Some New Simple t-Designs

D. L. Kreher, Y. M. Chee, D. de Caen, C. J. Colbourn and E. S. Kramer

ABSTRACT

The concept of using basis reduction for finding $t-(v,k,\lambda)$ designs without repeated blocks was introduced by D. L. Kreher and S. P. Radziszowski at the Seventeenth Southeastern International Conference on Combinatorics, Graph Theory and Computing. This tool and other algorithms were packaged into a system of programs that was called the design theory toolchest. It was distributed to several researchers at different institutions. This paper reports the many new open parameter situations that were settled using this toolchest.

1. Introduction

A $t-(v,k,\lambda)$ design $(X,\mathcal{D})$ is a family of k-element subsets $\mathcal{D}$ from a v-element set X such that every t–element subset $T \subseteq X$ is contained in exactly λ of the k-element subsets in $\mathcal{D}$. A current listing of the settled parameter situations for $t-(v,k,\lambda)$ designs is provided in [CCK]. A group $G \leq Sym(X)$ is an automorphism group of a $t-(v,k,\lambda)$ design $(X,\mathcal{D})$ if $\mathcal{D}$ is a union of orbits of k-element subsets under G. For each G-orbit Δ of t-element subsets and for each G-orbit Γ of k-element subsets define $A_{tk}[\Delta,\Gamma]$ to be $|\{K \in \Gamma : K \supseteq T\}|$, where $T \in \Delta$. This value is independent of the choice of T. If N_i is the number of G-orbits of i-element subsets, then A_{tk} is an N_t by N_k nonnegative integer valued matrix. In 1973 Kramer and Mesner [KM] made the following observation:

A $t-(v,k,\lambda)$ design exists with $G \leq Sym(X)$ as an automorphism group if and only if there is a (0,1)-solution U to the matrix equation

$$A_{tk}U = \lambda J, \tag{1}$$

where: $J=[1,1,1,...,1]^T$.

Several attempts were made to design a computer program for finding solutions to equation (1) among the most successful is the so called Basis Reduction algorithm designed and implemented by Kreher and Radziszowski [KR1,KR2]. The central idea of this algorithm is to find a (0,1)-vector U such that:

$$\begin{bmatrix} I & 0 \\ A_{tk} & -\lambda J \end{bmatrix} \begin{bmatrix} U \\ d \end{bmatrix} = [U^T,0,\cdots,0]^T.$$

Such a U gives a $t-(v,k,d\cdot\lambda)$ design with automorphism group G for some non-negative integer d. They observe that if $B=\begin{bmatrix} I & 0 \\ A_{tk} & -\lambda J \end{bmatrix}$ and Γ is the lattice obtained as the integer span of the columns of B then

$$\mathbf{U} = [U^T,0,\cdots,0]^T \text{ is a short vector of } \Gamma \ (i.e.\ \ \|\mathbf{U}\|^2 < N_k).$$

Finally they implemented several methods of efficiently transforming the basis B to a new basis B' of Γ such that

$$\sum\{\|V\|^2 : V\in B\} \geq \sum\{\|V\|^2 : V\in B'\}.$$

Repeated application of these methods to the basis causes basis vectors to become shorter and shorter and a solution to eqn. (1) very often appear in the basis. Using these methods and other tools found in the design theory toolchest we were able to settle all of the parameter situations found in Table I.

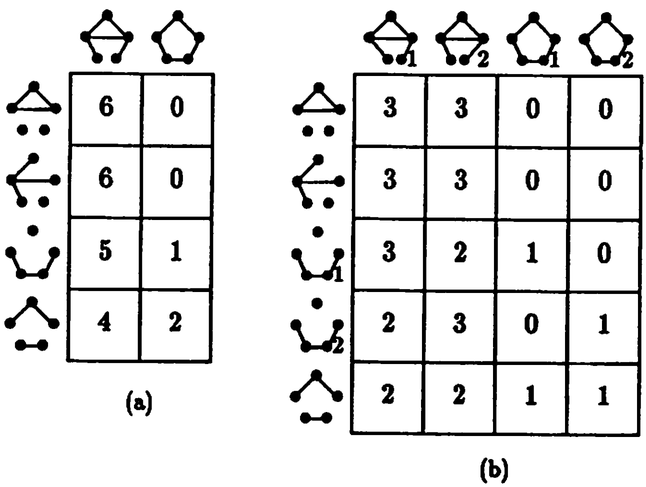
TABLE I

Parameter Situation		Automorphism group
2-(18,7,λ)	$\lambda\equiv 0\ (mod\, 336)$	$SAF(17)_\infty$
2-(20,4,λ)	$\lambda\equiv 0\ (mod\, 3)$	$SAF(19)_\infty$
3-(16,7,λ)	$\lambda=10$	Frobenius of order 16·5
3-(19,7,λ)	all possible λ's	$AF(19)$
3-(19,9,λ)	$\lambda\in\{112,196,280,364,924,1204,$ $1764,2044,2604,2884,3444,3724\}$	$AF(19)$
3-(20,5,λ)	$\lambda\in\{18,28,48,58\}$	Hypergraphical
	$\lambda\in\{24,54\}$	Semi-hypergraphical
	$\lambda\in\{12,22,34,42,52,64\}$	H_∞ where H is Frobenius of order 19·6.
	$\lambda\in\{50,56\}$	$D_4\ wr\ A_5$
3-(21,5,λ)	$\lambda\in\{15,39,48,69,75\}$	Semi-graphical
	$\lambda\in\{30,33,39,69,75\}$	Graphical
3-(21,6,λ)	$\lambda\in\{40,68,108,120,136,$ $160,208,220,236,248,268,$ $280,296,320,328,340,356,$ $376,388,400,168,176,256,$ $288,336,368\}$.	Semi-graphical
3-(23,8,8s)	$s\geq 2$	$AF(23)$
3-(23,9,24s)	$s\geq 2$	$AF(23)$
3-(25,4,λ)	$\lambda\in\{2,8,10\}$	$C_5\ wr\ A_5$
3-(26,6,λ)	$\lambda\equiv 0\ or\ 1\ (mod\, 10)\ \lambda\notin\{10,11\}$	$PSL_2(25)$
4-(20,5,λ)	$\lambda=4$	$AF(19)_\infty$
4-(20,6,λ)	$\lambda=30$	$AF(19)_\infty$
4-(21,6,λ)	$\lambda\in\{36,40,60\}$	$PSL_2(19)_\infty$
4-(23,5,λ)	$\lambda\in\{2,4,5,6,7,8,9\}$	$AF(23)$
4-(29,5,λ)	$\lambda=5$	$AF(29)$
5-(24,6,λ)	all possible λ's	$PSL_2(23)_\infty$
5-(24,7,λ)	all possible λ's	$PSL_2(23)_\infty$

In Table I the following notation is used for describing automorphism groups. If $q=p^e$ where p is a prime, then $AF(q)=\{x\rightarrow\alpha\cdot x+\beta : \alpha,\beta\in GF(q),\ \alpha\neq 0\}$ is the so called affine group and has order $q\cdot(q-1)$. The representation of this group we use is the natural action on the elements of $GF(q)$. We denote by $SAF(q)=\{x\rightarrow\alpha^2\cdot x+\beta:\alpha,\beta\in GF(q),\ \alpha\neq 0\}$ the special affine group a subgroup of $AF(q)$. Any other transitive subgroup of $AF(q)$ of order $q\cdot n$, $n\mid(q-1)$ is referred to as Frobenius of order $q\cdot n$. $PSL_2(p)$ is the projective special linear group acting on the projective line. The terms hypergraphical, graphical, semi-graphical and semi-hypergraphical are described in the next section. If G is a group acting on a set Y with $\infty\notin Y$, then we denote by G_∞ the representation of G on $X=Y\cup\{\infty\}$ obtained by adding the point ∞ fixed by all group elements. Let G and H be permutation groups acting on sets A and B respectively; $G\ wr\ H$ denotes the wreath product of G by H acting on $A\times B$.

2. Graphical, Semi-Graphical, Hypergraphical and Semi-Hypergraphical designs

A $t-(\binom{p}{2},k,\lambda)$ design $(X,\mathcal{D})$ is said to be *graphical* if X is the set of all $v=\binom{p}{2}$ labeled edges of the undirected complete graph K_p and if $B\in\mathcal{D}$, than all subgraphs of K_p isomorphic to B are also in $\mathcal{D}$. Thus $(X,\mathcal{D})$ has the full symmetric group S_p as an automorphism group. If the $t-(\binom{p}{2},k,\lambda)$ design $(X,\mathcal{D})$ only has the alternating group A_p as an automorphism group then we say that it is semi-graphical. An example of these designs are given in Figure 1 and the graphical and semi-graphical designs we found are presented in the appendix. Two orbits under A_p whose union is a single isomorphism class of graphs is indicated by adding the subscripts 1 and 2 to the graph.



(a)

	6	0
	6	0
	5	1
	4	2

(b)

	1	2	1	2
	3	3	0	0
	3	3	0	0
1	3	2	1	0
2	2	3	0	1
	2	2	1	1

FIGURE 1:(a) incidence matrix of a graphical 3-(10,5,6) design.
(b) incidence matrix of the design in (a) partitioned into two semi-graphical 3-(10,5,3) designs.

The generalization from graphical to *hypergraphical* designs is straight forward. We simply consider the action of the full symmetric group on $X = \binom{P}{d}$ the collection of all d-element subsets of the p-element set P. Many of the 3-designs on $20 = \binom{6}{3}$ points were found this way. They appear in the appendix.

3. Concluding remarks

Although we found many solutions in several of the parameter situations given in Table I, space prohibited the inclusion of more than one in the appendix. During this investigation we have realized that many improvements to the tools in the design theory toolchest can be made. Research is planned to make these improvements in the near future.

4. Acknowledgements

The graphical 3-(21,5,3) in section A9 first appeared in [K] we included it again in this paper because it appears as a subdesign of a graphical 3-(21,5,33) we construct.

This research was encouraged by many of our colleagues and we thank them all. In particular we would like to express our appreciation to Jeff Dinitz whose respect for block designers is unsurpassed. Also the author ESK wishes to thank Spyros S. Magliveras for some software used in obtaining some designs.

Finally, this work was supported by the National Science Foundation under grant CCR-8711229 (DLK) by the National Security Agency under grant MSP-076-89 (ESK) and by the Natural Science and Engineering Research Council of Canada under grants URF0038628 (DdeC) and A0579 (CJC).

Author Addresses

D. de Caen
Dept. of Mathematics
Queens University
Kingston, Ontario K7L 3N6
CANADA

Y.M. Chee
Dept. of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1
CANADA

C.J. Colbourn
Dept. of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario N2L 3G1
CANADA

E.S. Kramer
Dept. of Mathematics and Statistics
University of Nebraska - Lincoln
Lincoln, Nebraska 68588
U.S.A.

D.L. Kreher
Dept. of Computer Science
Rochester Institute of Technology
Rochester, New York 14623
U.S.A.

References

[CCK] Y.M. Chee, C.J. Colbourn and D.L. Kreher, Simple t-designs with $v \leq 30$, *submitted simultaneously with this paper.*

[K] E.S. Kramer, An $S_3(3,5,21)$ using graphs, to appear in *Discrete Mathematics.*

[KM] E.S. Kramer and D.M. Mesner, t-designs on hypergraphs, *Discrete Mathematics* Vol. 15 (1976), 263-296.

[KR1] D.L. Kreher and S.P. Radziszowski, Constructing 6-(14,7,4) designs, *AMS series Contemporary Mathematics* to appear.

[KR2] D.L. Kreher and S.P. Radziszowski, Finding t-designs by using basis reduction, *Congressus Numerantium,* 55 (1986) 235-244.

APPENDIX

A.1. 2−(18,7,λ) Designs with $\lambda \equiv 0 \ (mod\, 336)$

In Table III is a convenient listing of the orbit representatives of 7-element subsets under the action of $SAF(17)_\infty$. Develop each of the 7-element subsets indicated in Table II with the automorphisms in $SAF(17)_\infty$ to obtain a 2−(18,7,λ) design.

TABLE II

λ	row and column entry of Table III
336	23D 24D 24E 27B 27F 27H 28B 13G 13H 16H 17B 18C 14B 2C 14F 14H 15G 2D
672	23D 24D 24E 27B 27F 27H 28B 28H 29B 29C 29D 29H 30A 30D 13G 13H 16H 17B 18C 2F 19E 19F 3A 3C 14B 2C 14F 14H 15G 2D 17D 17H 3B 3E 3G 4B
1008	18G 21G 22C 23C 23E 23F 24F 24G 25A 25B 25C 25D 26E 26C 26G 26H 27E 28A 28E 29F 31G 14B 2C 14F 14H 15G 2D 17D 17H 3B 3E 3G 4B 5A 5B 5C 5E 5F 5G 1D 6E 6F 6H 7B 7G 8B 8F 14A 14C 14D 14E 15A 15C 17E
1344	18G 18H 20H 21D 21E 21H 22A 22B 21G 22C 23C 23E 23F 24F 24G 25A 25B 25C 25D 25E 26C 26G 26H 27E 28A 28E 29F 31G 14B 2C 14F 14H 15G 2D 17D 17H 3B 3E 3G 4B 5A 5B 5C 5E 5F 5G 1D 6E 6F 6H 7B 7G 8B 8F 9B 1H
1680	18G 18H 20H 21D 21E 21H 22A 22B 22E 22F 23A 23B 24A 24H 25H 21G 22C 23C 23E 23F 24F 24G 25A 25B 25C 25D 25E 26C 26G 26H 27E 28A 28E 29F 31G 14B 2C 14F 14H 15G 2D 17D 17H 3B 3E 3G 4B 5A 5B 5C 5E 5F 5G 1D 6E 6F 6H 7B 7G 8B 8F 9B 1H 11E 12C 14A 14C 14D 14E 15A 15C 17E 17F 18A 18B 18D 18E 19H 1B 3D 5D 6C 6D 6G 8A 8E 8G 8H 9A 9D 10A 10D 11B 11H 12F 13A
2016	18G 18H 20H 21D 21E 21H 21G 22C 23C 23E 23F 24F 24G 25A 25B 25C 25D 25E 26C 26G 26H 27E 28A 28E 29F 31G 23D 24D 24E 27B 27F 27H 28B 28H 29B 29C 29D 29H 30A 30D 31A 31F 13G 13H 16H 17B 18C 2F 19E 19F 3A 3C 3F 4D 4E 1C 4F 6A 6B 14B 2C 14F 14H 15G 2D 17D 17H 3B 3E 3G 4B 5A 5B 5C 5E 5F 5G 1D 6E 6F 6H 7B 7G 8B 8F 9B 1H 11E 12C 14A 14C 14D 14E 15A 15C 17E 17F 18A 18B 18D 18E 19H 1B 3D 5D 6C 6D 6G

TABLE III

	A	B	C	D	E	F	G	H
1	0 1 3 6 7 8 13	0 1 3 5 6 8 10	0 1 3 5 6 7 9	0 2 3 5 6 7 8	0 1 3 4 5 6 7	0 1 2 3 5 6 7	0 1 2 3 4 5 6	0 1 3 4 6 7 8
2	0 1 2 3 4 7 8	0 1 2 3 4 6 8	0 1 2 3 6 7 8	0 1 3 5 6 7 8	0 2 3 4 6 7 8	0 1 2 3 4 7 9	0 1 2 3 4 5 9	0 3 4 5 6 7 8
3	0 1 2 3 5 6 9	0 1 2 3 6 7 9	0 2 3 4 5 7 9	0 1 3 4 6 7 9	0 2 3 4 6 7 9	0 1 2 3 5 6 10	0 1 3 4 6 8 9	0 1 3 4 5 8 9
4	0 1 2 3 4 8 9	0 1 2 3 6 8 9	0 1 2 3 4 5 10	0 2 3 4 7 8 9	0 1 3 4 7 8 9	0 3 5 6 7 8 9	0 1 2 3 4 6 10	0 1 3 4 6 7 10
5	0 1 2 3 4 7 10	0 1 3 4 5 6 10	0 1 2 3 6 7 10	0 1 3 5 6 7 10	0 2 3 4 6 7 10	0 1 3 4 6 8 10	0 1 3 4 5 8 10	0 1 3 5 6 10 11
6	0 2 3 4 5 7 11	0 1 2 3 4 6 11	0 3 5 6 7 8 10	0 1 3 4 7 8 10	0 3 4 5 6 8 10	0 2 3 4 7 8 10	0 3 5 6 8 9 10	0 1 3 5 6 9 10
7	0 1 2 3 4 5 11	0 1 2 3 5 6 11	0 1 2 3 4 7 11	0 2 3 5 6 8 11	0 2 3 4 6 7 11	0 1 3 4 6 7 11	0 1 2 3 6 7 11	0 1 3 4 6 8 11
8	0 1 3 4 5 8 11	0 2 3 5 6 7 11	0 1 3 5 6 8 11	0 1 3 6 7 8 11	0 1 3 4 7 8 11	0 1 2 3 6 9 11	0 3 5 6 7 8 11	0 1 3 5 6 9 11
9	0 3 5 6 7 8 12	0 1 3 4 6 8 12	0 1 2 3 6 7 12	0 1 2 3 5 6 12	0 1 2 3 4 6 12	0 2 3 4 5 7 12	0 1 2 3 4 8 12	0 1 3 5 6 7 12
10	0 1 2 3 6 8 12	0 2 3 5 6 8 12	0 1 3 5 6 8 12	0 1 2 3 7 8 12	0 3 4 5 6 8 12	0 1 3 6 7 8 12	0 1 2 3 6 8 13	0 1 2 3 4 6 13
11	0 1 3 5 6 10 12	0 2 3 4 6 9 12	0 3 5 6 8 11 12	0 1 2 3 6 7 13	0 1 2 3 4 7 13	0 1 2 3 4 8 13	0 1 3 5 6 8 13	0 1 3 4 6 8 13
12	0 1 2 3 7 8 13	0 1 3 6 8 13 15	0 1 3 5 6 10 14	0 1 2 3 4 7 14	0 1 3 6 8 9 13	0 3 5 6 7 8 13	0 2 3 6 7 8 13	0 2 3 4 6 9 13
13	0 1 2 3 6 9 13	0 1 3 6 8 12 13	0 1 3 6 8 10 13	0 1 3 5 6 10 13	0 1 3 6 8 11 13	0 3 5 6 8 11 13	0 1 2 3 5 6 14	0 1 2 3 4 6 14
14	0 1 3 4 6 8 14	0 1 3 5 6 7 14	0 1 2 3 6 7 14	0 2 3 4 6 7 14	0 1 2 3 6 8 14	0 1 3 4 5 8 14	0 1 3 4 7 8 14	0 3 4 5 6 8 14
15	0 1 3 5 6 8 14	0 1 2 3 7 8 14	0 3 5 6 7 8 14	0 2 3 4 7 8 14	0 3 5 6 8 9 14	0 2 3 4 7 9 14	0 1 3 5 6 7 15	0 1 3 6 8 13 14
16	0 3 5 6 8 10 14	0 1 3 4 7 10 14	0 2 3 4 7 10 14	0 2 3 6 7 11 14	0 2 3 4 7 11 14	0 3 5 6 8 11 14	0 3 4 7 8 11 14	0 1 2 3 5 6 15
17	0 1 2 3 4 6 15	0 1 2 3 6 7 15	0 2 3 4 6 7 15	0 3 4 5 6 8 15	0 1 3 4 6 8 15	0 1 3 4 5 8 15	0 1 3 5 6 8 15	0 3 5 6 7 8 15
18	0 1 3 6 7 8 15	0 1 3 4 7 8 15	0 2 3 6 7 8 15	0 2 3 4 5 10 15	0 1 3 4 8 9 15	0 3 4 5 8 10 15	0 2 3 4 7 9 ∞	0 1 2 3 6 7 ∞
19	0 1 3 6 8 13 16	0 1 3 5 6 7 16	0 2 3 4 6 7 16	0 1 2 3 4 6 16	0 3 5 6 7 8 16	0 1 3 4 6 8 16	0 1 3 4 7 8 16	0 1 3 5 6 10 16
20	0 1 2 3 4 7 ∞	0 1 2 3 4 6 ∞	0 1 2 3 4 5 ∞	0 1 2 3 5 6 ∞	0 2 3 4 5 7 ∞	0 3 4 5 6 8 ∞	0 1 3 4 5 8 ∞	0 1 3 5 6 7 ∞
21	0 1 3 4 6 7 ∞	0 2 3 4 6 7 ∞	0 1 2 3 4 8 ∞	0 1 3 4 6 8 ∞	0 1 2 3 6 8 ∞	0 2 3 5 6 8 ∞	0 1 3 5 6 8 ∞	0 3 5 6 7 8 ∞
22	0 1 3 4 7 8 ∞	0 1 2 3 7 8 ∞	0 1 3 6 7 8 ∞	0 2 3 4 7 8 ∞	0 2 3 6 7 8 ∞	0 2 3 4 6 9 ∞	0 2 3 4 5 9 ∞	0 1 2 3 4 9 ∞
23	0 1 2 3 6 9 ∞	0 1 3 5 6 9 ∞	0 1 3 4 6 12 ∞	0 3 5 6 8 10 ∞	0 1 3 4 6 10 ∞	0 3 5 6 8 9 ∞	0 1 3 4 8 9 ∞	0 2 3 6 7 9 ∞
24	0 2 3 4 5 10 ∞	0 1 2 3 4 10 ∞	0 1 2 3 6 10 ∞	0 1 3 6 7 10 ∞	0 1 3 4 7 10 ∞	0 1 3 5 6 10 ∞	0 2 3 4 7 10 ∞	0 1 3 4 8 10 ∞
25	0 2 3 6 7 10 ∞	0 3 4 5 8 10 ∞	0 1 3 6 8 10 ∞	0 1 3 4 8 11 ∞	0 1 3 4 7 11 ∞	0 1 2 3 7 11 ∞	0 2 3 4 5 11 ∞	0 2 3 6 7 11 ∞
26	0 2 3 4 7 11 ∞	0 3 5 6 8 11 ∞	0 3 4 6 8 11 ∞	0 1 3 6 8 11 ∞	0 2 3 4 5 12 ∞	0 3 4 7 8 11 ∞	0 1 2 3 6 12 ∞	0 2 3 4 8 14 ∞
27	0 1 3 6 8 13 ∞	0 3 6 8 9 12 ∞	0 1 2 3 8 12 ∞	0 1 3 6 7 12 ∞	0 1 2 3 7 12 ∞	0 2 3 6 7 12 ∞	0 1 3 4 9 12 ∞	0 1 3 6 8 12 ∞
28	0 1 2 3 6 13 ∞	0 3 6 8 11 12 ∞	0 2 3 4 5 13 ∞	0 1 2 3 7 13 ∞	0 1 3 4 6 13 ∞	0 2 3 4 7 13 ∞	0 3 6 8 10 13 ∞	0 3 5 6 8 13 ∞
29	0 3 4 6 8 13 ∞	0 2 3 6 8 13 ∞	0 3 6 8 9 13 ∞	0 3 6 7 8 13 ∞	0 1 3 4 7 14 ∞	0 1 3 4 6 14 ∞	0 1 2 3 6 14 ∞	0 2 3 4 7 14 ∞
30	0 1 3 6 8 15 ∞	0 3 6 8 13 14 ∞	0 3 6 8 9 14 ∞	0 3 5 6 8 14 ∞	0 3 6 8 11 14 ∞	0 2 3 6 7 15 ∞	0 2 3 4 5 15 ∞	0 1 3 4 6 16 ∞
31	0 3 6 8 13 15 ∞	0 2 3 6 8 15 ∞	0 1 3 4 8 16 ∞	0 1 3 4 7 16 ∞	0 1 3 6 7 16 ∞	0 3 6 8 13 16 ∞	0 3 5 6 8 16 ∞	0 2 3 4 15 16 ∞

A.2. 2−(20,4,λ) Designs with $\lambda \equiv 0 \pmod 3$

Let H be the Frobenius group of order 3·19 generated by $\alpha: X \to X+1$ and $\beta: X \to 7 \cdot X$. In Table V is a convenient listing of all the orbit representatives of 4-element subsets under the action of $G_1 = H_\infty$. Developing each of the 4-element subsets in Table IV with the automorphisms in G_1 constructs a 2−(20,4,λ) design, for each $\lambda \equiv 0 \, (mod\ 3)$.

TABLE IV

λ	row and column entry of Table V
3	6A 7G 10E 8B 7H
6	10E 12E 7H 10B 5A 6H
9	11A 10E 12E 5A 2F 3H 5C
12	11A 12A 10E 12E 7H 5A 3A 2F 3H 5C
15	11H 11A 12A 10E 12E 7H 10B 5A 3A 2F 3H 5C 5E
18	1A 2G 9E 10G 3A 3B 2F 3H 5C 5E
21	10E 7H 1A 2G 3C 9E 10G 3A 3B 4C 2F 3H 5C
24	10E 12E 7H 10B 1A 2G 3C 3F 9E 10G 3A 3B 4C 4H 2F 3H
27	9D 1A 2G 9E 10G 3A 3B 2F 3H 5C 5E 5F 6C 6D 6G
30	10E 7H 9D 1A 2G 3C 9E 10G 3A 3B 4C 2F 3H 5C 5E 5F 6C 6D
33	10E 12E 7H 10B 9D 1A 2G 3C 3F 9E 10G 3A 3B 4C 4H 2F 3H 5C 5E 5F 6C
36	9D 11B 1A 2G 9E 10G 3A 3B 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G
39	10E 7H 9D 11B 1A 2G 3C 9E 10G 3A 3B 4C 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D
42	10E 12E 7H 10B 9D 11B 1A 2G 3C 3F 9E 10G 3A 3B 4C 4H 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C
45	9D 11B 11C 1A 2G 3C 9E 10G 3A 3B 4C 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 2B
48	10E 7H 9D 11B 11C 1A 2G 3C 9E 10G 3A 3B 4C 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2A
51	10E 12E 7H 10B 9D 11B 11C 1A 2G 3C 3F 9E 10G 3A 3B 4C 4H 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A
54	9D 11B 11C 11D 1A 2G 3C 3F 9E 10G 3A 3B 4C 4H 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2B 2C
57	10E 7H 9D 11B 11C 11D 1A 2G 3C 3F 4B 9E 10G 3A 3B 4C 4H 6B 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 2B 2C
60	10E 12E 7H 10B 9D 11B 11C 11D 1A 2G 3C 3F 4B 9E 10G 3A 3B 4C 4H 6B 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2A 2B
63	9D 11B 11C 11D 11G 1A 2G 3C 3F 4B 9E 10G 3A 3B 4C 4H 6B 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2A 2B 2C 4D
66	10E 7H 9D 11B 11C 11D 11G 1A 2G 3C 3F 4B 4G 9E 10G 3A 3B 4C 4H 6B 7A 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 2B 2C 10A 4D
69	10E 12E 7H 10B 9D 11B 11C 11D 11G 1A 2G 3C 3F 4B 4G 9E 10G 3A 3B 4C 4H 6B 7A 8E 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2D 2B 2C
72	9D 11B 11C 11D 11G 12C 1A 2G 3C 3F 4B 4G 9E 10G 3A 3B 4C 4H 6B 7A 8E 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2D 2B 2C 4D 5D
75	10E 7H 9D 11B 11C 11D 11G 12C 1A 2G 3C 3F 4B 4G 9E 10G 3A 3B 4C 4H 6B 7A 8E 2F 3H 5C 5E 5F 6C 6D 6G 7B 8C 8D 8G 9G 10A 2A 2D 3D 3G 2B 2C

TABLE V

	A	B	C	D	E	F	G	H
1	0 2 9 10	0 1 2 9	0 2 3 6	0 2 3 5	0 1 2 4	0 1 2 3	0 1 2 5	0 2 4 5
2	0 1 2 6	0 2 3 8	0 2 3 7	0 1 2 7	0 2 4 6	0 2 6 7	0 1 6 7	0 1 2 8
3	0 2 6 8	0 2 4 8	0 1 4 8	0 2 7 8	0 1 7 8	0 2 8 9	0 2 5 9	0 2 3 9
4	0 2 4 9	0 1 7 9	0 2 6 9	0 2 7 9	0 2 5 10	0 2 3 10	0 1 2 10	0 2 4 10
5	0 1 7 10	0 2 6 10	0 2 8 10	0 2 9 14	0 2 9 12	0 2 9 11	0 2 3 11	0 1 2 11
6	0 1 7 11	0 2 6 11	0 2 7 11	0 2 3 12	0 1 2 12	0 2 6 12	0 2 4 12	0 2 8 12
7	0 2 4 13	0 2 3 13	0 1 2 13	0 2 9 13	0 1 7 13	0 2 6 13	0 1 9 13	0 2 3 14
8	0 2 9 16	0 2 5 15	0 2 3 15	0 1 2 15	0 2 4 15	0 2 9 15	0 2 7 15	0 2 6 15
9	0 2 3 16	0 2 11 15	0 2 6 16	0 2 8 19	0 2 3 19	0 2 4 18	0 2 9 17	0 2 10 16
10	0 2 9 18	0 2 7 18	0 1 2 19	0 2 6 19	0 2 5 19	0 2 4 19	0 2 7 19	0 1 7 19
11	0 1 8 19	0 2 12 19	0 2 10 19	0 2 9 19	0 1 9 19	0 1 10 19	0 2 11 19	0 4 10 19
12	0 1 12 19	0 2 15 19	0 2 13 19	0 4 13 19	0 2 16 19			

A.3. A 3−(16,7,10)

Let G_2 be the representation of the Frobenius group of order 80 generated by the permutations in table VI. Then developing the 7-element subsets

0 2 3 4 5 9 15 and 0 1 2 4 5 10 15

into 160 blocks with the members of G_2 gives a 3-(16,7,10) design.

TABLE VI

(0,1)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15)
(0,2)(1,3)(4,6)(5,7)(8,10)(9,11)(12,14)(13,15)
(0,4)(1,5)(2,6)(3,7)(8,12)(9,13)(10,14)(11,15)
(0,8)(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)
(0)(1,8,15,5,3)(2,9,7,10,6)(4,11,14,13,12)

A.4. 3−(19,7,λ) designs from $AF(19)$.

Using the elements of $AF(19)$ the orbit representatives given in Table VII can be developed into seven disjoint 3-(19,7,λ) designs for $\lambda = 35$, 35, 105, 210, 210, 210 and 210 respectively. Taking unions of appropriate combinations of these designs yield 3-(19,7,λ) designs for each possible λ.

TABLE VII

λ	Orbit representatives			
35	0 1 2 3 7 11 14	0 1 2 3 4 5 12	0 1 2 3 5 6 7	0 1 3 6 7 8 12
35	0 1 3 6 7 15 17	0 1 2 3 4 5 6	0 1 2 3 4 5 7	0 1 2 3 6 7 8
105	0 1 2 3 4 5 9	0 1 2 3 5 8 9	0 1 3 5 6 9 11	0 1 3 4 6 7 12
	0 1 2 3 5 8 13	0 1 2 3 4 11 12	0 1 3 4 5 8 13	0 1 3 4 5 8 15
	0 1 3 4 8 9 18			
210	0 1 3 6 7 11 14	0 1 2 3 7 11 16	0 1 2 3 4 5 10	0 1 2 5 6 9 17
	0 1 3 6 7 8 10	0 1 3 4 6 8 12	0 1 3 6 7 9 15	0 1 2 3 4 7 8
	0 1 2 5 6 7 11	0 1 3 6 8 9 13	0 1 3 4 5 8 18	0 1 3 6 7 8 11
	0 1 2 3 4 8 11	0 1 3 4 6 9 18	0 1 3 6 8 9 10	0 1 2 3 4 7 10
	0 1 3 6 10 11 12			
210	0 1 3 4 5 9 11	0 1 2 3 8 11 16	0 1 3 4 6 8 9	0 1 2 3 5 8 17
	0 1 2 5 6 9 13	0 1 3 4 6 7 11	0 1 2 3 4 11 13	0 1 2 5 6 7 10
	0 1 2 3 5 11 16	0 1 2 3 5 8 18	0 1 3 4 7 9 14	0 1 2 3 5 8 16
	0 1 3 4 6 8 14	0 1 3 6 7 10 11	0 1 3 5 6 9 18	0 1 2 3 6 10 11
	0 1 3 6 8 9 12			
210	0 1 2 3 7 8 17	0 1 3 6 7 8 13	0 1 3 6 10 11 15	0 1 3 10 11 13 18
	0 1 3 4 5 6 9	0 1 3 4 5 10 14	0 1 2 5 6 7 9	0 1 2 3 4 5 11
	0 1 3 4 5 9 17	0 1 2 3 4 5 8	0 1 3 6 10 11 16	0 1 2 3 7 8 13
	0 1 3 6 10 11 13	0 1 3 4 6 8 18	0 1 3 5 6 8 18	0 1 3 4 6 8 16
	0 1 3 4 7 9 18			
210	0 1 3 5 6 8 16	0 1 2 3 4 11 16	0 1 3 4 8 9 15	0 1 3 4 6 7 9
	0 1 2 3 4 6 7	0 1 3 4 5 9 10	0 1 3 6 10 11 18	0 1 3 5 6 7 11
	0 1 2 5 6 9 12	0 1 2 3 4 8 10	0 1 3 5 6 9 14	0 1 2 3 7 10 11
	0 1 2 3 5 8 10	0 1 2 3 5 6 8	0 1 3 4 5 8 17	0 1 3 4 6 8 17
	0 1 2 3 7 8 14			

A.5. 3–(19,9,λ) designs from $AF(19)$.

Using the elements of $AF(19)$ the orbit representatives given in Table VIII can be developed into eleven disjoint 3-(19,7,λ) designs for $\lambda = 28$, 84, 84, 252, 252, 504, 504, 504, 504, 504 and 504 respectively. Taking unions of appropriate combinations of these designs yield 3-(19,9,λ) designs for many of the previously unreported values of λ in this situation. That is $\lambda = 112$, 196, 280, 364, 924, 1204, 1764, 2044, 2604, 2884, 3444 and 3724.

Table VIII

λ	Orbit representatives			
28	0 1 2 3 5 6 8 10 13	0 1 2 3 6 7 8 9 14	0 1 2 3 5 7 12 13 16	
84	0 1 2 3 4 5 8 9 13	0 1 2 3 4 5 8 10 13	0 1 3 4 5 6 8 11 13	0 1 2 3 4 5 6 9 16
84	0 1 3 4 5 6 8 10 15	0 1 3 4 6 7 8 10 18	0 1 2 3 4 5 6 7 8	0 1 3 4 5 7 9 14 17
252	0 1 2 3 4 5 6 8 12	0 1 2 3 6 7 8 9 12	0 1 2 3 4 6 7 9 13	0 1 2 3 4 6 8 10 13
	0 1 3 4 6 7 8 10 11	0 1 2 3 4 6 7 12 17	0 1 2 3 4 5 6 9 10	0 1 2 3 4 7 8 10 13
	0 1 3 4 5 6 8 9 14			
252	0 1 2 3 5 6 9 10 11	0 1 3 4 6 7 8 9 17	0 1 3 6 7 8 10 11 15	0 1 2 3 4 5 6 8 18
	0 1 3 4 5 6 8 9 11	0 1 2 3 4 5 7 11 14	0 1 2 3 4 5 6 7 10	0 1 2 3 4 5 7 10 15
	0 1 3 4 5 7 8 9 18			
504	0 1 2 5 6 7 9 11 12	0 1 3 4 6 7 8 9 13	0 1 2 3 4 5 9 10 17	0 1 2 3 4 6 7 14 15
	0 1 2 3 4 6 7 8 11	0 1 2 3 4 5 6 10 14	0 1 2 3 4 6 7 11 13	0 1 2 3 4 6 8 9 10
	0 1 2 3 4 5 8 9 12	0 1 2 3 4 6 7 9 12	0 1 3 5 6 7 8 10 11	0 1 2 3 5 8 9 11 15
	0 1 2 3 4 6 8 11 15	0 1 2 3 4 5 8 9 17	0 1 2 3 6 7 8 10 11	0 1 2 3 4 7 8 11 13
	0 1 2 3 5 6 7 9 14			
504	0 1 3 6 7 9 10 11 15	0 1 2 3 5 6 7 8 10	0 1 2 3 4 5 6 8 13	0 1 3 4 6 7 8 11 13
	0 1 3 4 6 8 9 10 12	0 1 2 3 4 7 8 9 11	0 1 2 3 4 6 7 10 15	0 1 3 4 5 7 8 9 17
	0 1 2 3 4 5 7 10 17	0 1 3 6 8 9 10 11 12	0 1 2 3 4 5 7 10 14	0 1 3 4 5 7 8 9 15
	0 1 2 3 4 5 7 12 13	0 1 3 4 5 6 8 11 18	0 1 2 3 4 5 6 8 16	0 1 3 4 5 7 8 9 11
	0 1 2 3 5 8 9 10 13			
504	0 1 2 3 4 6 8 10 15	0 1 2 3 4 7 8 10 14	0 1 2 3 4 7 8 9 12	0 1 2 5 6 8 9 10 13
	0 1 3 4 6 8 9 11 18	0 1 2 3 5 6 7 9 13	0 1 3 4 5 6 7 8 12	0 1 2 3 6 7 8 11 17
	0 1 2 3 5 6 7 9 11	0 1 2 3 6 8 9 11 15	0 1 2 3 4 6 7 8 13	0 1 2 3 4 6 7 8 14
	0 1 2 3 4 6 7 10 11	0 1 3 4 6 8 9 10 17	0 1 3 4 6 7 8 10 12	0 1 2 3 4 5 6 8 10
	0 1 2 3 4 6 7 8 12			
504	0 1 2 3 4 5 8 9 15	0 1 2 3 4 5 6 7 11	0 1 3 4 5 7 8 9 16	0 1 2 3 4 6 7 8 17
	0 1 2 3 4 5 6 9 11	0 1 2 3 5 6 7 9 15	0 1 2 3 5 8 9 10 11	0 1 2 3 5 6 7 10 12
	0 1 2 3 6 7 8 10 14	0 1 3 4 6 7 8 10 15	0 1 3 5 6 7 8 11 18	0 1 2 3 4 5 8 10 15
	0 1 3 4 5 6 8 10 12	0 1 2 3 4 7 8 11 17	0 1 2 3 4 6 8 9 15	0 1 3 4 5 7 9 10 11
	0 1 3 4 5 6 7 8 11			
504	0 1 2 3 4 6 7 8 15	0 1 3 4 5 6 7 8 14	0 1 2 3 4 5 9 10 16	0 1 2 3 4 5 7 9 12
	0 1 3 4 6 7 8 9 18	0 1 2 3 5 6 8 10 11	0 1 2 3 4 6 8 9 11	0 1 2 3 4 7 8 9 17
	0 1 2 3 5 6 7 9 12	0 1 2 3 4 7 8 10 11	0 1 3 4 5 6 8 9 10	0 1 2 3 4 5 6 9 14
	0 1 3 4 5 6 7 8 13	0 1 2 3 4 7 8 10 16	0 1 2 3 6 7 9 10 14	0 1 3 5 6 7 8 11 15
	0 1 3 4 5 6 8 10 16			
504	0 1 3 6 7 8 10 11 18	0 1 2 3 4 5 8 9 16	0 1 2 3 4 5 8 10 16	0 1 3 4 6 7 8 12 17
	0 1 2 3 6 8 10 11 15	0 1 2 3 6 7 8 10 17	0 1 2 3 7 8 10 11 14	0 1 3 4 5 6 8 9 12
	0 1 2 3 6 8 10 11 13	0 1 3 4 5 6 8 10 17	0 1 2 3 4 6 8 11 16	0 1 2 3 4 5 7 11 12
	0 1 2 3 5 6 7 9 10	0 1 2 3 6 7 8 10 12	0 1 2 3 4 5 8 11 14	0 1 3 6 7 8 10 11 16
	0 1 2 3 4 7 8 9 13			

A.6. Hypergraphical 3−(20,5,λ) designs, $\lambda \in \{18,28,48,58\}$

A representation of S_6 on 20 points is generated by the permutations α and β below.

$$\alpha = (0,10,16,7,2)(1,11,4,13,5)(3,12,17,19,9)(6,14,18,8,15)$$

$$\beta = (2,3)(5,6)(7,8)(11,12)(13,14)(16,17)$$

Let $G_3 = \langle \alpha,\beta \rangle$.

A.6.1. 3-(20,5,18)

To obtain a 3-(20,5,18) design develop each of the 5-element sets below with the automorphisms in G_3

0 3 7 9 10	0 3 4 7 9	0 1 2 4 5	0 3 6 7 10	0 1 2 4 16	0 1 3 7 18

A.6.2. 3-(20,5,28)

To obtain a 3-(20,5,28) design develop each of the 5-element sets below with the automorphisms in G_3

0 3 7 9 10	0 3 4 5 7	0 1 2 4 10	0 3 5 7 10	0 3 6 7 12	0 1 2 4 16
0 1 2 3 16	0 1 7 11 16	0 1 3 7 18	0 1 7 14 18	0 3 7 16 19	

A.6.3. 3-(20,5,48)

To obtain a 3-(20,5,48) design develop each of the 5-element sets below with the automorphisms in G_3

0 1 3 4 7	0 3 4 5 7	0 3 5 7 10	0 3 7 10 12	0 3 4 7 12	0 1 2 4 11
0 3 7 10 16	0 1 3 7 16	0 1 7 11 16	0 3 7 10 18	0 3 7 10 17	

A.6.4. 3-(20,5,58)

To obtain a 3-(20,5,58) design develop each of the 5-element sets below with the automorphisms in G_3

0 1 2 3 4	0 1 3 4 7	0 1 2 4 10	0 3 5 7 10	0 3 4 7 10	0 3 4 7 12
0 3 6 7 12	0 3 4 7 13	1 3 7 13	0 1 2 4 15	0 3 7 10 16	0 1 2 3 16
0 1 3 7 16	0 1 7 11 16	0 3 7 10 18	0 1 3 7 18	0 1 7 14 18	0 1 4 11 19

A.7. Semi-Hypergraphical 3−(20,5,λ) designs, $\lambda \in \{24,54\}$.

A representation of A_6 on 20 points is generated by the permutations γ and δ below.

$$\gamma = (0,10,16,7,2)(1,11,4,13,5)(3,12,17,19,9)(6,14,18,8,15)$$

$$\delta = (1,2,3)(4,5,6)(7,9,8)(10,11,12)(13,15,14)(16,18,17)$$

Let $G_4 = \langle \gamma,\delta \rangle$.

A.7.1. 3-(20,5,24)

To obtain a 3-(20,5,24) design develop each of the 5-element sets below with the automorphisms in G_4

0 2 6 7 8	0 1 2 7 10	0 5 6 7 10	0 2 3 7 12	0 1 6 7 12	0 5 7 10 14
0 2 3 7 15	0 1 6 7 15	0 2 3 7 18	0 6 7 10 19		

A.7.2. 3-(20,5,54)

To obtain a 3-(20,5,54) design develop each of the 5-element sets below with the automorphisms in G_4

0 2 4 6 7	0 1 5 7 10	0 2 6 7 8	0 2 5 6 7	0 1 6 7 10
0 1 6 7 12	0 5 7 10 15	0 5 7 10 14	0 2 3 7 15	0 1 7 10 15
0 2 3 7 16	0 1 2 7 17	0 2 6 7 17	0 1 2 7 18	0 2 3 7 18
0 6 7 10 19	0 3 7 10 19			

A.8. 3−(20,5,λ) designs with $\lambda \in \{12,22,34,42,52,64\}$.

Let $X = Z_{19} \cup \{\infty\}$ and let $\omega \in Z_{19}$ be a primitive root. Let $\epsilon : X \to X$ and $\varsigma : X \to X$ be given by

$$\epsilon(x) = \begin{cases} x+1 \text{ if } x \in Z_{19} \\ x \text{ if } x = \infty \end{cases}$$

$$= (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18) \text{ and}$$

$$\varsigma(x) = \begin{cases} \omega^3 \cdot x+1 \text{ if } x \in Z_{19} \\ x \text{ if } x = \infty \end{cases}$$

$$= (1,12,11,18,7,8)(2,5,3,17,14,16)(4,10,6,15,9,13)$$

Then $G_5 = <\epsilon,\varsigma>$ has order 114 .

A.8.1. 3-(20,5,12)

To obtain a 3-(20,5,12) design develop each of the 5-element sets below with the automorphisms in G_5

0 1 2 6 8	0 2 3 4 9	0 2 5 8 9	0 1 2 3 10	0 2 3 6 10	0 2 5 6 10
0 2 4 8 12	0 2 5 8 12	0 2 5 6 13	0 2 3 9 19	0 2 3 15 19	0 2 8 18 19

A.8.2. 3-(20,5,42)

To obtain a 3-(20,5,42) design develop each of the 5-element sets below with the automorphisms in G_5

0 2 5 8 10	0 2 5 6 8	0 2 5 6 7	0 1 2 3 6	0 1 2 3 7	0 2 3 6 7
0 2 3 5 8	0 2 4 5 8	0 1 2 7 8	0 2 3 6 9	0 2 3 8 9	0 1 4 8 9

0 2 7 8 9	0 1 2 8 10	0 2 5 6 10	0 1 4 8 10	0 2 5 8 11	0 1 2 8 11
0 2 7 8 11	0 2 3 9 11	0 2 4 8 12	0 1 2 8 12	0 2 5 6 12	0 1 6 7 12
0 2 3 8 12	0 2 6 8 12	0 2 3 4 13	0 2 4 8 13	0 2 5 8 19	0 2 3 4 15
0 2 5 6 15	0 2 3 8 18	0 2 3 4 19	0 2 3 9 18	0 1 2 8 19	0 2 4 8 19
0 2 3 9 19	0 1 2 10 19	0 2 4 11 19	0 2 8 13 19	0 2 3 15 19	0 2 8 17 19
0 2 8 15 19					

A.8.3. 3-(20,5,22)

To obtain a 3-(20,5,22) design develop each of the 5-element sets below with the automorphisms in G_5

0 2 8 10 14	0 1 2 3 6	0 1 2 3 7	0 1 2 5 8	0 2 5 7 8	0 2 3 6 9
0 1 2 8 9	0 2 7 8 9	0 2 5 6 10	0 2 3 6 12	0 1 2 8 11	0 2 5 6 12
0 2 4 6 12	0 2 3 8 12	0 2 6 10 12	0 2 4 8 13	0 2 3 8 15	0 2 5 6 19
0 2 8 9 19	0 2 4 11 19	0 1 7 11 19	0 2 8 13 19	0 2 5 15 19	0 2 3 14 19
0 2 3 15 19					

A.8.4. 3-(20,5,52)

To obtain a 3-(20,5,52) design develop each of the 5-element sets below with the automorphisms in G_5

0 2 8 10 14	0 2 5 6 8	0 2 5 6 7	0 1 2 3 6	0 1 2 3 7	0 2 3 5 8
0 2 3 4 8	0 1 4 5 8	0 2 3 4 10	0 1 2 7 8	0 2 5 7 8	0 2 3 6 9
0 2 5 8 9	0 2 3 8 9	0 2 7 8 9	0 1 2 8 10	0 2 4 6 10	0 2 5 6 10
0 1 4 8 10	0 2 3 6 12	0 1 2 8 11	0 2 3 4 11	0 1 6 7 11	0 2 7 8 11
0 2 6 8 11	0 2 3 4 12	0 1 2 8 12	0 2 5 6 12	0 2 4 6 12	0 2 3 8 12
0 2 8 9 12	0 2 5 6 13	0 2 4 8 13	0 1 2 8 14	0 2 5 8 18	0 2 3 4 15
0 2 5 6 15	0 2 3 8 18	0 2 3 9 18	0 2 3 6 19	0 2 3 8 19	0 1 2 8 19
0 2 4 8 19	0 2 8 11 19	0 2 5 10 19	0 1 2 10 19	0 2 4 11 19	0 2 3 11 19
0 1 7 11 19	0 2 6 12 19	0 2 5 15 19	0 2 3 14 19	0 2 3 15 19	0 2 8 15 19

A.8.5. 3-(20,5,34)

To obtain a 3-(20,5,34) design develop each of the 5-element sets below with the automorphisms in G_5

0 2 8 10 14	0 1 2 3 6	0 2 3 6 7	0 2 3 5 8	0 1 2 5 8	0 2 5 7 8
0 2 3 4 9	0 2 5 8 9	0 2 3 8 9	0 1 4 8 9	0 2 7 8 9	0 2 3 6 10
0 1 4 8 10	0 2 5 8 11	0 1 2 8 11	0 2 3 9 11	0 2 5 6 12	0 2 6 8 12
0 2 3 4 13	0 1 4 8 13	0 1 2 8 13	0 2 4 8 13	0 2 3 6 15	0 2 3 4 15
0 2 3 8 15	0 2 3 9 15	0 2 4 8 19	0 2 8 11 19	0 2 8 9 19	0 2 5 10 19
0 2 4 11 19	0 2 3 11 19	0 1 7 11 19	0 2 8 13 19	0 2 5 15 19	0 2 3 14 19
0 2 8 15 19					

A.8.6. 3-(20,5,64)

To obtain a 3-(20,5,64) design develop each of the 5-element sets below with the automorphisms in G_5

0 2 5 6 7	0 1 2 3 6	0 2 3 4 8	0 1 2 3 8	0 1 2 5 8	0 2 6 9 19
0 1 2 6 8	0 2 3 4 10	0 2 5 7 8	0 2 3 4 9	0 2 3 8 9	0 2 6 10 19
0 1 4 8 9	0 2 7 8 9	0 2 6 8 9	0 2 4 6 10	0 2 5 6 10	0 1 2 10 19
0 1 6 7 10	0 1 4 8 10	0 2 3 8 10	0 2 5 8 11	0 2 4 8 11	0 2 4 11 19
0 2 6 8 11	0 2 3 9 11	0 2 3 4 12	0 2 8 10 12	0 2 4 8 12	0 1 7 11 19
0 1 2 8 12	0 2 5 6 12	0 1 6 7 12	0 1 4 8 12	0 2 3 9 12	0 2 8 12 19
0 2 5 8 13	0 2 3 6 13	0 2 3 4 13	0 2 8 10 13	0 1 2 8 14	0 2 5 15 19
0 1 2 10 14	0 2 5 8 19	0 2 5 8 18	0 2 3 6 16	0 2 3 6 15	0 2 3 14 19
0 2 3 4 15	0 2 5 6 15	0 2 8 9 15	0 2 3 4 16	0 2 5 8 16	0 2 8 17 19
0 2 5 6 17	0 2 3 8 18	0 2 3 9 18	0 2 8 9 18	0 2 3 6 19	0 2 8 16 19
0 2 3 8 19	0 1 4 8 19	0 2 7 8 19	0 2 3 9 19	0 1 2 9 19	0 2 8 18 19

A.9. 3−(20,5,λ) designs with $\lambda \in \{50,56\}$.

A representation of the wreath product of D_4 *wr* A_5 on 20 points is generated by the permutations ϵ, ς, η and θ below.

$$\epsilon = (0,16,17,18,19)(1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)$$

$$\varsigma = (0,5,10,15)(1,6,11,16)(2,7,12,17)(3,8,13,18)(4,9,14,19)$$

$$\eta = (0,15)(1,6)(2,7)(3,8)(4,9)(5,10)(11,16)(12,17)(13,18)(14,19)$$

$$\theta = (0,19)(1,2)(4,5)(6,7)(9,10)(11,12)(14,15)(16,17)$$

Let $G_6 = \langle \epsilon, \varsigma, \eta, \theta \rangle$.

A.9.2. 3-(20,5,50)

To obtain a 3-(20,5,50) design develop each of the 5-element sets below with the automorphisms in G_6

0 5 6 10 11	0 1 2 3 9	0 1 2 5 6	0 2 5 6 8	0 2 5 6 7	0 1 2 3 10
0 1 2 8 9	0 6 7 8 9	0 1 2 3 11	0 2 5 6 10	0 1 2 5 11	0 1 2 8 11
0 2 5 6 13	0 1 5 10 12	0 1 2 5 13	0 1 2 13 14	0 5 6 7 15	0 5 6 7 16
0 1 2 3 19					

A.9.3. 3-(20,5,56)

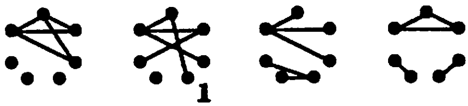
To obtain a 13-(20,5,56) design develop each of the 5-element sets below with the automorphisms in G_6

0 1 5 6 10	0 2 3 5 6	0 1 2 5 6	0 2 5 6 8	0 2 5 6 7	0 1 2 3 10
0 6 7 8 9	0 1 2 5 10	0 1 2 3 11	0 5 6 7 10	0 1 2 8 11	0 2 5 6 13
0 5 6 10 12	0 1 2 5 13	0 1 2 11 12	0 1 2 11 13	0 5 6 7 15	0 2 5 6 16
0 5 6 7 18	0 16 17 18 19				

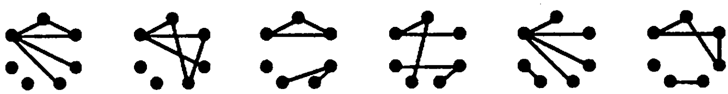
A10. Graphical and Semi-graphical 3-(21,k,λ) designs, $k \in \{5,6\}$.

A10.1. A 3-(21,5,3) design.

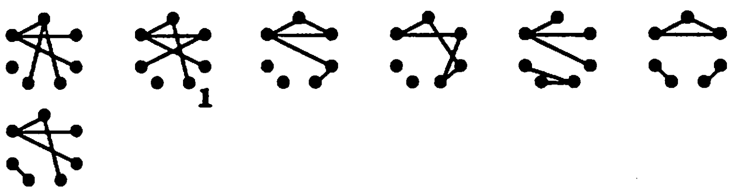
A10.2. A 3-(21,5,15) design.



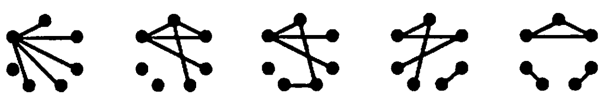
A10.3. A 3-(21,5,30) design.



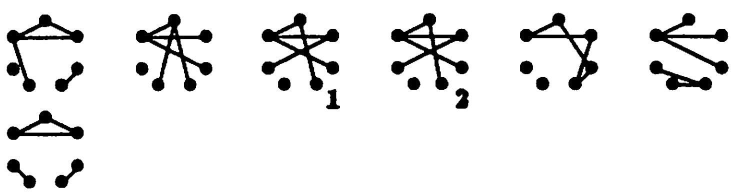
A10.4. A 3-(21,5,39) design.



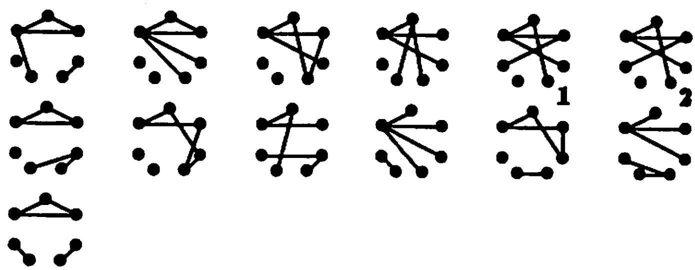
A10.5. A 3-(21,5,39) design.



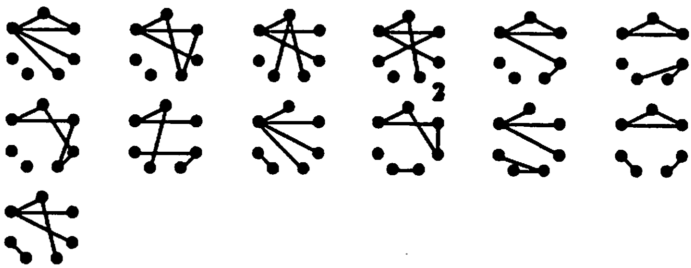
A10.6. A 3-(21,5,39) design.



A10.7. A 3-(21,5,69) design.



A10.8. A 3-(21,5,69) design.

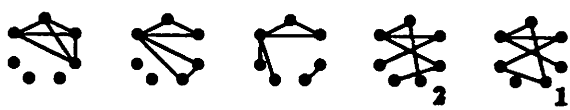


A10.9. 3-(21,5,λ) designs, $\lambda \in \{33,45,69,75\}$

These designs are obtained by using the construction indicated below.

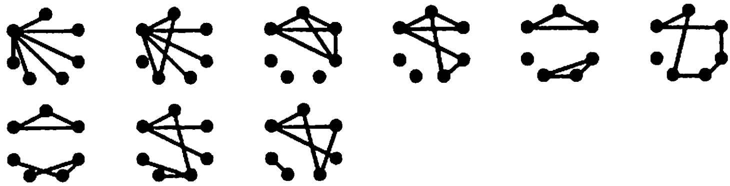
3-(21,5,33): Union of designs A10.1 and A10.3.
3-(21,5,45): Union of designs A10.2 and A10.3.
3-(21,5,69): Union of designs A10.4 and A10.3.
3-(21,5,78): Union of designs A10.6 and A10.3.
3-(21,5,75): Complement of the above design .

A10.10. A 3-(21,6,40) design.

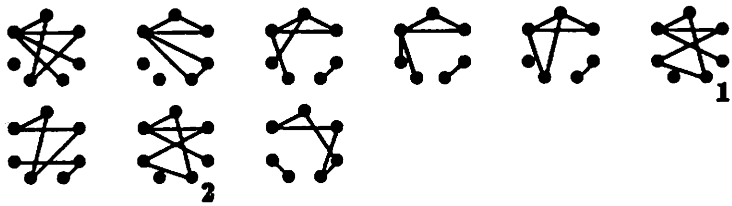


A10.11. A 3-(21,6,40) design.

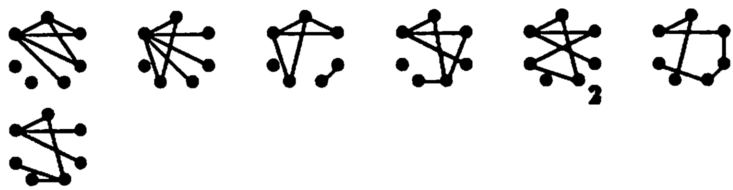
A10.12. A 3-(21,6,68) design.



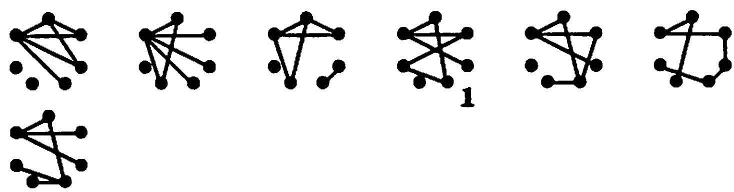
A10.13. A 3-(21,6,108) design.



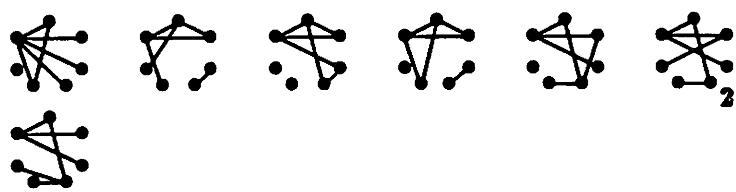
A10.14. A 3-(21,6,120) design.



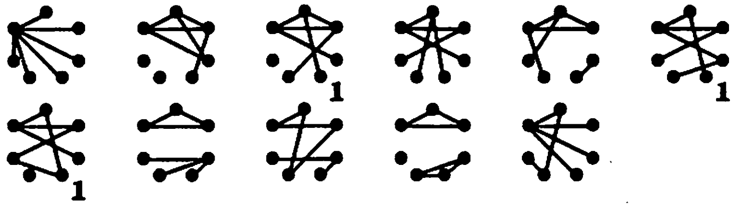
A10.15. A 3-(21,6,120) design.



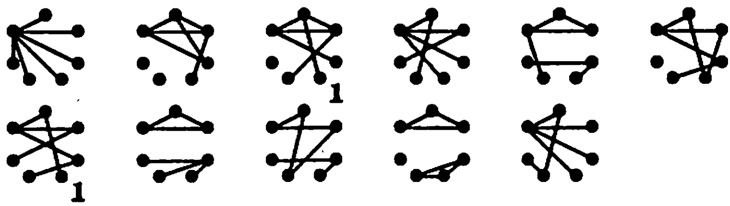
A10.16. A 3-(21,6,120) design.



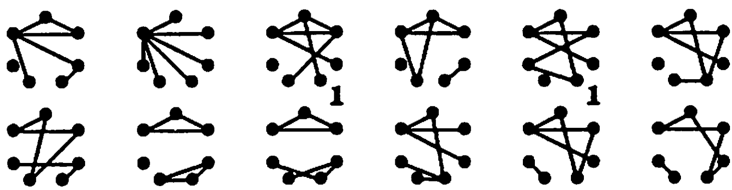
A10.17. A 3-(21,6,128) design.



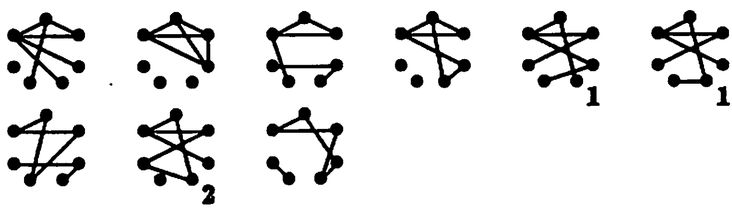
A10.18. A 3-(21,6,128) design.



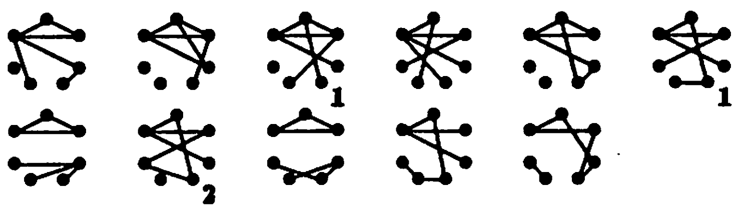
A10.19. A 3-(21,6,136) design.



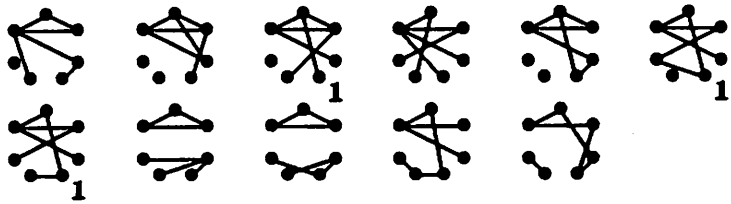
A10.20. A 3-(21,6,148) design.



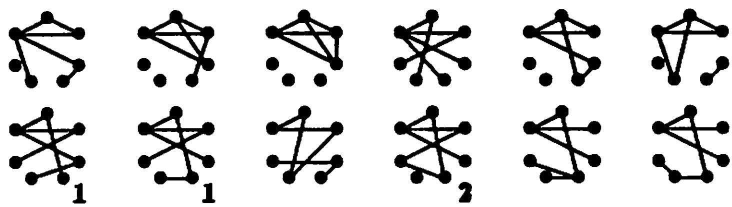
A10.21. A 3-(21,6,208) design.



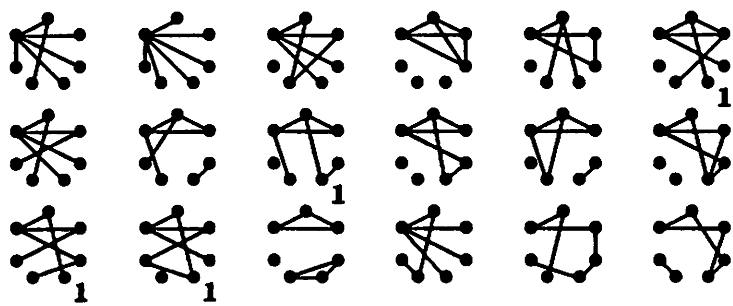
A10.22. A 3-(21,6,208) design.



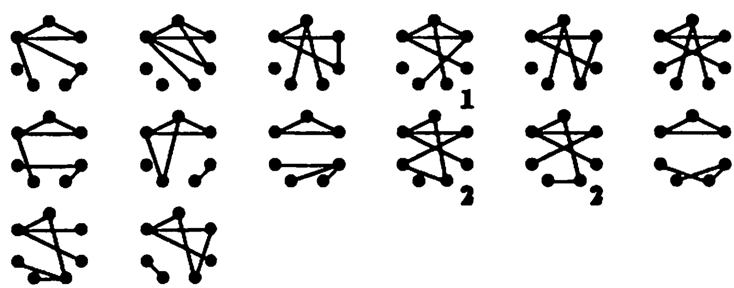
A10.23. A 3-(21,6,220) design.



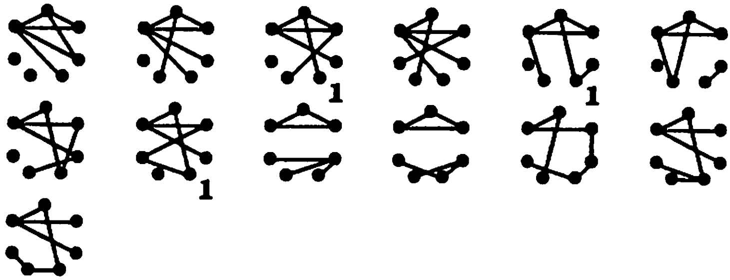
A10.24. A 3-(21,6,236) design.



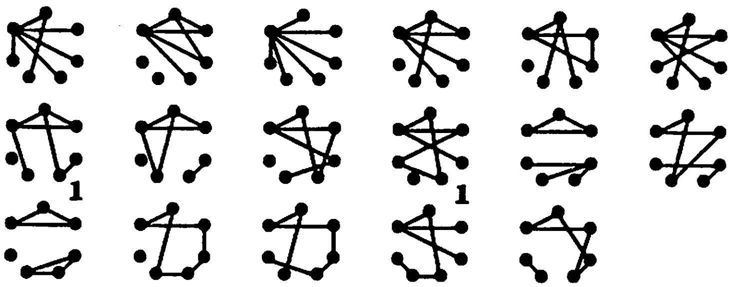
A10.25. A 3-(21,6,280) design.



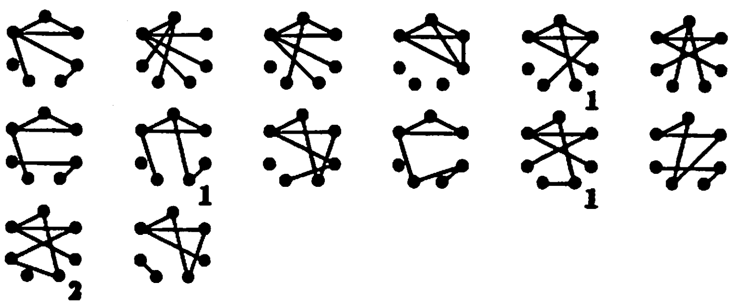
A10.26. A 3-(21,6,280) design.



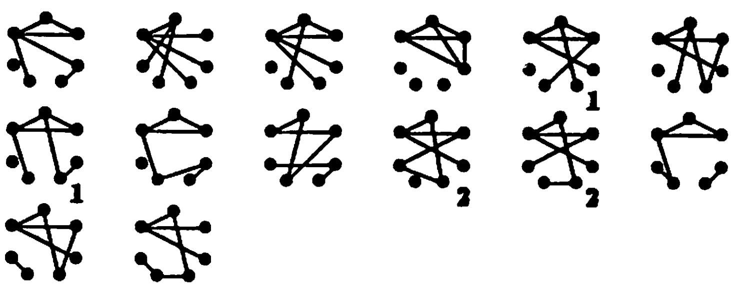
A10.27. A 3-(21,6,296) design.



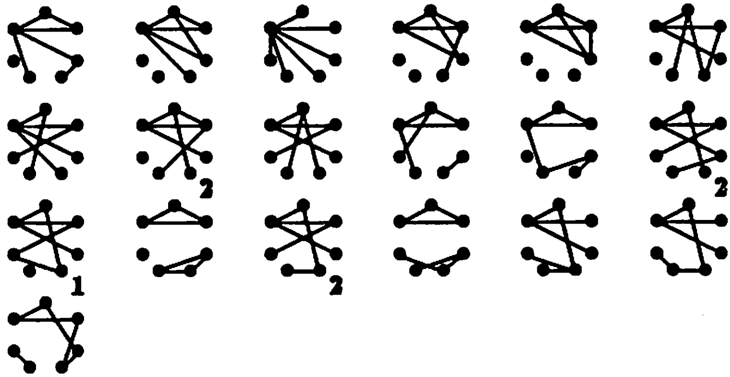
A10.28. A 3-(21,6,340) design.



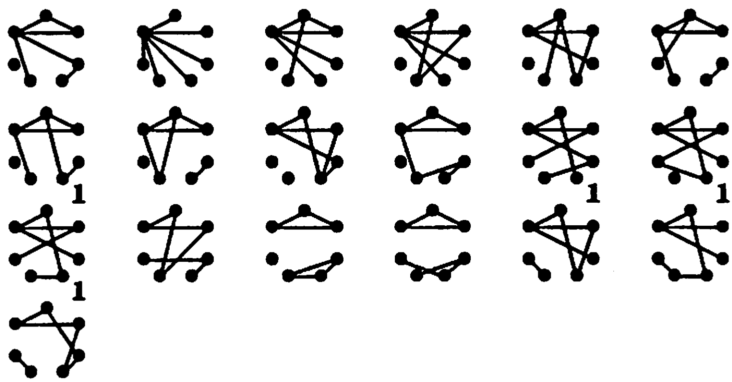
A10.29. A 3-(21,6,340) design.



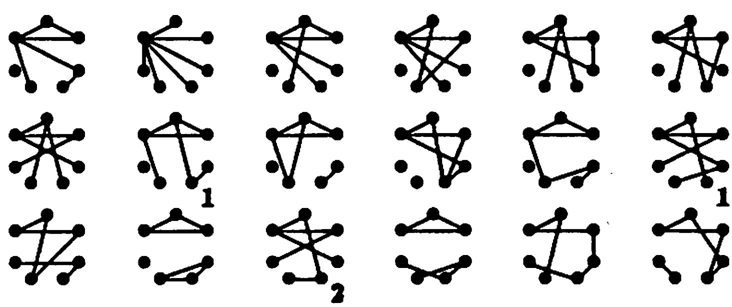
A10.30. A 3-(21,6,356) design.



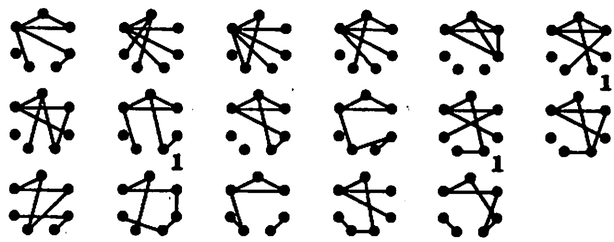
A10.31. A 3-(21,6,376) design.



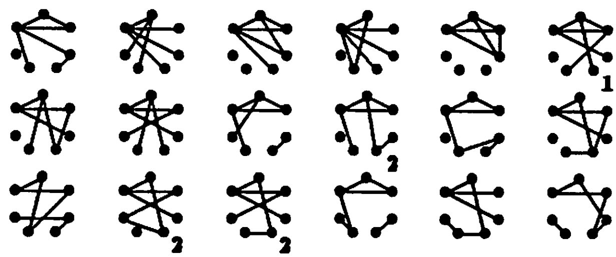
A10.32. A 3-(21,6,376) design.



A10.33. A 3-(21,6,388) design.



A10.34. A 3-(21,6,388) design.



A10.35. 3-(21,6,λ) designs obtained by union and complement.

We obtain 3-(21,6,λ) designs with $\lambda \in \{160,168,176,248,256,268,288,320,328,336,356,368,400\}$ by the constructions indicated below.

3-(21,6,148): Union of designs A10.16 and A10.18.
3-(21,6,160): Union of designs A10.10 and A10.16.
3-(21,6,168): Union of designs A10.10 and A10.17.
3-(21,6,176): Union of designs A10.10 and A10.19.
3-(21,6,248): Union of designs A10.10 and A10.21.
3-(21,6,256): Union of designs A10.13 and A10.20.
3-(21,6,268): Union of designs A10.20 and A10.14.
3-(21,6,288): Union of designs A10.10, A10.16 and A10.18.
3-(21,6,496): Union of designs A10.13 and A10.33.
3-(21,6,320): Complement of the above design.
3-(21,6,328): Union of designs A10.14 and A10.21.
3-(21,6,336): Union of designs A10.10 and A10.27.
3-(21,6,460): Union of designs A10.16 and A10.28.
3-(21,6,356): Complement of the above design.
3-(21,6,448): Union of designs A10.13 and A10.28.
3-(21,6,368): Complement of the above design.
3-(21,6,416): Union of designs A10.10 and A10.31.
3-(21,6,400): Complement of the above design.

A.11. 3−(23,8,8s) designs for $s \geq 2$.

Generating the orbit of {0,1,2,3,5,7,12,16} under the group $AF(23)$ constructs a 3-(23,8,16) design and the union of the orbits of {0,1,3,4,6,7,8,22} and {0,1,2,3,5,7,12,17} forms a 3-(23,8,24) design. These two designs are disjoint. In each box of Tables IXa, IXb and IXc is displayed two 6-element subsets, A and B. The union of the orbits of $A \cup \{0,1\}$ and $B \cup \{0,1\}$ under $AF(23)$ is for each box a 3-(23,8,32) design. Furthermore the 241 designs generated from these tables are pairwise disjoint and are each disjoint from the 3-(23,8,16) and 3-(23,8,24) designs given above. Thus by taking unions of combinations of these 243 pairwise disjoint designs we can construct a 3−(23,8,λ) design for each $\lambda = 8s \leq (15504)/2 = 7752$ except $\lambda = 8$.

TABLE IXa

3 5 6 10 11 20	3 5 8 9 11 17	2 3 4 5 6 8	3 4 6 9 10 11	2 3 5 6 8 9	3 5 6 7 10 13
2 3 5 6 7 9	3 4 5 6 9 10	2 3 4 5 7 9	2 5 7 9 10 12	2 3 4 5 8 9	2 3 5 6 10 11
3 4 5 7 8 9	2 3 5 6 9 10	2 3 4 5 7 10	3 4 6 7 9 11	2 3 4 5 6 10	3 4 7 8 9 11
2 3 4 5 8 10	3 4 5 7 10 12	2 3 5 6 7 10	3 4 6 9 11 19	2 3 4 6 8 10	3 4 5 8 10 14
2 3 6 8 9 10	2 3 5 7 12 15	2 3 6 7 8 10	2 3 5 7 10 13	3 4 5 6 8 10	2 3 4 8 10 14
3 5 6 7 8 10	2 5 7 9 10 13	3 4 6 7 8 10	2 3 4 8 9 15	2 3 4 6 9 10	2 3 6 7 11 13
2 3 5 7 9 10	2 5 6 7 9 18	2 3 4 7 9 10	3 4 5 6 7 15	2 3 6 7 9 10	2 3 5 7 13 20
3 5 6 7 9 10	3 4 5 7 8 13	2 5 6 7 9 10	2 3 5 7 12 13	3 4 5 6 7 11	3 4 5 7 9 12
2 3 4 5 7 11	3 4 5 8 9 13	2 3 4 5 6 11	3 5 6 8 10 14	3 4 6 8 9 10	2 3 5 7 9 18
2 3 4 6 8 11	3 5 6 10 11 14	2 3 4 5 8 11	3 4 5 6 9 11	2 3 5 6 8 11	2 3 5 6 9 14
3 5 6 7 9 11	3 4 5 10 11 14	2 3 5 6 9 11	2 3 5 8 9 18	3 5 6 7 8 11	2 3 4 6 9 11
2 3 6 7 8 11	2 3 5 7 9 13	3 4 5 7 8 11	3 4 5 7 9 13	3 4 6 7 8 11	2 3 4 8 10 13
3 4 5 7 9 11	2 3 6 9 11 13	2 3 5 7 9 11	2 3 6 7 8 13	2 3 6 7 9 11	2 3 6 8 10 13
2 5 6 7 9 11	3 4 5 6 9 14	3 4 5 8 9 11	2 3 4 5 6 12	2 3 4 8 9 11	2 3 4 5 7 20
4 5 6 7 9 11	2 3 5 8 9 13	2 3 5 8 9 11	2 3 4 8 12 14	3 4 6 8 9 11	2 3 4 6 10 19
2 3 6 8 9 11	2 3 4 8 12 21	2 3 4 5 10 11	3 4 6 8 13 15	3 5 7 8 9 11	3 5 6 7 8 13
2 5 7 8 9 11	2 3 6 8 12 21	4 5 7 8 9 11	2 3 5 8 9 14	2 3 6 8 10 11	3 4 6 8 12 15
2 5 6 7 10 11	2 3 4 6 9 14	2 3 5 8 10 11	3 4 5 6 7 17	3 4 5 9 10 11	2 3 4 5 8 13
4 5 7 9 10 11	2 3 4 9 12 14	2 5 7 9 10 11	2 3 4 6 13 15	2 3 5 8 9 12	2 3 4 5 8 12
3 5 6 7 8 12	2 3 4 8 9 17	3 4 5 6 8 12	3 5 6 10 13 14	3 4 5 6 7 12	2 3 5 6 8 12
2 3 5 6 7 12	3 5 6 7 9 21	2 3 6 7 8 12	2 3 4 5 10 16	2 3 5 7 8 12	3 4 6 8 12 13
3 4 5 7 8 12	2 3 4 8 10 15	3 4 6 7 8 12	3 4 5 8 10 16	2 3 6 7 9 12	3 6 7 9 11 13
2 3 5 7 9 12	2 3 5 6 8 14	2 3 5 6 9 12	3 5 8 9 12 18	2 3 4 7 9 12	3 5 6 7 12 19
3 5 6 7 9 12	3 6 7 10 12 14	2 5 6 7 9 12	2 3 5 9 10 14	2 3 4 8 10 12	3 4 6 8 11 17
2 3 5 6 10 12	2 3 5 8 10 14	2 3 4 5 10 12	2 3 5 6 11 18	3 5 7 8 9 12	3 4 5 6 7 14

TABLE IXb

2 3 5 7 10 12	2 5 6 7 9 16	2 3 4 7 10 12	2 3 5 6 8 13	3 4 5 6 10 12	3 4 6 8 12 21
2 3 6 8 10 12	2 3 5 8 12 14	3 4 5 8 10 12	3 6 7 8 12 16	2 3 5 8 10 12	2 3 4 5 7 19
3 4 6 8 10 12	2 3 6 7 8 15	4 5 7 9 10 12	3 5 6 7 9 13	2 3 5 6 11 12	2 3 6 8 10 14
3 4 5 6 11 12	3 5 7 8 11 18	3 4 6 8 11 12	3 5 7 8 14 15	3 4 6 7 11 12	2 3 6 7 10 13
3 4 5 7 11 12	3 4 5 8 9 16	2 3 6 8 11 12	3 5 6 7 10 17	3 4 5 10 11 12	3 4 5 7 9 18
3 6 7 8 11 12	3 4 5 7 9 20	3 5 6 8 11 12	2 3 4 8 10 21	3 4 5 9 11 12	3 5 6 7 11 22
3 4 7 9 11 12	2 3 4 5 8 16	3 6 8 10 11 12	2 3 6 7 13 15	3 5 6 10 11 12	2 3 4 8 9 20
3 4 5 6 7 13	3 6 8 9 11 15	2 3 5 6 7 13	2 3 6 7 9 20	3 4 5 6 8 13	2 5 7 9 11 14
3 4 6 7 8 13	2 3 6 9 10 14	2 3 6 7 9 13	2 3 6 10 12 21	2 3 5 6 9 13	3 4 5 8 11 15
2 5 6 7 9 13	2 3 5 7 10 15	2 3 5 6 10 13	3 4 6 8 12 14	3 4 7 8 9 13	2 3 5 6 10 15
3 4 6 8 9 13	3 5 6 10 11 17	2 3 4 5 10 13	4 5 7 9 11 20	3 5 7 8 9 13	3 7 10 11 12 14
2 3 4 6 10 13	3 6 7 8 13 14	3 6 7 8 10 13	2 3 4 7 9 17	3 4 5 8 10 13	3 4 5 7 9 15
2 3 5 8 10 13	3 4 6 8 10 21	3 4 6 8 10 13	2 3 6 8 14 21	3 5 7 8 10 13	3 4 6 8 13 17
2 3 5 9 10 13	3 5 6 7 8 14	3 6 8 9 10 13	3 10 11 12 13 15	3 5 6 7 12 13	3 4 6 7 12 13
3 4 5 9 11 13	2 3 6 10 12 19	2 5 7 10 11 13	2 3 5 7 13 16	2 5 6 7 12 13	3 4 6 8 10 16
3 4 7 8 12 13	2 3 5 7 12 17	3 5 7 8 12 13	2 3 5 8 9 20	3 4 8 10 12 13	3 4 5 7 8 14
3 6 8 10 12 13	3 4 5 10 12 14	3 6 7 8 10 14	2 3 5 7 9 16	2 3 6 7 9 14	2 3 6 9 11 18
2 3 4 6 8 14	3 5 6 7 10 21	3 9 10 11 12 13	3 4 7 8 14 20	3 4 10 11 12 13	2 5 6 7 10 15
3 5 10 11 12 13	3 4 6 7 8 14	3 8 10 11 12 13	2 3 6 7 8 14	2 3 4 5 7 14	3 5 8 9 11 19
2 3 5 6 7 14	2 3 5 9 12 19	2 3 5 6 10 14	2 3 6 7 9 22	3 4 5 8 9 14	3 5 6 7 11 19
3 4 6 7 9 14	3 5 6 7 10 15	2 5 6 7 9 14	3 4 5 7 10 16	3 4 7 8 9 14	3 5 6 7 10 18
2 3 4 5 10 14	3 4 6 7 8 20	3 5 7 8 9 14	3 5 6 8 10 19	2 3 5 7 10 14	3 4 5 7 12 17
2 3 4 7 10 14	2 3 5 6 10 16	2 5 6 7 10 14	2 3 4 6 11 17	3 4 6 7 10 14	3 4 6 8 12 19
3 5 6 7 10 14	2 3 5 9 12 16	3 4 6 8 10 14	2 3 4 6 9 15	3 5 7 8 10 14	2 3 4 8 12 19
3 4 6 9 11 14	2 3 5 7 9 15	3 4 6 9 10 14	4 5 7 9 11 14	2 5 7 9 10 14	2 3 4 5 7 17
3 4 7 9 10 14	2 3 5 7 12 20	3 4 6 8 11 14	2 3 4 10 12 17	2 3 6 7 11 14	3 4 8 10 12 16
3 4 5 8 11 14	2 3 4 9 10 21	3 4 5 9 11 14	2 4 5 7 12 18	3 6 8 10 11 14	3 6 7 9 11 16
3 6 7 9 11 14	2 3 5 9 10 19	2 5 7 10 11 14	3 10 11 12 13 17	3 4 6 7 12 14	2 3 5 6 9 17
3 4 5 7 12 14	2 3 5 9 10 16	2 5 6 7 12 14	2 3 4 5 11 19	3 4 5 8 12 14	3 10 11 12 13 20
3 4 7 10 12 14	2 3 4 6 9 17	3 5 6 10 12 14	3 4 6 7 12 17	3 4 6 8 13 14	2 3 4 5 7 12
2 3 4 5 7 15	3 4 7 9 10 19	2 3 5 6 7 15	3 4 6 11 12 16	2 3 5 6 8 15	3 4 6 10 12 17
2 3 4 7 9 15	3 6 7 8 10 17	3 5 6 7 9 15	3 4 5 9 11 16	3 4 6 7 9 15	2 3 5 6 12 18
2 3 5 8 9 15	2 3 4 9 10 15	3 4 7 8 9 15	2 3 5 7 9 17	2 3 6 8 9 15	2 3 5 7 10 19
3 5 7 8 9 15	3 5 6 7 11 21	2 3 6 7 10 15	2 3 4 7 9 18	3 6 7 8 10 15	3 6 8 11 12 15
2 3 5 8 10 15	3 4 6 8 10 20	3 5 6 8 10 15	3 4 6 7 9 16	3 6 8 9 10 15	3 4 6 7 10 17

Table IXc

2 3 4 8 12 15	3 4 6 8 10 18	3 5 6 8 11 15	3 5 7 8 9 20	3 5 8 9 11 15	2 3 4 7 9 16
3 5 6 10 11 15	2 3 4 9 13 20	3 5 6 7 12 15	2 3 4 9 11 16	3 4 5 10 12 15	2 3 4 5 10 19
2 3 6 8 12 15	3 4 5 7 11 21	3 4 6 7 13 15	2 3 5 9 10 20	2 3 5 6 13 15	2 3 4 8 9 21
3 4 7 8 9 16	2 3 10 11 12 20	3 5 6 7 8 16	3 4 6 9 11 16	2 3 4 5 7 16	2 3 5 10 12 16
3 4 5 7 14 15	3 4 7 9 10 20	2 3 5 7 14 15	3 5 7 8 9 19	3 4 5 6 7 16	3 4 6 9 10 16
3 4 5 6 8 16	3 4 7 9 11 16	2 3 6 7 8 16	3 4 5 7 9 21	3 4 5 7 9 16	2 3 6 7 8 20
3 5 6 7 11 16	2 3 5 6 8 18	2 3 6 7 10 16	2 3 5 6 10 20	3 5 7 8 9 16	3 6 7 8 11 20
2 3 5 7 10 16	3 6 8 11 12 21	2 5 6 7 10 16	2 3 4 6 11 18	2 3 4 9 10 16	2 3 5 8 12 19
2 3 4 5 11 16	3 4 5 8 9 18	2 3 4 7 11 16	2 3 6 7 8 22	3 4 6 7 11 16	2 3 6 8 9 17
4 5 7 9 11 16	3 4 5 7 12 22	3 4 5 8 9 17	3 4 5 7 10 17	3 4 5 9 14 16	3 4 7 9 10 22
3 6 7 10 15 16	3 4 5 7 13 18	2 3 4 5 8 17	3 4 7 8 12 18	2 3 6 7 9 17	2 3 5 7 12 18
3 4 5 7 9 17	3 4 5 8 14 17	3 5 6 7 9 17	3 5 6 7 8 21	3 4 6 7 9 17	2 3 4 5 7 18
2 3 5 6 10 17	3 4 6 9 10 21	2 3 4 5 10 17	3 4 5 8 9 21	3 5 7 8 9 17	2 3 6 8 11 17
2 3 6 7 10 17	3 4 5 8 10 17	2 3 5 7 10 17	3 6 8 10 13 19	2 3 6 8 10 17	3 5 6 10 11 21
3 4 7 9 10 17	2 3 4 7 12 19	3 4 6 9 10 17	2 3 4 7 11 18	2 3 4 8 11 17	3 4 6 8 11 21
3 4 6 9 11 17	3 4 5 7 10 19	3 5 6 8 11 17	3 4 5 9 11 19	3 4 5 9 11 17	3 4 5 7 13 20
2 5 7 10 11 17	3 4 6 8 12 17	3 4 7 9 12 17	3 4 5 6 7 18	3 6 8 9 13 17	3 4 8 10 13 19
3 4 6 9 13 17	2 3 5 7 12 21	3 4 6 7 8 18	2 3 5 7 10 18	2 3 5 6 9 18	3 6 7 8 12 21
3 4 7 8 9 18	2 3 5 6 10 19	2 3 4 7 10 18	3 5 8 9 11 18	3 4 5 7 10 18	2 3 6 8 9 19
2 3 5 9 11 18	2 3 4 8 10 20	3 4 5 6 8 11	2 3 4 8 11 22	2 3 4 10 12 18	2 3 6 9 11 21
2 3 5 7 13 18	3 6 8 9 10 21				

A.12. 3−(23,9,12s) designs for $s \geq 2$.

Generating the orbit of {0,1,3,4,5,6,7,11,12} under the group $AF(23)$ constructs a 3-(23,9,24) design and the union of the orbits of {0,1,2,3,5,7,8,9,10} and {0,1,3,4,5,6,7,8,12} forms a 3-(23,9,36) design. These two designs are disjoint. In each box of Tables Xa, Xb, Xc, Xd and Xe is displayed two 6-element subsets, A and B. The union of the orbits of $A \cup \{0,1\}$ and $B \cup \{0,1\}$ under $AF(23)$ is for each box a 3-(23,9,32) design. Furthermore the 403 designs generated from these tables are pairwise disjoint and are each disjoint from the 3-(23,9,24) and 3-(23,9,36) designs given above. Thus by taking unions of combinations of these 405 pairwise disjoint designs we can construct a 3−(23,9,λ) design for each $\lambda = 12s \leq (38760)/2 = 19380$ except $\lambda = 12$.

TABLE Xa

2 3 5 7 8 10 13	3 4 6 8 9 10 15	3 4 5 6 9 10 16	3 4 5 6 9 11 17	3 6 7 8 11 12 18	3 4 5 8 11 15 19
3 4 6 8 11 15 20	2 3 6 7 9 12 21	3 4 6 8 10 12 21	2 3 5 6 10 12 22	3 10 11 12 13 14 17	2 3 6 8 9 11 15
3 6 10 11 12 13 14	2 5 6 7 9 11 20	2 3 4 5 7 12 13	4 5 6 7 9 11 20	3 4 5 6 7 8 13	3 4 5 6 7 10 12
2 3 4 5 7 10 12	2 3 5 8 9 10 13	2 3 4 5 8 10 11	2 3 4 5 7 10 14	3 4 5 6 7 8 11	2 3 6 8 9 10 12
2 3 5 6 7 8 10	2 3 4 6 8 11 12	2 3 5 6 7 8 9	3 5 6 7 9 10 11	2 3 4 5 6 8 10	2 3 5 8 9 10 12
2 3 4 5 6 7 10	3 4 6 7 9 11 13	3 5 6 7 8 9 10	3 4 5 6 9 10 14	2 3 5 6 8 9 10	2 3 5 6 7 9 12
2 3 5 6 7 9 10	2 3 6 8 9 12 15	3 4 5 6 7 8 10	3 4 5 6 8 9 13	2 3 4 5 8 9 10	2 3 5 7 9 11 14
3 4 5 6 8 9 10	3 5 6 7 9 10 13	2 3 4 5 6 8 11	3 4 5 9 10 11 14	2 3 4 5 6 7 11	2 5 6 7 12 13 16
2 3 5 6 7 8 11	3 10 11 12 13 14 16	3 4 5 6 8 9 11	2 3 4 6 8 11 19	2 3 5 6 7 9 11	3 4 6 7 8 12 13
2 3 4 5 7 9 11	2 3 5 6 7 8 13	2 3 5 6 8 9 11	2 3 4 5 8 11 14	2 3 4 5 8 9 11	2 3 4 5 6 13 15
3 5 6 7 8 9 11	2 3 5 6 8 10 12	3 4 5 7 8 9 11	2 3 4 5 10 12 15	2 3 5 7 8 9 11	3 4 5 6 9 12 14
2 3 6 7 8 9 11	2 3 4 5 6 8 12	2 5 6 7 8 9 11	2 3 4 5 7 12 14	2 3 4 5 7 10 11	2 3 4 5 8 10 14
2 3 4 5 6 10 11	2 3 5 6 8 11 14	2 3 4 5 6 7 12	3 4 6 9 11 15 17	2 3 5 6 9 10 11	2 3 4 5 8 9 18
2 3 5 6 8 10 11	2 3 4 5 6 11 14	2 3 6 7 8 10 11	2 3 4 5 8 14 19	2 3 4 6 9 10 11	3 6 8 9 10 13 17
2 3 4 5 9 10 11	3 4 5 6 7 14 22	3 4 5 6 9 10 11	2 3 4 6 9 14 15	2 3 6 7 9 10 11	2 3 4 5 8 9 14
2 5 6 7 9 10 11	2 3 5 6 7 9 16	2 3 6 8 9 10 11	2 3 4 7 8 13 14	3 4 6 8 9 10 11	2 3 4 6 8 12 19
2 3 5 6 8 9 12	3 4 5 8 11 14 17	2 3 4 5 7 8 12	3 4 6 7 9 12 17	2 3 5 6 7 8 12	2 3 4 6 9 11 14
2 3 4 5 7 9 12	3 5 6 10 11 12 14	2 3 4 5 8 9 12	2 3 5 6 7 12 13	3 4 5 7 8 9 12	2 3 5 7 10 13 16
3 4 5 6 8 9 12	3 5 6 7 10 12 20	2 3 5 7 8 9 12	2 3 4 7 8 12 13	3 5 6 7 8 9 12	2 3 6 7 10 13 15
2 3 6 7 8 9 12	2 3 4 5 8 11 21	2 5 6 7 8 9 12	2 3 4 5 6 10 12	2 3 4 5 8 10 12	2 3 5 8 9 10 13
2 3 5 6 7 10 12	3 5 6 7 8 9 13	2 3 4 6 8 10 12	3 4 6 7 8 11 13	2 3 6 7 8 10 12	3 5 6 7 9 11 14
3 4 5 7 8 10 12	2 3 5 8 9 13 20	3 4 5 6 8 10 12	3 4 5 6 9 11 14	3 4 6 7 8 10 12	3 4 5 8 9 14 15
2 3 4 6 9 10 12	3 4 5 6 7 8 14	2 3 4 7 9 10 12	2 3 4 5 6 7 14	3 4 5 6 9 10 12	3 7 10 11 12 13 14
3 5 6 7 9 10 12	2 5 7 9 10 13 15	2 5 6 7 9 10 12	2 3 4 6 8 10 13	2 3 5 6 10 11 12	3 4 5 7 8 9 14
3 5 6 7 8 11 12	2 3 5 6 9 12 16	2 5 7 8 9 10 12	2 3 6 7 9 12 16	3 4 6 8 9 10 12	2 3 5 7 11 12 20
3 5 6 8 9 10 12	2 3 6 7 10 13 14	2 3 4 5 6 11 12	3 6 7 8 9 11 13	2 3 5 6 7 11 12	2 5 7 9 10 11 13
3 4 5 6 8 11 12	3 4 5 7 8 10 13	2 3 4 5 8 11 12	2 5 6 7 9 12 13	2 3 5 6 8 11 12	2 3 4 6 8 11 14
3 4 6 7 8 11 12	2 5 7 8 9 11 14	2 3 4 6 9 11 12	2 3 5 6 7 14 16	3 4 5 6 9 11 12	4 5 7 9 10 11 14
2 3 6 7 9 11 12	3 4 5 6 7 13 17	3 4 5 7 9 11 12	3 4 5 6 7 8 18	3 4 6 7 9 11 12	3 4 5 6 8 10 15
2 3 5 8 9 11 12	2 3 4 5 8 13 14	2 3 4 6 10 11 12	2 3 6 7 8 13 15	2 3 4 8 10 11 12	3 5 6 7 11 14 17
2 3 5 7 10 11 12	3 4 5 8 9 13 18	2 3 6 8 10 11 12	2 4 5 6 7 10 13	3 4 5 8 10 11 12	2 3 5 7 10 13 14
2 3 5 8 10 11 12	2 3 4 6 9 12 14	3 5 6 8 10 11 12	3 4 5 8 9 13 16	3 4 6 8 10 11 12	2 3 5 6 8 14 15
2 3 4 5 6 7 13	3 4 6 9 10 11 20	4 5 7 9 10 11 12	2 3 4 7 9 13 14	2 3 5 6 8 10 13	2 3 5 6 9 10 14
2 3 5 6 8 9 13	2 3 5 6 9 10 15	2 3 5 6 7 9 13	2 3 5 6 11 12 15	2 3 4 5 7 9 13	3 4 6 7 8 11 20
2 3 4 5 8 9 13	4 5 7 8 9 11 15	3 4 5 7 8 9 13	2 3 5 7 10 12 15	2 3 5 7 8 9 13	2 3 6 7 10 11 13

Table Xb

2 3 6 7 8 9 13	3 4 5 6 7 10 15	2 3 5 6 7 10 13	2 3 5 6 7 9 19	2 3 4 5 7 10 13	3 4 6 7 10 12 14
2 3 4 5 6 10 13	3 4 5 7 8 11 13	2 3 4 5 8 10 13	3 5 6 7 10 11 13	3 4 5 6 7 10 13	2 3 6 7 8 9 15
3 4 6 7 8 10 13	2 3 5 6 8 10 14	3 4 5 6 8 10 13	2 3 4 5 8 12 18	2 3 6 7 8 10 13	2 3 5 8 10 12 13
3 5 6 7 8 10 13	2 3 4 8 9 11 14	2 5 6 7 8 10 13	2 3 5 6 9 12 13	2 3 4 6 9 10 13	3 6 7 8 10 13 15
2 3 5 7 9 10 13	2 3 5 6 8 10 21	3 4 5 6 9 10 13	2 3 5 6 7 12 18	3 4 6 7 9 10 13	2 3 4 5 7 14 20
2 3 6 7 9 10 13	3 4 5 9 12 14 20	2 5 6 7 9 10 13	3 6 10 11 12 13 15	3 4 5 6 7 11 13	2 3 6 8 10 12 [illegible]
2 3 4 5 6 11 13	3 4 5 6 8 14 19	2 3 6 8 9 10 13	2 3 5 7 9 12 15	3 5 6 8 9 10 13	3 5 6 7 9 10 17
2 5 7 8 9 10 13	2 3 4 5 7 11 17	2 3 4 5 7 11 13	2 3 4 8 10 12 14	2 3 5 6 7 11 13	2 3 4 6 9 10 14
2 3 4 6 8 11 13	3 5 6 7 10 11 20	2 3 4 5 8 11 13	3 4 5 7 12 14 17	2 3 6 7 8 11 13	2 3 4 6 10 12 17
4 5 7 8 9 11 13	3 4 5 6 8 10 17	2 3 4 8 9 11 13	4 5 7 9 11 14 22	2 3 5 6 9 11 13	4 5 6 7 9 11 14
2 3 4 6 9 11 13	3 4 5 6 8 10 19	2 3 6 7 9 11 13	2 3 4 6 9 10 16	3 4 5 8 9 11 13	2 3 4 5 7 10 15
2 3 5 8 9 11 13	3 4 5 6 7 14 18	2 3 6 8 9 11 13	3 4 5 6 10 12 19	2 5 7 8 9 11 13	2 3 5 7 8 11 14
3 4 7 8 9 11 13	3 5 6 7 10 15 20	2 3 5 8 10 11 13	2 3 4 6 8 12 14	2 3 5 7 10 11 13	2 3 6 7 8 12 16
2 3 5 6 10 11 13	3 5 6 7 8 9 15	2 3 4 6 10 11 13	2 3 5 6 8 14 21	3 4 5 6 10 11 13	2 3 4 5 7 13 20
2 3 5 9 10 11 13	3 4 5 6 8 9 15	2 3 6 9 10 11 13	2 3 5 6 7 9 14	3 6 7 9 10 11 13	2 3 4 6 8 11 15
2 3 6 10 11 12 13	3 4 6 8 9 11 14	3 5 6 7 8 12 13	2 3 6 8 9 10 16	3 4 5 6 8 12 13	2 3 5 6 10 12 14
3 4 5 6 7 12 13	3 6 7 10 12 13 14	2 3 5 6 8 12 13	2 3 6 8 11 12 21	3 4 5 7 8 12 13	2 3 4 6 8 11 17
2 3 5 7 8 12 13	2 3 6 7 8 10 17	2 3 6 7 9 12 13	2 3 5 6 8 12 15	2 3 4 8 10 12 13	2 3 6 7 8 10 16
2 3 5 8 9 12 13	3 5 6 7 8 11 18	3 5 6 7 9 12 13	3 5 6 8 10 11 14	2 3 6 8 9 12 13	2 3 5 6 7 10 15
3 5 7 8 9 12 13	3 4 5 6 7 11 14	2 3 6 7 10 12 13	2 3 5 6 8 10 20	2 3 4 7 10 12 13	2 3 5 7 8 10 14
3 4 5 7 10 12 13	3 4 5 8 11 14 22	3 5 6 7 10 12 13	2 3 4 7 9 12 14	3 4 5 7 11 12 13	2 3 5 6 7 13 16
3 4 6 8 10 12 13	2 3 5 6 7 8 14	3 4 5 8 10 12 13	2 3 5 8 10 12 17	2 3 6 8 10 12 13	3 4 5 7 8 13 14
2 3 5 9 10 12 13	3 4 5 10 12 14 20	2 5 7 9 10 12 13	2 3 5 7 11 12 15	3 4 5 8 11 12 13	3 4 5 6 7 13 16
3 4 6 7 11 12 13	3 4 5 6 8 10 16	2 3 5 6 8 9 14	2 3 5 6 10 12 15	2 3 6 7 8 9 14	2 3 5 7 13 16 21
2 3 4 5 6 10 14	3 4 5 6 10 11 21	2 4 5 6 7 10 14	2 3 6 7 9 10 17	2 3 5 6 7 10 14	2 3 4 5 8 10 22
3 4 6 7 8 10 14	3 4 5 6 7 14 21	3 4 5 6 8 10 14	3 8 10 11 12 13 14	2 3 6 7 8 10 14	2 3 4 5 6 11 17
3 5 6 7 8 10 14	3 4 5 7 9 13 16	2 3 6 8 9 10 14	2 3 4 5 8 11 17	2 3 4 7 9 10 14	2 3 5 6 7 10 18
2 3 5 7 9 10 14	2 5 7 9 10 14 18	2 3 6 7 9 10 14	2 3 4 5 7 13 16	3 5 6 7 9 10 14	3 4 8 10 12 13 15
2 5 6 7 9 10 14	3 6 7 9 11 13 16	2 3 5 8 9 10 14	3 5 6 7 9 10 15	2 3 4 5 7 11 14	3 4 5 6 11 12 15
3 4 6 8 9 10 14	2 3 5 6 10 12 17	3 4 5 6 8 12 14	2 3 4 6 9 10 18	2 3 5 6 10 11 14	2 3 5 6 9 10 18

Table Xc

3 4 5 7 8 11 14	2 3 5 6 9 10 20	3 4 6 7 8 11 14	2 3 4 6 10 11 21	3 4 5 7 9 11 14	3 4 5 9 10 11 20
3 4 6 7 9 11 14	3 4 5 6 7 11 16	2 3 6 7 9 11 14	2 3 4 6 9 10 21	2 5 6 7 9 11 14	2 3 5 7 9 14 18
3 4 7 8 9 11 14	2 3 4 6 10 11 17	3 4 5 8 9 11 14	3 5 6 7 8 12 18	2 3 5 8 9 11 14	2 3 6 8 9 10 15
4 5 7 8 9 11 14	3 4 7 8 9 11 16	3 5 7 8 9 11 14	3 4 5 9 10 11 18	2 3 4 6 10 11 14	3 6 7 9 11 12 14
2 3 6 8 10 11 14	3 4 5 6 8 11 16	2 3 5 7 10 11 14	2 3 4 5 6 10 16	2 3 4 7 10 11 14	3 9 10 11 12 13 16
3 4 5 6 10 11 14	2 3 4 6 10 13 17	3 5 6 7 10 11 14	2 3 4 6 9 11 19	3 4 5 7 10 11 14	2 3 4 5 8 10 18
2 3 5 8 10 11 14	2 3 5 6 7 11 16	2 3 4 8 10 11 14	2 3 4 5 7 13 15	3 4 5 8 10 11 14	2 5 7 9 10 13 16
3 4 6 9 10 11 14	2 3 4 6 8 10 15	3 4 6 8 10 11 14	2 3 4 5 8 10 15	3 4 5 6 7 12 14	3 4 5 7 9 11 16
2 3 4 5 8 12 14	2 3 4 7 9 10 15	2 3 5 6 8 12 14	2 3 4 6 8 10 16	2 3 6 7 10 12 14	3 5 7 8 9 11 22
2 3 5 8 9 12 14	3 4 5 6 7 12 19	2 3 6 7 9 12 14	2 3 5 6 7 12 17	3 5 6 7 8 12 14	2 3 4 5 7 13 18
3 4 6 7 8 12 14	2 3 4 8 10 13 19	3 4 5 7 9 12 14	3 4 5 6 9 10 15	3 5 6 7 9 12 14	4 5 7 9 10 11 15
2 5 6 7 9 12 14	2 3 4 8 10 13 21	2 3 4 5 10 12 14	2 3 6 9 10 12 14	3 4 5 7 10 12 14	2 3 4 5 7 9 17
3 5 6 7 10 12 14	3 4 6 8 9 16 17	3 4 7 9 10 12 14	2 3 4 5 6 10 15	3 5 6 9 10 12 14	2 3 4 7 9 10 16
2 3 6 8 11 12 14	3 5 6 7 9 10 18	3 4 6 8 11 12 14	4 5 7 8 9 11 16	3 4 6 10 11 12 14	3 10 11 12 13 14 15
3 4 7 9 11 12 14	2 3 5 8 10 13 19	3 4 7 10 11 12 14	3 5 6 7 9 12 19	3 6 8 10 11 12 14	3 6 7 8 10 16 17
2 3 6 7 8 13 14	3 6 8 10 11 12 15	2 3 4 6 9 13 14	2 3 6 7 8 11 17	3 4 6 7 8 13 14	2 3 4 6 9 10 17
2 3 5 6 10 13 14	2 3 5 6 8 12 19	3 4 7 8 9 13 14	3 7 10 11 12 13 15	3 4 6 8 9 13 14	3 5 6 10 11 12 17
2 3 4 6 10 13 14	3 4 5 6 8 14 18	2 3 4 7 10 13 14	3 4 6 9 10 11 16	3 5 6 7 10 13 14	2 3 4 7 9 10 18
2 5 6 7 10 13 14	2 3 5 7 9 10 17	3 5 7 8 12 13 14	2 3 6 8 10 14 17	2 3 7 8 11 13 14	3 4 6 7 10 12 17
3 4 8 10 12 13 14	3 4 5 6 11 12 18	3 6 8 10 12 13 14	3 4 6 7 8 11 16	3 4 10 11 12 13 14	2 3 6 7 9 11 18
3 5 10 11 12 13 14	2 3 5 6 7 10 16	2 3 5 6 9 10 16	3 4 6 8 9 10 17	2 3 5 6 8 9 15	2 3 4 6 9 10 15
3 9 10 11 12 13 14	2 4 5 7 12 14 17	2 3 5 6 7 9 15	3 4 5 7 9 11 19	3 4 5 6 7 8 15	2 3 4 8 9 11 19
2 3 4 5 7 9 15	3 4 6 8 9 11 15	2 3 4 5 8 9 15	2 3 5 6 10 11 16	2 5 6 7 8 9 15	2 3 5 7 9 10 16
2 3 5 6 8 10 15	3 4 5 7 10 12 16	2 3 6 7 8 10 15	3 4 5 8 9 16 17	3 5 6 7 8 10 15	3 4 6 7 9 12 19
2 3 5 7 9 10 15	2 3 4 8 9 13 15	3 4 5 6 9 11 15	3 4 5 7 10 12 19	2 3 4 5 6 11 15	3 4 6 7 8 13 21
3 5 6 8 9 10 15	3 6 8 10 11 12 20	3 4 5 6 7 11 15	3 4 6 7 9 11 19	2 3 4 5 8 11 15	3 4 5 9 10 11 19
2 3 5 6 8 11 15	2 3 5 6 10 14 18	2 3 5 6 9 11 15	2 3 4 6 9 12 21	2 3 4 6 9 11 15	2 3 5 6 8 10 19
3 4 6 7 9 11 15	2 5 7 8 9 11 18	2 3 6 7 9 11 15	2 3 5 6 7 9 20	2 3 4 8 9 11 15	2 3 4 6 8 10 19
2 3 5 8 9 11 15	2 5 7 9 10 13 17	2 3 4 5 10 11 15	3 4 7 8 13 14 22	2 3 5 7 10 11 15	3 4 5 8 9 14 16
3 4 5 6 10 11 15	2 3 4 5 8 14 18	2 3 6 8 10 11 15	3 5 6 7 8 9 17	3 4 5 8 10 11 15	3 4 6 8 12 13 22

TABLE Xd

2 3 5 8 10 11 15	3 6 8 10 12 13 22	3 4 5 9 10 11 15	2 3 4 8 12 14 19	2 3 5 8 9 12 15	2 3 4 7 9 10 17
3 4 5 6 7 12 15	2 3 4 6 9 11 17	2 3 5 6 7 12 15	2 3 4 6 9 10 20	2 3 4 6 8 12 15	2 3 5 7 10 11 21
2 3 4 5 8 12 15	3 4 6 7 9 10 16	3 5 6 7 8 12 15	3 4 6 7 8 12 16	2 3 4 8 9 12 15	2 3 6 7 9 12 20
3 5 7 8 9 12 15	3 4 5 6 8 13 18	3 5 6 7 10 12 15	2 3 5 6 8 9 16	3 4 5 8 10 12 15	3 6 8 10 11 12 21
3 4 6 8 10 12 15	3 4 6 7 8 11 18	3 4 6 8 11 12 15	3 4 5 6 10 11 17	3 6 8 9 11 12 15	2 3 5 6 7 10 19
3 5 6 10 11 12 15	2 3 6 8 9 13 17	3 4 5 6 7 14 15	2 3 5 7 9 10 19	3 4 7 8 9 13 15	3 4 5 6 11 12 17
2 3 5 6 8 13 15	3 5 6 7 9 10 16	3 4 5 6 7 13 15	2 3 4 6 8 10 20	3 4 6 7 8 13 15	3 4 7 8 10 12 16
3 4 5 8 9 13 15	2 5 6 7 9 11 16	2 3 6 8 9 13 15	2 3 4 6 10 13 15	2 3 4 5 10 13 15	3 4 6 7 9 11 20
2 3 4 8 10 13 15	3 6 7 8 11 12 19	3 6 7 10 12 13 15	2 3 4 6 9 13 19	2 3 4 5 7 14 15	3 4 5 8 9 11 19
2 3 5 8 9 14 15	2 3 6 7 9 10 20	3 4 5 6 9 14 15	2 5 7 8 9 11 21	2 3 4 8 9 14 15	3 4 5 8 10 11 16
2 3 5 7 10 14 15	2 3 4 8 10 12 18	3 4 5 6 7 8 16	2 3 4 8 9 11 22	2 3 4 5 8 9 16	3 4 5 9 10 11 16
2 3 5 6 8 10 16	3 4 6 7 11 12 19	2 3 4 5 8 10 16	2 3 4 5 10 11 21	3 4 6 7 8 10 16	3 4 5 7 9 12 22
3 4 5 6 8 12 16	2 3 5 6 7 10 21	2 3 4 5 7 11 16	2 5 6 7 9 13 20	3 10 11 12 13 14 17	2 3 6 8 9 11 15
3 6 10 11 12 13 14	2 5 6 7 9 11 20	2 3 4 5 7 12 13	4 5 6 7 9 11 20	3 4 5 6 7 8 13	3 4 5 6 7 10 12
2 3 4 5 7 10 12	2 3 5 8 9 10 13	2 3 4 5 8 10 11	2 3 4 5 7 10 14	3 4 5 6 7 8 11	2 3 6 8 9 10 12
2 3 5 6 7 8 10	2 3 4 6 8 11 12	2 3 5 6 7 8 9	3 5 6 7 9 10 11	2 3 4 5 6 8 10	2 3 5 8 9 10 12
2 3 4 5 6 7 10	3 4 6 7 9 11 13	3 5 6 7 8 9 10	3 4 5 6 9 10 14	2 3 5 6 8 9 10	2 3 5 6 7 9 12
2 3 5 6 7 9 10	2 3 6 8 9 12 15	3 4 5 6 7 8 10	3 4 5 6 8 9 13	2 3 4 5 8 9 10	2 3 5 7 9 11 14
3 4 5 6 8 9 10	3 5 6 7 9 10 13	2 3 4 5 6 8 11	3 4 5 9 10 11 14	2 3 4 5 6 7 11	2 5 6 7 12 13 16
2 3 5 6 7 8 11	3 10 11 12 13 14 16	3 4 5 6 8 9 11	2 3 4 6 8 11 19	2 3 5 6 7 9 11	3 4 6 7 8 12 13
2 3 4 5 7 9 11	2 3 5 6 7 8 13	2 3 5 6 8 9 11	2 3 4 5 8 11 14	2 3 4 5 8 9 11	2 3 4 5 6 13 15
3 5 6 7 8 9 11	2 3 5 6 8 10 12	3 4 5 7 8 9 11	2 3 4 5 10 12 15	2 3 5 7 8 9 11	3 4 5 6 9 12 14
2 3 6 7 8 9 11	2 3 4 5 6 8 12	2 5 6 7 8 9 11	2 3 4 5 7 12 14	2 3 4 5 7 10 11	2 3 4 5 8 10 14
2 3 4 5 6 10 11	2 3 5 6 8 11 14	2 3 4 5 6 7 12	3 4 6 9 11 15 17	2 3 5 6 9 10 11	2 3 4 5 8 9 18
2 3 5 6 8 10 11	2 3 4 5 6 11 14	2 3 6 7 8 10 11	2 3 4 5 8 14 19	2 3 4 6 9 10 11	3 6 8 9 10 13 17
2 3 4 5 9 10 11	3 4 5 6 7 14 22	3 4 5 6 9 10 11	2 3 4 6 9 14 15	2 3 6 7 9 10 11	2 3 4 5 8 9 14
2 5 6 7 9 10 11	2 3 5 6 7 9 16	2 3 6 8 9 10 11	2 3 4 7 8 13 14	3 4 6 8 9 10 11	2 3 4 6 8 12 19
2 3 5 6 8 9 12	3 4 5 8 11 14 17	2 3 4 5 7 8 12	3 4 6 7 9 12 17	2 3 5 6 7 8 12	2 3 4 6 9 11 14
2 3 4 5 7 9 12	3 5 6 10 11 12 14	2 3 4 5 8 9 12	2 3 5 6 7 12 13	3 4 5 7 8 9 12	2 3 5 7 10 13 16
3 4 5 6 8 9 12	3 5 6 7 10 12 20	2 3 5 7 8 9 12	2 3 4 7 8 12 13	3 5 6 7 8 9 12	2 3 6 7 10 13 15

TABLE Xe

2 3 6 7 8 9 12	2 3 4 5 8 11 21	2 5 6 7 8 9 12	2 3 4 5 6 10 12	2 3 4 5 8 10 12	2 3 5 6 9 10 13
2 3 5 6 7 10 12	3 5 6 7 8 9 13	2 3 4 6 8 10 12	3 4 6 7 8 11 13	2 3 6 7 8 10 12	3 5 6 7 9 11 14
3 4 5 7 8 10 12	2 3 5 8 9 13 20	3 4 5 6 8 10 12	3 4 5 6 9 11 14	3 4 6 7 8 10 12	3 4 5 8 9 14 15
2 3 4 6 9 10 12	3 4 5 6 7 8 14	2 3 4 7 9 10 12	2 3 4 5 6 7 14	3 4 5 6 9 10 12	3 7 10 11 12 13 14
3 5 6 7 9 10 12	2 5 7 9 10 13 15	2 5 6 7 9 10 12	2 3 4 6 8 10 13	2 3 5 6 10 11 12	3 4 5 7 8 9 14
3 5 6 7 8 11 12	2 3 5 6 9 12 16	2 5 7 8 9 10 12	2 3 6 7 9 12 16	3 4 6 8 9 10 12	2 3 5 7 11 12 20
3 5 6 8 9 10 12	2 3 6 7 10 13 14	2 3 4 5 6 11 12	3 6 7 8 9 11 13	2 3 5 6 7 11 12	2 5 7 9 10 11 13
3 4 5 6 8 11 12	3 4 5 7 8 10 13	2 3 4 5 8 11 12	2 5 6 7 9 12 13	2 3 5 6 8 11 12	2 3 4 6 8 11 14
3 4 6 7 8 11 12	2 5 7 8 9 11 14	2 3 4 6 9 11 12	2 3 5 6 7 14 16	3 4 5 6 9 11 12	4 5 7 9 10 11 14
2 3 6 7 9 11 12	3 4 5 6 7 13 17				

A.13. 3–(25,4,λ) **designs with** $\lambda \in \{2,8,10\}$.

Let G_7 be the representation of the wreath product C_5 *wr* A_5 generated by the permutations in Table XI. Then a 3–(25,4,λ) design for each $\lambda \in \{2,8,10\}$ can be obtained by developing the 4-element subsets in the appropriate table below.

TABLE XI

(1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,0)
(1,6,11,16,21)(2,7,12,17,22)(3,8,13,18,23)(4,9,14,19,24)(5,10,15,20,0)
(1,2)(3,4)(6,7)(8,9)(11,12)(13,14)(16,17)(18,19)(21,22)(23,24)

TABLE XI: A 3-(25,4,2) design.

0 1 2 8	0 1 6 10	0 1 5 11	0 5 6 16	0 5 10 15
0 6 7 18	0 1 10 21	0 5 11 21	0 21 22 23	

TABLE XI: A 3-(25,4,8) design.

0 1 2 5	0 6 7 8	0 5 6 10	0 1 5 11	0 1 2 11	0 6 7 10
0 1 7 12	0 1 10 20	0 1 2 15	0 6 10 12	0 1 10 15	0 1 10 17
0 5 6 17	0 6 7 18	0 1 2 18	0 6 7 20	0 5 6 22	0 5 16 20
0 5 11 21	0 1 5 22	0 1 10 22	0 1 7 23		

TABLE XI: A 3-(25,4,10) design.

0 1 5 6	0 1 2 5	0 1 10 11	0 1 2 11	0 6 7 10	0 5 6 11
0 6 7 11	0 1 10 20	0 1 2 15	0 1 7 13	0 1 2 13	0 6 10 12
0 1 10 15	0 5 10 15	0 1 10 17	0 1 5 17	0 5 10 16	0 6 10 16
0 5 6 17	0 1 2 18	0 5 6 20	0 6 7 20	0 5 6 22	0 6 7 21
0 1 5 22	0 1 10 22	0 1 7 23	0 1 22 23	0 21 22 23	

A.14. 3–(26,6,λ) designs with $\lambda \equiv 0$ *or* $1 \ (mod\ 10)$, $\lambda \notin \{10,11\}$

Let G_8 be the representation of $PSL_2(25)$ generated by

$$(1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,25)$$
$$(1,6,11,16,21)(2,7,12,17,22)(3,8,13,18,23)(4,9,14,19,24)(5,10,15,20,25)$$
$$(1)(2,7,14,3,13,22,5,25,18,4,19,10)(6,8,20,11,15,9,21,24,12,16,17,23)$$
and
$$(0,1)(3,4)(6,16)(7,10)(8,12)(9,15)(11,21)(13,18)(14,19)(17,23)(20,24)(22,25)$$

There are two orbits of 3-element subsets and orbit representatives for them are: $T_1 = \{0,1,2\}$ and $T_2 = \{0,1,6\}$. There are forty-five orbits of 6-element subsets and orbit representatives for them are given in Table XII. Using tools in the design theory toolchest these representatives were obtained and the A_{36} matrix was constructed. The transpose of this matrix can be found in Table XIII. Note that many columns of A_{36} have exactly the same entries. We represent this in Table XIII by listing in a particular row all the orbits which yield the column entries given in that row. From this data it is relatively easy to construct a 3–(26,6,λ) design for each $\lambda \equiv 0$ *or* $1 \ (mod\ 10)$, $\lambda \notin \{10,11\}$.

TABLE XII

	A	B	C	D	E
1	0 1 2 7 9 12	0 1 2 6 7 9	0 1 2 3 6 9	0 1 2 5 6 7	0 1 2 3 6 7
2	0 1 2 3 4 5	0 1 2 3 4 9	0 1 2 3 6 8	0 1 2 4 7 9	0 1 2 3 7 9
3	0 1 2 5 7 9	0 1 2 3 9 11	0 1 2 7 9 10	0 1 2 7 8 9	0 1 2 6 8 9
4	0 1 2 6 7 11	0 1 2 7 9 11	0 1 2 6 9 11	0 1 2 3 9 12	0 1 2 6 9 12
5	0 1 2 7 9 18	0 1 2 7 9 15	0 1 2 6 7 14	0 1 2 6 9 13	0 1 2 3 9 13
6	0 1 2 9 12 13	0 1 2 3 9 14	0 1 2 3 9 15	0 1 2 3 9 16	0 1 2 6 7 16
7	0 1 2 9 11 15	0 1 2 3 9 17	0 1 2 6 7 18	0 1 2 3 9 23	0 1 2 3 9 20
8	0 1 2 3 9 19	0 1 2 6 7 19	0 1 2 4 9 19	0 1 2 3 6 21	0 1 2 9 18 20
9	0 1 6 11 16 21	0 1 2 7 9 23	0 1 2 4 9 23	0 1 2 3 9 24	0 1 2 4 9 24

TABLE XIII

A_{36}^{T} T_1	T_2	Row and column entries of Table XII
0	1	9A
1	0	2A
20	20	5B 5D
8	12	8E
12	8	6D
30	30	5E 9B 9E
12	18	6E 8D
18	12	2C 8B
60	60	1B 1C 1D 1E 3A 3D 4D 5C 7D 7E
24	36	1A 3B 4C 6A 7A 9D
36	24	3C 3E 6C 7B 8C 9C
48	72	4A 4B 4E 5A
72	48	2D 2E 6B 8A
84	36	2B
36	84	7C

A.15. A 4-(20,5,4) Design.

Developing each of the thirteen 5-element subsets in Table XIV with the automorphisms in $AF(19)_\infty$ constructs a 4-(20,5,4) design.

TABLE XIV

0 1 2 3 4	0 1 3 7 8	0 1 2 3 10	0 1 3 6 11	0 1 3 4 11
0 1 3 11 13	0 1 3 6 14	0 1 3 6 15	0 1 3 5 19	0 1 3 11 17
0 1 3 4 19		0 1 3 8 19		0 1 3 10 19

A.16. A 4-(20,6,30) Design.

Developing each of the thirty one 6-element subsets in Table XV with the automorphisms in $AF(19)_\infty$ constructs a 4-(20,6,30) design.

TABLE XV

0 1 3 10 11 14	0 1 3 6 10 11	0 1 3 6 7 8	0 1 2 3 4 5	0 1 2 3 7 8	0 1 3 5 6 9
0 1 3 4 5 9	0 1 3 4 8 9	0 1 2 3 5 11	0 1 2 3 7 11	0 1 3 4 8 11	0 1 3 6 9 11
0 1 3 10 11 12	0 1 2 3 5 12	0 1 3 4 5 14	0 1 4 5 11 13	0 1 3 10 11 18	0 1 3 4 5 15
0 1 2 3 10 15	0 1 3 6 9 15	0 1 3 4 5 16	0 1 3 6 8 16	0 1 3 6 9 17	0 1 2 3 5 17
0 1 3 10 11 17	0 1 2 3 4 19	0 1 3 4 9 19	0 1 3 8 11 19	0 1 3 6 14 19	0 1 3 11 17 19
0 1 3 6 7 17					

A.17. 4−(21,6,λ) Designs from $PSL_2(19)_\infty$.

Let G_9 be the representation of $PSL_2(19)_\infty$ generated by

$$(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18)(19)(\infty)$$
and
$$(0,19,1)(2,10,18)(3,7,9)(4,15,6)(5,16,14)(8)(11,13,17)(12)(\infty)$$

A 4−(21,6,λ) design for each λ ∈ {36,40,60} can be obtained by developing the 5-element subsets in the appropriate table below with the

TABLE XVI:4-(21,6,36) design.

0 1 2 3 4 11	0 1 2 3 5 7	0 1 2 4 5 11	0 1 2 4 7 ∞	0 1 2 3 9 ∞

TABLE XVII:4-(21,6,40) design.

0 1 2 3 4 11	0 1 2 3 4 5	0 1 2 3 5 7	0 1 2 4 7 9	0 1 2 4 7 ∞	0 1 2 4 11 ∞

TABLE XVIII:4-(21,6,60) design.

0 1 2 3 4 11	0 1 2 3 4 7	0 1 2 3 5 7	0 1 2 4 7 9	0 1 2 4 5 11	0 1 2 3 4 ∞
			0 1 2 4 7 ∞		

A.18. 4−(23,5,λ) Designs from $AF(23)$.

A 4−(23,5,λ) design for each λ ∈ {2,4,5,6,7,8,9} can be obtained by developing the 5-element subsets in the appropriate table below.

TABLE XIX:A 4-(23,5,2) design.

0 1 3 7 8	0 1 3 4 11	0 1 3 5 12	0 1 3 12 13	0 1 4 5 13	0 1 3 5 20	0 1 2 5 21

TABLE XX:A 4-(23,5,4) design.

0 1 2 3 5	0 1 3 4 6	0 1 3 6 7	0 1 3 5 8	0 1 3 8 11
0 1 3 4 12	0 1 3 7 12	0 1 3 11 12	0 1 3 8 13	0 1 3 5 14
0 1 3 12 19	0 1 3 6 17	0 1 3 15 18	0 1 3 8 22	

TABLE XXI:A 4-(23,5,5) design.

0 1 3 5 6	0 1 2 3 4	0 1 3 5 8	0 1 3 6 9	0 1 3 8 11
0 1 2 5 12	0 1 3 4 12	0 1 3 5 12	0 1 3 7 12	0 1 3 4 13
0 1 2 3 13	0 1 3 8 13	0 1 3 12 14	0 1 3 7 14	0 1 2 5 16
0 1 3 6 17	0 1 3 5 19	0 1 2 5 20	0 1 3 12 21	0 1 3 10 21

TABLE XXII: A 4-(23,5,6) design.

0 1 2 3 5	0 1 3 4 6	0 1 3 6 7	0 1 3 5 8	0 1 3 8 11
0 1 3 4 12	0 1 3 7 12	0 1 3 11 12	0 1 3 8 13	0 1 3 5 14
0 1 3 12 19	0 1 3 6 17	0 1 3 15 18	0 1 3 8 22	

TABLE XXIII: A 4-(23,5,7) design.

0 1 3 5 6	0 1 2 3 4	0 1 3 4 6	0 1 2 5 7	0 1 3 5 8	0 1 3 4 9
0 1 3 7 10	0 1 2 3 12	0 1 4 5 11	0 1 3 8 11	0 1 3 4 12	0 1 3 5 12
0 1 3 7 12	0 1 3 12 15	0 1 3 12 13	0 1 2 3 13	0 1 3 8 13	0 1 3 7 14
0 1 3 5 15	0 1 2 5 16	0 1 3 6 17	0 1 3 5 18	0 1 2 5 20	0 1 3 12 21
0 1 3 10 21	0 1 3 8 22	0 1 3 6 22			

TABLE XXIV: A 4-(23,5,8) design.

0 1 3 4 5	0 1 2 3 5	0 1 2 5 7	0 1 3 4 7	0 1 4 5 7	0 1 3 6 9
0 1 3 7 8	0 1 2 3 11	0 1 3 7 10	0 1 3 6 12	0 1 3 5 12	0 1 3 8 12
0 1 3 9 12	0 1 3 4 13	0 1 4 5 13	0 1 3 8 13	0 1 3 12 14	0 1 3 6 14
0 1 3 7 14	0 1 3 6 15	0 1 3 12 19	0 1 3 12 17	0 1 3 6 16	0 1 3 6 17
0 1 3 12 18	0 1 3 5 19	0 1 3 15 18	0 1 3 6 20		

TABLE XXV: A 4-(23,5,9) design.

0 1 3 6 8	0 1 3 5 6	0 1 2 3 4	0 1 3 4 6	0 1 3 5 8	0 1 3 4 10
0 1 3 6 9	0 1 3 7 8	0 1 2 3 11	0 1 3 7 10	0 1 3 4 11	0 1 4 5 11
0 1 3 6 12	0 1 2 5 12	0 1 3 4 12	0 1 3 12 13	0 1 2 3 13	0 1 4 5 13
0 1 3 12 14	0 1 3 5 14	0 1 3 8 14	0 1 3 7 14	0 1 3 5 15	0 1 3 12 19
0 1 3 12 17	0 1 3 12 16	0 1 3 6 17	0 1 3 5 21	0 1 3 12 20	0 1 3 5 20
0 1 2 5 20	0 1 2 5 21	0 1 3 12 21	0 1 3 8 22		

A.19. A 4-(29,5,5) Design from $AF(29)$

Developing each of the thirty three 5-element subsets in Table XXVI with the automorphisms in $AF(13)$ constructs a 4-(29,5,5) design.

TABLE XXVI

0 1 2 3 4	0 1 2 3 6	0 1 3 5 6	0 1 2 5 8	0 1 2 7 9	0 1 3 4 10
0 1 3 7 10	0 1 3 6 10	0 1 2 5 11	0 1 3 4 11	0 1 2 5 12	0 1 3 11 13
0 1 4 5 13	0 1 2 9 13	0 1 2 3 16	0 1 4 5 16	0 1 3 5 17	0 1 2 7 16
0 1 5 6 16	0 1 4 5 17	0 1 3 6 18	0 1 2 13 18	0 1 3 8 19	0 1 3 5 22
0 1 3 11 23	0 1 3 21 22	0 1 3 6 23	0 1 2 7 24	0 1 3 11 25	0 1 3 5 26
	0 1 2 5 26	0 1 3 7 26		0 1 3 11 27	

A.20. 5−(24,k,λ) Designs, $k=6$ and $k=7$, from $PSL_2(23)$.

Let $G_{10} = \langle\alpha,\beta\rangle$ be the representation of $PSL_2(23)$ in its action on the projective line given by:

$$\alpha = (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)(23)$$

$$\beta = (0,23,1)(2,12,22)(3,16,11)(4,18,15)(5,10,17)(6,20,9)(7,14,19)(8,21,13)$$

Then 5−(24,k,λ) designs for $k=6$ and $k=7$ and each admissible λ can be obtained from G_{10} by developing the orbit representatives given in the appropriate table below

TABLE XXVII: A 5-(24,6,1) design.

0 1 2 4 6 8	0 1 2 3 6 10	0 1 2 4 9 20

TABLE XXVIII: A 5-(24,6,2) design.

0 1 2 4 5 6	0 1 2 4 7 8	0 1 2 4 7 13	0 1 2 4 9 17

TABLE XXIX: A 5-(24,6,3) design.

0 1 2 4 6 8	0 1 2 3 4 10	0 1 2 4 9 11
0 1 2 4 7 12	0 1 2 4 6 17	0 1 2 4 9 13

TABLE XXX: A 5-(24,6,4) design.

0 1 2 3 4 5	0 1 2 4 6 8	0 1 2 4 7 8
0 1 2 4 9 10	0 1 2 3 6 10	0 1 2 4 6 13
0 1 2 4 6 16	0 1 2 4 14 17	0 1 2 4 9 17

TABLE XXXI: A 5-(24,6,5) design.

0 1 2 4 7 8	0 1 2 4 5 9	0 1 2 3 4 10	0 1 2 4 9 11	0 1 2 4 6 12
0 1 2 4 7 12	0 1 2 4 6 17	0 1 2 4 9 18	0 1 2 4 14 17	0 1 2 4 9 22

TABLE XXXII: A 5-(24,6,6) design.

0 1 2 3 4 5	0 1 2 4 6 8	0 1 2 4 7 8	0 1 2 4 6 9	0 1 2 4 9 10
0 1 2 3 6 10	0 1 2 4 6 12	0 1 2 4 7 13	0 1 2 4 6 14	0 1 2 4 9 18
0 1 2 4 14 17	0 1 2 4 9 17	0 1 2 4 6 18	0 1 2 4 16 18	

TABLE XXXIII: A 5-(24,6,7) design.

0 1 2 4 7 8	0 1 2 4 5 9	0 1 2 4 6 9	0 1 2 3 4 10	0 1 2 3 6 10
0 1 2 4 6 17	0 1 2 4 9 13	0 1 2 4 6 13	0 1 2 4 14 17	0 1 2 4 6 18
0 1 2 4 9 20	0 1 2 4 6 19	0 1 2 4 16 18	0 1 2 4 9 22	

TABLE XXXIV: A 5-(24,6,8) design.

0 1 2 3 4 5	0 1 2 4 6 8	0 1 2 4 7 8	0 1 2 4 9 10	0 1 2 4 8 9	0 1 2 4 9 11
0 1 2 4 6 11	0 1 2 4 6 12	0 1 2 4 7 12	0 1 2 4 9 14	0 1 2 3 4 17	0 1 2 4 9 18
0 1 2 4 14 17	0 1 2 4 9 17	0 1 2 4 6 18	0 1 2 4 6 19	0 1 2 4 16 18	

TABLE XXXV: A 5-(24,6,9) design.

0 1 2 4 6 8	0 1 2 4 6 9	0 1 2 4 8 9	0 1 2 4 6 12	0 1 2 4 9 14	0 1 2 4 9 13
0 1 2 4 7 13	0 1 2 4 6 14	0 1 2 4 6 16	0 1 2 3 4 17	0 1 2 4 9 18	0 1 2 4 14 17
0 1 2 4 9 17	0 1 2 4 6 19	0 1 2 4 16 18	0 1 2 4 9 22		

TABLE XXXVI: A 5-(24,7,3) design.

0 1 2 4 9 11 13

TABLE XXXVII: A 5-(24,7,6) design.

0 1 2 3 4 7 9	0 1 2 3 4 8 9

TABLE XXXVIII: A 5-(24,7,9) design.

0 1 2 3 4 9 16	0 1 2 4 6 9 18	0 1 2 4 9 12 23

TABLE XXXIX: A 5-(24,7,12) design.

0 1 2 4 6 7 17	0 1 2 4 7 9 18	0 1 2 4 5 6 18	0 1 2 4 9 12 19

TABLE XL: A 5-(24,7,15) design.

0 1 2 4 7 9 17	0 1 2 4 7 9 22	0 1 2 4 6 7 21	0 1 2 4 5 9 22	0 1 2 4 9 12 23

TABLE XLI: A 5-(24,7,18) design.

0 1 2 4 6 7 16	0 1 2 4 9 12 20	0 1 2 3 4 9 19
0 1 2 4 5 6 20	0 1 2 4 7 9 22	0 1 2 4 5 9 22

TABLE XLII: A 5-(24,7,21) design.

0 1 2 4 6 7 16	0 1 2 4 6 7 17	0 1 2 4 7 9 19	0 1 2 3 4 9 18
0 1 2 4 5 6 20	0 1 2 4 9 12 19	0 1 2 4 9 12 23	

TABLE XLIII: A 5-(24,7,24) design.

0 1 2 3 4 9 16	0 1 2 4 7 9 16	0 1 2 4 7 9 18	0 1 2 4 5 6 18
0 1 2 4 6 9 18	0 1 2 3 4 9 19	0 1 2 4 9 12 19	0 1 2 4 5 9 22

TABLE XLIV: A 5-(24,7,27) design.

0 1 2 4 6 7 16	0 1 2 4 7 9 16	0 1 2 4 6 7 17	0 1 2 4 7 9 19	0 1 2 4 7 9 18
0 1 2 4 5 6 18	0 1 2 4 5 6 20	0 1 2 4 9 12 19	0 1 2 4 5 9 22	

TABLE XLV: A 5-(24,7,30) design.

0 1 2 4 6 7 16	0 1 2 4 7 9 16	0 1 2 3 4 9 17	0 1 2 4 7 9 18	0 1 2 4 5 6 18
0 1 2 4 6 9 18	0 1 2 4 9 12 19	0 1 2 4 7 9 21	0 1 2 4 6 7 21	0 1 2 3 4 9 23

TABLE XLVI: A 5-(24,7,33) design.

0 1 2 4 7 9 16	0 1 2 4 7 9 19	0 1 2 4 6 9 18	0 1 2 3 6 10 18	0 1 2 4 5 6 20
0 1 2 4 9 12 19	0 1 2 4 5 9 20	0 1 2 4 7 9 22	0 1 2 4 7 9 21	0 1 2 3 4 9 22
0 1 2 4 6 7 16				

TABLE XLVII: A 5-(24,7,36) design.

0 1 2 4 7 9 16	0 1 2 4 6 7 17	0 1 2 4 7 9 17	0 1 2 4 9 12 20	0 1 2 4 6 9 18
0 1 2 4 5 6 20	0 1 2 4 9 12 19	0 1 2 4 7 9 22	0 1 2 3 4 9 22	0 1 2 4 9 12 23
0 1 2 3 4 9 16	0 1 2 3 6 10 18			

TABLE XLVIII: A 5-(24,7,39) design.

0 1 2 4 7 9 16	0 1 2 4 7 9 19	0 1 2 3 4 9 18	0 1 2 3 4 9 19	0 1 2 3 6 10 18
0 1 2 4 9 12 19	0 1 2 4 7 9 22	0 1 2 4 7 9 21	0 1 2 4 6 7 21	0 1 2 4 9 12 23
0 1 2 4 6 7 16	0 1 2 4 5 6 20	0 1 2 3 4 9 23		