This finally implies $n < B \cdot k^{T-1}$. ∎

Theorem 7. *Let be $k \geq 3$. Then there does not exist any ideal IF for H_n^k, if k/n and: $n \geq \left(1 + \frac{2}{3k-2}\right)k^4$ or $\left(\frac{3}{2} + \frac{3}{4k-2}\right)k^3 \leq n \leq k^4$ or $3k^2 \leq n \leq k^3$.*

Proof: We prove that for these values of n the necessary condition of Lemma 5 is not fulfilled.

If $n > k^5$, then $\lceil \log n \rceil \geq 6$ and $\mathcal{B} \leq \frac{3}{3(1-1/k)+1} \leq 1$.
But $\log n > \lceil \log n \rceil - 1$, that is, $n > k^{\lceil \log n \rceil - 1} \geq \mathcal{B} \cdot k^{\lceil \log n \rceil - 1}$.
If $k^4 < n \leq k^5$, then $\lceil \log n \rceil = 5$ and $\mathcal{B} = \frac{3}{2(1-1/k)+1} = \frac{3k}{3k-2} = 1 + \frac{2}{3k-2}$.
If $k^3 < n \leq k^4$, then $\lceil \log n \rceil = 4$ and $\mathcal{B} = \frac{3}{1-1/k+1} = \frac{3k}{2k-1} = \frac{3}{2} + \frac{3}{4k-2}$.
If $k^2 < n \leq k^3$, then $\lceil \log n \rceil = 3$ and $\mathcal{B} = 3$.
So in all these cases $n \geq \mathcal{B}k^{\lceil \log n \rceil - 1}$. ∎

Without any gap we therefore proved the non-existence of ideal IF for $n \geq k^4 + \frac{6}{7}k^3$.

In the last part of our paper we investigate the question for the "classical" case $k = 2$. For a well-ordered arrangement we share the proof into some propositions. We should remark that an ideal IF has $2n - 4$ calls and $T = \lceil \log_2 n \rceil$ steps.

Proposition 2. *For even $n \geq 14$, $n \neq 18$, there is no ideal IF for $K_n = H_n^2$.*

Proof: For $n > 2^6 = 64, T \geq 7$, and in Lemma 5: $\mathcal{B} \leq 1$. Therefore, $n > 2^T \geq \mathcal{B} \cdot k^{T-1}$, and the assertion follows from Lemma 5. If there is some ideal IF on K_n, then inequality (3) of the proof of Lemma 5 holds. Here it reads as

$$\frac{T-1}{2^T}n^2 - 3n + 8 \leq 0.$$

We are interested in its solution

$$n \leq \frac{3}{T-1} \cdot 2^{T-1} \cdot \left(1 + \sqrt{1 - \frac{8(T-1)}{9 \cdot 2^{T-2}}}\right) =: N.$$

We have

$$\lfloor N \rfloor = \begin{cases} 12 & \text{if} \quad T = 4 \\ 20 & \text{if} \quad T = 5 \\ 35 & \text{if} \quad T = 6, \end{cases}$$

that is, our assertion also holds for $14 \leq n \leq 16$, $22 \leq n \leq 32, 36 \leq n \leq 64$. In the remaining cases the estimate of Theorem 2 is not strong enough. Therefore we compute the exact minimum $l_2(n)$, which has been introduced in the proof of Theorem 2. One gets

$$L \geq l_2(n) = n + n \cdot 2^{-T+1} \left\{ \sum_{j=0}^{T-4} 2^j \lceil 2^{-j-2}n - 2 \rceil + \left\lceil \frac{n}{2} - 2 \right\rceil \right\}$$

$$= \frac{n^2}{2^T} + \frac{n}{2} + \frac{n}{2^{T-1}} \sum_{j=0}^{T-4} 2^j \left\lceil \frac{n}{2^{j+2}} \right\rceil.$$

It is $l_2(34) = 68 > 64 = 2 \cdot 34 - 4$ and $l_2(20) = 36 + \frac{1}{4} > 36 = 2 \cdot 20 - 4$, that is, the proof is completed for $n = 34$ and $n = 20$. ∎

For the next open value $n = 18$, even the computation of $l_2(n)$ does not suffice. But an estimate of $l_1(n)$ yields

Proposition 3. *There is no ideal IF on* K_{18}.

Proof: For any complete IF on K_{18} with $T = \lceil \log_2 18 \rceil = 5$ we have in (OP): $p_2(v) \geq 1, p_2(v) + p_3(v) \geq 3, p_2(v) + p_3(v) + p_4(v) \geq 7$ for $v = 1, \ldots, 18$. For $t = 2, 3, 4$ let us consider the sums

$$s_t := \sum_{v=1}^{18} p_t(v)$$

and formulate a new problem (OP′) in these new variables:

$$\begin{gathered} s_2 \geq 18 \qquad s_3 \geq 54 - s_2 \qquad s_4 \geq 126 - s_2 - s_3 \\ 18 + \left\lceil \frac{1}{4} s_2 \right\rceil + \left\lceil \frac{1}{8} s_3 \right\rceil + \left\lceil \frac{1}{16} s_4 \right\rceil =: l_1(s_2, s_3, s_4) \to \text{MIN.} \end{gathered} \tag{4}$$

In the following we solve (OP′). For this we need the fact that for arbitrary real u and v,

$$\lceil u - 1 \rceil + \left\lceil v + \frac{1}{2} \right\rceil = \lceil u \rceil + \left\lceil v + \frac{1}{2} - 1 \right\rceil \leq \lceil u \rceil + \lceil v \rceil. \tag{5}$$

The inequalities of (OP′) have solutions, for instance the triple $(18, 36, 72)$. So among all those, which minimize $l_1(s_2, s_3, s_4)$ we may select one triple with minimal first coordinate, say (s_2', s_3', s_4'). Assume $s_2' \geq 22$. Then even $(s_2' - 4, s_3' + 4, s_4')$ fulfills (4), but by (5), $l_1(s_2' - 4, s_3' + 4, s_4') \leq l_1(s_2', s_3', s_4')$. This contradicts to the choice of (s_2', s_3', s_4'), that is, $s_2' \leq 21$.

Now among all solutions of (OP′) with minimal first coordinate we select one with minimal second coordinate, let it be denoted by (s_2', s_3'', s_4''). Then the analogous way as above leads to $s_3'' \leq 61 - s_2'$. Finally from (4) it immediately follows that $(s_2', s_3'', 126 - s_2' - s_3'')$ is solution of (4) and $l_1(s_2', s_3'', s_4'') \geq l_1(s_2', s_3'', 126 - s_2' - s_3'')$.

Consequently, among all triples which fulfill (4) and minimize $l_1(s_2, s_3, s_4)$, we find one with $18 \leq s_2 \leq 21$, $54 - s_2 \leq s_3 \leq 61 - s_2$ and $s_4 = 126 - s_2 - s_3$. But for this we have:

$$\begin{gathered} \left\lceil \frac{1}{4} s_2 \right\rceil \geq 5; \quad s_3 \geq 33, \text{ that is, } \left\lceil \frac{1}{8} s_3 \right\rceil \geq 5 \text{ and} \\ s_4 = 126 - (s_2 + s_3) \geq 65, \text{ that is, } \left\lceil \frac{1}{16} s_4 \right\rceil \geq 5. \end{gathered}$$

Therefore $l_1(s_2, s_3, s_4) \geq 18 + 3 \cdot 5 = 33$. Since each solution of our original problem (OP) generates an according solution of (OP′), the minimal value $l_1(18)$ is no smaller than the minimum of $l_1(s_2, s_3, s_4)$, hence $l_1(18) \geq 33 > 32 = 2 \cdot 18 - 4$. ∎

The investigation of the remaining cases $n = 10$ and $n = 12$ is somewhat harder. For both we would get $l_1(n) = 2n - 4$, but indeed ideal IF do not exist, as the following propositions show.

Proposition 4. *Any complete IF on K_{12} with 4 steps contains at least 21 calls.*

Proof: From Lemma 3 c) we have $|S_1| = |S_4| = 6$. Without loss of generality we may assume $|S_2| \leq |S_3|$, because otherwise the inverse IF can be considered. The inequalities of (OP) read as $p_2(v) \geq 1, p_2(v) + p_3(v) \geq 4$ for all $v \in V$. On the other hand, $p_2(v) \leq 2$ because after the first step at most 2 points know v's information. Let us define

$$W_i := \{v \in V:\ p_2(v) = i\} \text{ for } i = 1, 2.$$

As we already showed in the course of the proof of Lemma 2, for each step t,

$$\sum_{v \in V} p_t(v) \leq 2^t |S_t|. \tag{6}$$

Consequently, $|S_2| \geq \frac{1}{4} \cdot \sum_{v=1}^{12} p_2(v) \geq 3$.

Case 1: $|S_2| = 3$. We have $\sum_{v=1}^{12} p_2(v) \leq 4 \cdot 3 = 12$ by (6), that is, $p_2(v) = 1$ for all $v \in V$. For any $v \in W_1$, $p_3(v) \geq 4 - p_2(v) = 3$. On the other hand, v's information is known to at most 2 other points before the third step because it has been transmitted in one call of the first resp. second step. Consequently, v itself takes part in a call of S_3 (and so do both the other points which know that unit of information). Since $W_1 = V$, all points participate in calls of S_3, that is $|S_3| = 6$ and $L = 6 + 3 + 6 + 6 = 21$.

Case 2: $|S_2| = 4$. Then analogously

$$16 = 2^2 \cdot 4 \geq \sum_{v=1}^{12} p_2(v) = |W_1| + (12 - |W_1|) \cdot 2 = 24 - |W_1|, \text{ that is, } |W_1| \geq 8.$$

Case 2.1: S_1 or S_2 contains a call between a point $v \in W_1$ and a point $w \in W_2$. Then w knows v's information after the second step, and as we have shown in Case 1, all the points of W_1 and w have to take part in calls of S_3. Hence $|S_3| \geq 5$, because these are at least 9 points.

Case 2.2: There is no such call. Then the information of points of W_2 cannot be conveyed by points of W_1 in S_3. But for every $w \in W_2$, $p_3(w) \geq 4 - p_2(w) = 2$, that is, at least one point of W_2 has to take part in a call of S_3 besides the 8 points of W_1. So here $|S_3| \geq 5$, too. Thus, in our second case $L \geq 6 + 4 + 5 + 6 = 21$.

Case 3: $|S_2| \geq 5$. Then by $|S_3| \geq |S_2| \geq 5$, $L \geq 6 + 5 + 5 + 6 = 22$. ∎

Figure 3 shows a complete IF with 4 steps and 21 calls on a graph on 12 points. Hence the bound from Proposition 5 indeed can be achieved.

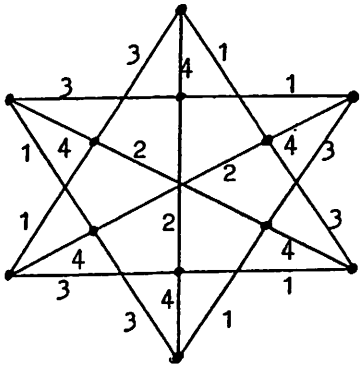


Figure 3

Proposition 5. *Any complete IF on K_{10} with 4 steps contains at least 17 calls.*

Proof: We already know that it has to contain at least $2 \cdot 10 - 4 = 16$ calls. Assume, there is a complete IF on K_{10} with exactly 16 calls. By Lemma 3 c) we have $|S_1| = |S_4| = 5$, and again we may assume $|S_2| \leq |S_3|$ without loss of generality. Then $16 = 10 + |S_2| + |S_3| \geq 10 + 2 \cdot |S_2|$, that is, $|S_2| \leq 3$. On the other hand we get from (OP): $p_2(v) \geq 1$ and by Lemma 6: $|S_2| \geq 2^{-2} \cdot 10 > 2$. Hence $|S_2| = |S_3| = 3$. We continue the proof with the help of the minimal order of the calls introduced in [2]. The definition can be found there, all the properties used in our proof are immediate consequences of this definition. Let $\trianglelefteq$ or $\triangleleft$ denote the order or its covering relation, respectively. Because all points take part in calls of S_1, each call of S_2 has exactly 2 predecessors in S_1. Conversely, by $p_2(v) \geq 1$, each call of S_1 has at least one successor in S_2. The analogous statements hold for S_4 (instead of S_1) and S_3 (instead of S_2). Thus the minimal order contains the relations shown in Figure 4, of course, up to isomorphisms.

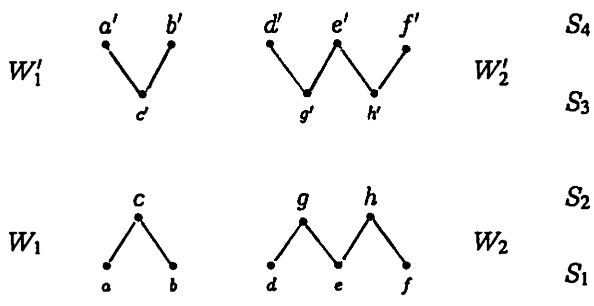


Figure 4

In the following we use the notations given in Figure 4, where

$$W_1 = \{a, b, c\} \qquad W_2 = \{d, e, f, g, h\}$$
$$W_1' = \{a', b', c'\} \qquad W_2' = \{d', e', f', g', h'\}.$$

Furthermore we use the two following basic properties of the minimal order:
(I) Every call of S_1 is smaller than every call of S_4.
(II) Every call has at most 2 immediate predecessors and at most 2 immediate successors.
Because W_2 does not contain any greatest element, d' and g' have to cover at least 2 elements of W_2, that $d, e, f \trianglelefteq d'$. But d' and g' cover at most 3 elements, that is, at most one element of W_1. Because $a, b \trianglelefteq d'$, this has to be c. Analogously, $c \triangleleft h'$ or $c \triangleleft f'$. Therefore c cannot have any successor in W_1'. Because, except of c, a has no more than one immediate successor, but $a \trianglelefteq a'$ and $a \trianglelefteq b'$, we found $a \triangleleft c'$ and analogously $b \triangleleft c'$. Thus c' has no predecessors in W_1. Consequently a' has to cover exactly one element of W_2, such that $d, e, f \trianglelefteq a'$. Since W_2 does not contain any greatest element, this is impossible. Thus we got a contradiction to (I) or, equivalently, to the completeness of our IF, and $L > 16$ is proved. ∎

Figure 5 shows a complete IF with 4 steps and 17 calls on a graph on 10 points. Hence the bound from Proposition 6 can be achieved.

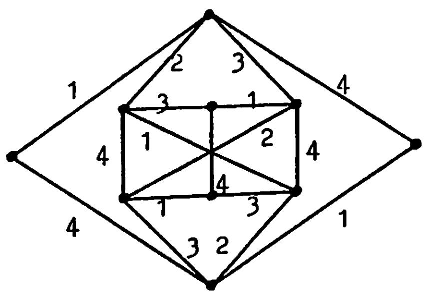


Figure 5

So we altogether proved

Theorem 8. *For even n, there is an ideal IF on K_n if and only if $n \leq 8$.*

This result means that the demand for minimum length and time is too strong. To find "good" IF with respect to both parameters, one should introduce and minimize mixed parameters. The function $F = L \cdot k^T$ is one possibility, for which our Theorem 1 gives a first lower bound.

References

1. B. Baker and R. Shostak, *Gossips and telephones*, Discrete Math. **2** (1972), 191-193.
2. R.T. Bumby, *A problem with telephones*, SIAM J. Algebraic Discrete Methods **2** (1981), 13-18.
3. A. Hajnal, E.C. Milner and E. Szemeredi, *A cure for the telephone disease*, Canad. Math. Bull. **15** (1972), 447-450.
4. D.J. Kleitman and J.B. Shearer, *Further gossip problems*, Discrete Math. **30** (1980), 151-156.
5. W. Knödel, *New gossips and telephones*, Discrete Math. **13** (1975), 95.
6. R. Labahn, *Information flows on hypergraphs*. submitted to publication.
7. K. Lebensold, *Efficient communication by phone calls*, Studies in Appl. Math. **52** (1973), 345-358.
8. P. Schmitt, *Spreading information by conferences*, Discrete Math. **15** (1976), 305-306.
9. R. Tijdeman, *On a telephone problem*, Nieuw Arch. Wisk. **19** (1971), 188-192.
10. D.B. West, *A class of solutions to the gossip problem, Part I*, Discrete Math. **39** (1982), 307-326.