On the Products of Hadamard Matrices, Williamson Matrices and Other Orthogonal Matrices using M-Structures

Jennifer Seberry* and Mieko Yamada†

Abstract

The new concept of M-structures is used to unify and generalize a number of concepts in Hadamard matrices including Williamson matrices, Goethals-Seidel matrices, Wallis-Whiteman matrices and generalized quaternion matrices. The concept is used to find many new symmetric Williamson-type matrices, both in sets of four and eight, and many new Hadamard matrices. We give as corollaries "that the existence of Hadamard matrices of orders $4g$ and $4h$ implies the existence of an Hadamard matrix of order $8gh$" and "the existence of Williamson type matrices of orders u and v implies the existence of Williamson type matrices of order $2uv$". This work generalizes and utilizes the work of Masahiko Miyamoto and Mieko Yamada.

1 Definitions and Introduction

An *orthogonal design of order n and type* $(s_1, \ldots, s_u)$, s_i positive integers, is an $n \times n$ matrix X, with entries $\{0, \pm x_1, \ldots, \pm x_u\}$ (the x_i commuting indeterminates) satisfying

$$XX^T = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n. \tag{1}$$

We write this as OD$(n; s_1, s_2, \ldots, s_u)$.

Alternatively, each X has s_i entries of the type $\pm x_i$ and the distinct rows are orthogonal under the Euclidean inner product. We may view X as a matrix with entries in the field of fractions of the integral domain $Z[x_1, \ldots, x_u]$, (Z the rational integers), and then if we let $f = (\Sigma_{i=1}^{u} s_i x_i^2)$, X is an invertible matrix with inverse $\frac{1}{f}X^T$. Thus $XX^T = fI_n$, and so our alternative definition that the row vectors are orthogonal applies equally well to the column vectors of X.

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An orthogonal design with no zeros and in which each of the entries is replaced by +1 or −1 is called an *Hadamard matrix*. Alternatively an Hadamard matrix of order n, H has entries +1 or −1 and the distinct row vectors orthogonal so

$$HH^T = nI_n.$$

Orthogonal designs, Hadamard matrices and other definitions not given here are extensively described in Geramita and Seberry [8] and Jennifer Seberry Wallis [22].

A special orthogonal design, the OD$(4t;t,t,t,t)$, is especially useful in the construction of Hadamard matrices. An OD$(12;3,3,3,3)$ was first found by Baumert and M. Hall Jr [4] and an OD$(20;5,5,5,5)$ by Welch (see below). OD$(4t;t,t,t,t)$ are sometimes called Baumert-Hall arrays.

X and Y are said to be *amicable matrices* if

$$XY^T = YX^T. \tag{2}$$

Williamson matrices of order w are four circulant symmetric matrices, A, B, C, D which have entries +1 or −1 and which satisfy

$$AA^T + BB^T + CC^T + DD^T = 4wI_w. \tag{3}$$

(Symmetric) Williamson-type matrices of order w are four pairwise amicable (that is pairwise satisfy (2)) (symmetric) matrices, A, B, C, D which have entries +1 or −1 and which satisfy

$$AA^T + BB^T + CC^T + DD^T = 4wI_w. \tag{4}$$

(Symmetric) 8 Williamson-type matrices of order w are eight pairwise amicable (that is pairwise satisfy (2)) (symmetric) matrices, A_i, $i = 1, \ldots, 8$ which have entries +1 or −1 and which satisfy

$$\sum_{i=1}^{8} A_i A_i^T = 8wI_w. \tag{5}$$

The appropriate theorem for the construction of Hadamard matrices (it is implied by Williamson, Baumert-Hall, Welch, Cooper-J. Wallis, Turyn) is:

Theorem 1 *Suppose there exists an OD$(4t;t,t,t,t)$ and four suitable matrices A, B, C, D of order w which are pairwise amicable, have entries +1 or −1, and which satisfy*

$$AA^T + BB^T + CC^T + DD^T = 4wI_w.$$

Then there is an Hadamard matrix of order $4wt$.

Suitable matrices of order w for an $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$ are u pairwise amicable (that is pairwise satisfy (2)) matrices, A_i, $i = 1, \ldots, u$ which have entries $+1$ or -1 and which satisfy

$$\sum_{i=1}^{u} s_i A_i A_i^T = (\Sigma s_i) w I_w. \tag{6}$$

They are used in the following theorem.

Theorem 2 (Geramita-Seberry) *Suppose there exists an $OD(\Sigma s_i; s_1, \ldots, s_u)$ and u suitable matrices of order m. Then there is an Hadamard matrix of order $(\Sigma u_i)m$.*

If some of the *suitable matrices* have entries $0, +1, -1$, then *weighing matrices* rather than Hadamard matrices could have been constructed.

A set of 4 *T-matrices*, T_i, $i = 1, \ldots, 4$ of order t are four (4) circulant or type 1 matrices which have entries 0, $+1$ or -1 and which satisfy

(i) $T_i * T_j = 0$, $i \neq j$, ($*$ the Hadamard product)

(ii) $\sum_{i=1}^{4} T_i$ is a $(1, -1)$ matrix, (7)

(iii) $\sum_{i=1}^{4} T_i T_i^T = t I_t$,

(iv) $t = t_1^2 + t_2^2 + t_3^2 + t_4^2$ where t_i is the row(column) sum of T_i.

T-matrices are known (see Cohen, Rubie, Koukouvinos, Kounias, Seberry, Yamada [7] for a recent survey) for many orders including:
1, ..., 70, 72, 74, ..., 78, 80, ..., 82, 84, ..., 88, 90, ..., 96, 98, ..., 102, 104, ..., 106, 108, 110, ..., 112, 114, ..., 126, 128, ..., 130, 132, 136, 138, 140, ..., 148, 150, 152, ..., 156, 158, ..., 162, 164, ..., 166, 168, ...; 172, 174, ..., 178, 180, 182, 184, ..., 190, 192, 194, ..., 196, 198, 200, ..., 210, ...

The following result, in a slightly different form, was also discovered by R.J. Turyn.

Theorem 3 (Cooper-J. Wallis) *Suppose there exist T-matrices (T-sequences) X_i, $i = 1, \ldots, 4$ of order n. Let a, b, c, d be commuting variables. Then*

$$\begin{aligned}
A &= aX_1 + bX_2 + cX_3 + dX_4 \\
B &= -bX_1 + aX_2 + dX_3 - cX_4 \\
C &= -cX_1 - dX_2 + aX_3 + bX_4 \\
D &= -dX_1 + cX_2 - bX_3 + aX_4
\end{aligned}$$

can be used in the Goethal-Seidel (or J. Wallis-Whiteman) array to obtain an $OD(4n; n, n, n, n)$.

Example: Let

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad X_4 = 0.$$

Then X_1, X_2, X_3, X_4, are T-matrices of order 3, and the $OD(12; 3, 3, 3, 3)$ is:

a	b	c	-b	a	d	-c	-d	a	-d	c	-b
c	a	b	a	d	-b	-d	a	-c	c	-b	-d
b	c	a	d	-b	a	a	-c	-d	-b	-d	c
b	-a	-d	a	b	c	-d	-b	c	c	-a	d
-a	-d	b	c	a	b	-b	c	-d	-a	d	c
-d	b	-a	b	c	a	c	-d	-b	d	c	-a
c	d	-a	d	b	-c	a	b	c	-b	d	a
d	-a	c	b	-c	d	c	a	b	d	a	-b
-a	c	d	-c	d	b	b	c	a	a	-b	d
d	-c	b	-c	a	-d	b	-d	-a	a	b	c
-c	b	d	a	-d	-c	-d	-a	b	c	a	b
b	d	-c	-d	-c	a	-a	b	-d	b	c	a

We will not give the proof here which can be found in J. Wallis [22, p. 360] but will just quote the results given there. Cyclotomy may be used in constructing these arrays including the orders t = 13, 19, 25, 31, 37, 41, 61.

Such structures are not limited to constructing $OD(4t; t, t, t, t)$. For example it was shown in Geramita and Seberry [8] that the following matrices

$$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}, \quad B = \begin{bmatrix} a & -b & c \\ c & a & -b \\ -b & c & a \end{bmatrix},$$

$$C = \begin{bmatrix} a & b & -c \\ -c & a & b \\ b & -c & a \end{bmatrix}, \quad D = \begin{bmatrix} -a & b & c \\ c & -a & b \\ b & c & -a \end{bmatrix},$$

can be used as follows to give an $OD(12; 4, 4, 4)$

a	b	c	a	-b	c	a	b	-c	-a	b	c
c	a	b	-b	c	a	b	-c	a	b	c	-a
b	c	a	c	a	-b	-c	a	b	c	-a	b
-a	b	-c	a	b	c	-a	c	b	-a	c	-b
b	-c	-a	c	a	b	c	b	-a	c	-b	-a
-c	-a	b	b	c	a	b	-a	c	-b	-a	c
-a	-b	c	a	-c	-b	a	b	c	a	c	-b
-b	c	-a	-c	-b	a	c	a	b	c	-b	a
c	-a	-b	-b	a	-c	b	c	a	-b	a	c
a	-b	-c	a	-c	b	-a	-c	b	a	b	c
-b	-c	a	-c	b	a	-c	b	-a	c	a	b
-c	a	-b	b	a	-c	b	-a	-c	b	c	a

We now introduce some new terminology to unify some previous ideas.

2 M-structures

An orthogonal matrix of order $4t$ can be divided into sixteen (16) $t \times t$ blocks M_{ij}. This partitioned matrix is said to be an M-structure. If the orthogonal matrix can be partitioned into sixty-four (64) $s \times s$ blocks M_{ij} it will be called a 64 block M-structure.

An Hadamard matrix made from (symmetric) Williamson matrices W_1, W_2, W_3, W_4 is an M-structure with

$$\begin{aligned} W_1 &= M_{11} = M_{22} = M_{33} = M_{44}, \\ W_2 &= M_{12} = -M_{21} = M_{34} = -M_{43}, \\ W_3 &= M_{13} = -M_{31} = -M_{24} = M_{42}, \quad \text{and} \\ W_4 &= M_{14} = -M_{41} = M_{23} = -M_{32}. \end{aligned}$$

An Hadamard matrix made from four (4) circulant (or type 1) matrices A_1, A_2, A_3, A_4 of order n, where R is the matrix which makes all the A_iR back-circulant (or type 2), is an M-structure with

$$\begin{aligned} A_1 &= M_{11} = M_{22} = M_{33} = M_{44}, \\ A_2 &= M_{12}R = -M_{21}R = RM_{34}^T = -RM_{43}^T, \\ A_3 &= M_{13}R = -M_{31}R = -RM_{24}^T = RM_{42}^T, \quad \text{and} \\ A_4 &= M_{14}R = -M_{41}R = RM_{23}^T = -RM_{32}^T. \end{aligned}$$

In this paper we will mostly not be concerned with the structure of the M_{ij} but two interesting cases should first be mentioned.

Welch's OD(20; 5, 5, 5, 5) composed of block *circulant* matrices is:

-D B -C -C -B	C A -D -D -A	-B -A C -C -A	A -B -D D -B
-B -D B -C -C	-A C A -D -D	-A -B -A C -C	-B A -B -D D
-C -B -D B -C	-D -A C A -D	-C -A -B -A C	D -B A -B -D
-C -C -B -D B	-D -D -A C A	C -C -A -B -A	-D D -B A -B
B -C -C -B -D	A -D -D -A C	-A C -C -A -B	-B -D D -B A
-C A D D -A	-D -B -C -C B	-A B -D D B	-B -A -C C -A
-A -C A D D	B -D -B -C -C	B -A B -D D	-A -B -A -C C
D -A -C A D	-C B -D -B -C	D B -A B -D	C -A -B -A -C
D D -A -C A	-C -C B -D -B	-D D B -A B	-C C -A -B -A
A D D -A -C	-B -C -C B -D	B -D D B -A	-A -C C -A -B
B -A -C C -A	A B -D D B	-D -B C C B	-C A -D -D -A
-A B -A -C C	B A B -D D	B -D -B C C	-A -C A -D -D
C -A B -A -C	D B A B -D	C B -D -B C	-D -A -C A -D
-C C -A B -A	-D D B A B	C C B -D -B	-D -D -A -C A
-A -C C -A B	B -D D B A	-B C C B -D	A -D -D -A -C
-A -B -D D -B	B -A C -C -A	C A D D -A	-D B C C -B
-B -A -B -D D	-A B -A C -C	-A C A D D	-B -D B C C
D -B -A -B -D	-C -A B -A C	D -A C A D	C -B -D B C
-D D -B -A -B	C -C -A B -A	D D -A C A	C C -B -D B
-B -D D -B -A	-A C -C -A B	A D D -A C	B C C -B -D

Each M_{ij} in its M-structure is circulant. In fact it can be constructed using sixteen (16) circulant matrices with first rows using:

M_{11}	:	1	1	-1	-1	-1	M_{12}	:	1	-1	1	1	1;
M_{13}	:	-1	1	1	-1	1	M_{14}	:	-1	-1	1	-1	-1;
M_{21}	:	-1	-1	-1	-1	1	M_{22}	:	1	-1	-1	-1	1;
M_{23}	:	1	1	1	-1	1	M_{24}	:	-1	1	-1	1	1;
M_{31}	:	1	1	-1	1	1	M_{32}	:	-1	1	1	-1	1;
M_{33}	:	1	-1	1	1	1	M_{34}	:	-1	-1	1	1	1;
M_{41}	:	1	-1	1	-1	-1	M_{42}	:	1	1	1	-1	1;
M_{43}	:	1	-1	-1	-1	1	M_{44}	:	1	1	1	1	-1;

K. Yamamoto's [38] restructuring of Ono and Sawade's OD(36; 9, 9, 9, 9) [13] composed of blocks of type 1 (or block circulant) matrices. Each M_{ij} in its M-structure is type 1. In fact it can be constructed using sixteen (16) circulant matrices with first rows:

$$A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & c & d \\ d & 0 & c \\ c & d & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & c & -d \\ -d & 0 & c \\ c & -d & 0 \end{bmatrix},$$

viz

$$\begin{array}{llll}
M_{11} = & A & bI + C & -bI - C^T \\
M_{12} = & bI + aB^T & bI + D^T & bI - D^T \\
M_{13} = & cI + aB^T & -bI + C & bI + D^T \\
M_{14} = & dI + aB^T & bI - D & -bI + C^T \\
\\
M_{21} = & -bI + aB & -bI + D & -bI - D^T \\
M_{22} = & A & bI - C & -bI + C^T \\
M_{23} = & -dI + aB^T & bI + D & -bI - C^T \\
M_{24} = & cI + aB & bI + C & -bI + D^T \\
\\
M_{31} = & -cI + aB & -bI - D & bI - C^T \\
M_{32} = & dI + aB & bI + C & -bI - D^T \\
M_{33} = & A & -bI + D & bI - D^T \\
M_{34} = & -bI + aB^T & -bI + C & -bI - C^T \\
\\
M_{41} = & -dI + aB & bI - C & -bI + D^T \\
M_{42} = & -cI + aB^T & bI - D & -bI + C^T \\
M_{43} = & bI + aB & bI + C & bI - C^T \\
M_{44} = & A & -bI - D & bI + D^T
\end{array}$$

When written in full the Ono-Sawade-Yamamoto OD(36; 9, 9, 9, 9) is as on the following page.

The following theorem shows the power of M-structures comprising wholly circulant or type 1 blocks. The original version with circulant matrices was due to Turyn.

a	a	a	b	c	d	-b	-d	-c	b	-a	a	b	c	-d	b	d	-c	c	-a	a	-b	c	d	b	-d	c	d	-a	a	b	-c	d	-b	d	c
a	a	a	d	b	c	-c	-b	-d	a	b	-a	-d	b	c	-c	b	d	a	c	-a	d	-b	c	c	b	-d	a	d	-a	d	b	-c	c	-b	d
a	a	a	c	d	b	-d	-c	-b	-a	a	b	c	-d	b	d	-c	b	-a	a	c	c	d	-b	-d	c	b	-a	a	d	-c	d	b	d	c	-b
-b	-d	-c	a	a	a	b	c	d	b	d	-c	b	-a	a	b	c	-d	b	-d	c	c	-a	a	-b	c	d	-b	d	c	d	-a	a	b	-c	d
-c	-b	-d	a	a	a	d	b	c	-c	b	d	a	b	-a	-d	b	c	c	b	-d	a	c	-a	d	-b	c	c	-b	d	a	d	-a	d	b	-c
-d	-c	-b	a	a	a	c	d	b	d	-c	b	-a	a	b	c	-d	b	-d	c	b	-a	a	c	c	d	-b	d	c	-b	-a	a	d	-c	d	b
b	c	d	-b	-d	-c	a	a	a	b	c	-d	b	d	-c	b	-a	a	-b	c	d	b	-d	c	c	-a	a	b	-c	d	-b	d	c	d	-a	a
d	b	c	-c	-b	-d	a	a	a	-d	b	c	-c	b	d	a	b	-a	d	-b	c	c	b	-d	a	c	-a	d	b	-c	c	-b	d	a	d	-a
c	d	b	-d	-c	-b	a	a	a	c	-d	b	d	-c	b	-a	a	b	c	d	-b	-d	c	b	-a	a	c	-c	d	b	d	c	-b	-a	a	d
-b	a	-a	-b	c	-d	-b	d	-c	a	a	a	b	-c	-d	-b	d	c	-d	-a	a	b	c	-d	-b	-d	-c	c	a	-a	b	c	d	-b	-d	c
-a	-b	a	-d	-b	c	-c	-b	d	a	a	a	-d	b	-c	c	-b	d	a	-d	-a	-d	b	c	-c	-b	-d	-a	c	a	d	b	c	c	-b	-d
a	-a	-b	c	-d	-b	d	-c	-b	a	a	a	-c	-d	b	d	c	-b	-a	a	-d	c	-d	b	-d	-c	-b	a	-a	c	c	d	b	-d	c	-b
-b	d	-c	-b	a	-a	-b	c	-d	-b	d	c	a	a	a	b	-c	-d	-b	-d	-c	-d	-a	a	b	c	-d	-b	-d	c	c	a	-a	b	c	d
-c	-b	d	-a	-b	a	-d	-b	c	c	-b	d	a	a	a	-d	b	-c	-c	-b	-d	a	-d	-a	-d	b	c	c	-b	-d	-a	c	a	d	b	c
d	-c	-b	a	-a	-b	c	-d	-b	d	c	-b	a	a	a	-c	-d	b	-d	-c	-b	-a	a	-d	c	-d	b	-d	c	-b	a	-a	c	c	d	b
-b	c	-d	-b	d	-c	-b	a	-a	b	-c	-d	-b	d	c	a	a	a	b	c	-d	-b	-d	-c	-d	-a	a	b	c	d	-b	-d	c	c	a	-a
-d	-b	c	-c	-b	d	-a	-b	a	-d	b	-c	c	-b	d	a	a	a	-d	b	c	-c	-b	-d	a	-d	-a	d	b	c	c	-b	-d	-a	c	a
c	-d	-b	d	-c	-b	a	-a	-b	-c	-d	b	d	c	-b	a	a	a	c	-d	b	-d	-c	-b	-a	a	-d	c	d	b	-d	c	-b	a	-a	c
-c	a	-a	-b	-c	d	b	-d	-c	d	a	-a	b	c	d	-b	d	-c	a	a	a	-b	c	-d	b	d	-c	-b	-a	a	-b	c	d	-b	-d	-c
-a	-c	a	d	-b	-c	-c	b	-d	-a	d	a	d	b	c	-c	-b	d	a	a	a	-d	-b	c	-c	b	d	a	-b	-a	d	-b	c	-c	-b	-d
a	-a	-c	-c	d	-b	-d	-c	b	a	-a	d	c	d	b	d	-c	-b	a	a	a	c	-d	-b	d	-c	b	-a	a	-b	c	d	-b	-d	-c	-b
b	-d	-c	-c	a	-a	-b	-c	d	-b	d	-c	d	a	-a	b	c	d	b	d	-c	a	a	a	-b	c	-d	-b	-d	-c	-b	-a	a	-b	c	d
-c	b	-d	-a	-c	a	d	-b	-c	-c	-b	d	-a	d	a	d	b	c	-c	b	d	a	a	a	-d	-b	c	-c	-b	-d	a	-b	-a	d	-b	c
-d	-c	b	a	-a	-c	-c	d	-b	d	-c	-b	a	-a	d	c	d	b	d	-c	b	a	a	a	c	-d	-b	-d	-c	-b	-a	a	-b	c	d	-b
-b	-c	d	b	-d	-c	-c	a	-a	b	c	d	-b	d	-c	d	a	-a	-b	c	-d	b	d	-c	a	a	a	-b	c	d	-b	-d	-c	-b	-a	a
d	-b	-c	-c	b	-d	-a	-c	a	d	b	c	-c	-b	d	-a	d	a	-d	-b	c	-c	b	d	a	a	a	d	-b	c	-c	-b	-d	a	-b	-a
-c	d	-b	-d	-c	b	a	-a	-c	c	d	b	d	-c	-b	a	-a	d	c	-d	-b	d	-c	b	a	a	a	c	d	-b	-d	-c	-b	-a	a	-b
-d	a	-a	b	-c	-d	-b	-d	c	-c	-a	a	b	-c	d	-b	-d	-c	b	a	-a	b	c	d	b	-d	-c	a	a	a	-b	-c	d	b	-d	c
-a	-d	a	-d	b	-c	c	-b	-d	a	-c	-a	d	b	-c	-c	-b	-d	-a	b	a	d	b	c	-c	b	-d	a	a	a	d	-b	-c	c	b	-d
a	-a	-d	-c	-d	b	-d	c	-b	-a	a	-c	-c	d	b	-d	-c	-b	a	-a	b	c	d	b	-d	-c	b	a	a	a	-c	d	-b	-d	c	b
-b	-d	c	-d	a	-a	b	-c	-d	-b	-d	-c	-c	-a	a	b	-c	d	b	-d	-c	b	a	-a	b	c	d	b	-d	c	a	a	a	-b	-c	d
c	-b	-d	-a	-d	a	-d	b	-c	-c	-b	-d	a	-c	-a	d	b	-c	-c	b	-d	-a	b	a	d	b	c	c	b	-d	a	a	a	d	-b	-c
-d	c	-b	a	-a	-d	-c	-d	b	-d	-c	-b	-a	a	-c	-c	d	b	-d	-c	b	a	-a	b	c	d	b	-d	c	b	a	a	a	-c	d	-b
b	-c	-d	-b	-d	c	-d	a	-a	b	-c	d	-b	-d	-c	-c	-a	a	b	c	d	b	-d	-c	b	a	-a	-b	-c	d	b	-d	c	a	a	a
-d	b	-c	c	-b	-d	-a	-d	a	d	b	-c	-c	-b	-d	a	-c	-a	d	b	c	-c	b	-d	-a	b	a	d	-b	-c	c	b	-d	a	a	a
-c	-d	b	-d	c	-b	a	-a	-d	-c	d	b	-d	-c	-b	-a	a	-c	c	d	b	-d	-c	b	a	-a	b	-c	d	-b	-d	c	b	a	a	a

Figure 1: Ono-Sawade-Yamamoto OD(36; 9, 9, 9, 9)

Theorem 4 *Suppose there are T-matrices of order t. Further suppose there is an $OD(4s; u_1, \ldots, u_n)$ constructed of sixteen circulant (or type 1) $s \times s$ blocks on the variables $x_1, \ldots, x_n$. Then there is an $OD(4st; tu_1, \ldots, tu_n)$. In particular if there is an $OD(4s; s, s, s, s)$ constructed of sixteen circulant (or type 1) $s \times s$ blocks then there is an $OD(4st; st, st, st, st)$.*

Proof: We write the OD as (N_{ij}), $i, j = 1, 2, 3, 4$, where each N_{ij} is circulant (or type 1). Hence we are considering the OD purely as an M-structure. Since we have an OD

$$N_{i1}N_{j1}^T + N_{i2}N_{j2}^T + N_{i3}N_{j3}^T + N_{i4}N_{j4}^T = \begin{cases} \sum_{k=1}^{4} u_k x_k^2 I_s, & i = j, \\ 0, & i \neq j. \end{cases}$$

Suppose the T-matrices are T_1, T_2, T_3, T_4. Then form the matrices

$$\begin{aligned} A &= T_1 \times N_{11} + T_2 \times N_{21} + T_3 \times N_{31} + T_4 \times N_{41} \\ B &= T_1 \times N_{12} + T_2 \times N_{22} + T_3 \times N_{32} + T_4 \times N_{42} \\ C &= T_1 \times N_{13} + T_2 \times N_{23} + T_3 \times N_{33} + T_4 \times N_{43} \\ D &= T_1 \times N_{14} + T_2 \times N_{24} + T_3 \times N_{34} + T_4 \times N_{44}. \end{aligned}$$

Now

$$AA^T + BB^T + CC^T + DD^T = t \sum_{k=1}^{4} u_k x_k^2 I_{st},$$

and since A, B, C, D are type 1, they can be used in the J. Wallis-Whiteman generalization of the Goethals-Seidel array to obtain the result. □

Corollary 5 *Suppose the T-matrices are of order t. Then there are orthogonal designs $OD(20t; 5t, 5t, 5t, 5t)$ and $OD(36t; 9t, 9t, 9t, 9t)$.*

Proof: We use the Welch array for the $OD(20t; 5t, 5t, 5t, 5t)$ and the Yamamoto-Ono-Sawade array for the $OD(36t; 9t, 9t, 9t, 9t)$.

Note that to prove the Hadamard conjecture "there is an Hadamard matrix of order $4t$ for all $t > 0$" it would be sufficient to prove:

Conjecture 6 *There exists an $OD(4t; t, t, t, t)$ for every positive integer t.*

We also conjecture

Conjecture 7 *There exists an M-structure $OD(4t; t, t, t, t)$ for every $t \equiv 1 \pmod 4$ comprising sixteen circulant or type 1 blocks.*

3 Some properties of certain amicable orthogonal matrices

Lemma 8 *Suppose there exist two amicable $(0,+1,-1)$ matrices U, V of order u satisfying $UU^T + VV^T = (2u-1)I$. Then there exist matrices A, B, D of order u satisfying*

$$
\begin{aligned}
AA^T + BB^T = B^TB + D^TD &= (2u-1)I\\
A^T = (-1)^{\frac{1}{2}(u-1)}A,\ D^T &= (-1)^{\frac{1}{2}(u-1)}D,
\end{aligned}
$$

where A and D have zero diagonal.

Proof: By the properties of U and V we have

$$W = \begin{bmatrix} U & V \\ V & -U \end{bmatrix}$$

is a $(0,+1,-1)$ matrix of order $2u$ satisfying $WW^T = (2u-1)I_{2u}$.

Then by the Delsarte-Goethals-Seidel theorem (see [7] or [22, p. 306]) W is Hadamard equivalent (*i.e.* use the operations of multiplying rows or columns by -1 and rearranging rows or columns) to a $(0,+1,-1)$ matrix C with zero diagonal satisfying

$$CC^T = (2u-1)I_{2u}, \qquad C^T = (-1)^{\frac{1}{2}(u-1)}C.$$

Hence C can be written

$$C = \begin{bmatrix} A & B \\ \pm B^T & \pm D^T \end{bmatrix}$$

where $A^T = (-1)^{\frac{1}{2}(u-1)}A$, $D^T = (-1)^{\frac{1}{2}(u-1)}D$, and A and D have zero diagonal. □

Lemma 9 *Let $q+1$ be the order of a conference matrix. Then there exist four matrices C_1, C_2, C_3, C_4, of order $\frac{1}{2}(q-1)$ satisfying*

$$
\begin{gathered}
C_1C_1^T + C_2C_2^T = C_3C_3^T + C_4C_4^T = qI - 2J,\\
eC_1^T = eC_4^T = e, \quad eC_2^T = eC_3^T = 0,\\
C_1C_3^T - C_2C_4^T = 0, \quad C_1^T = C_1, \quad C_4^T = C_4, \quad C_3^T = C_2,
\end{gathered}
$$

where e is the $1 \times \frac{1}{2}(q-1)$ matrix of ones, C_1 and C_4 have zero diagonal elements ± 1, C_2 and C_4 have elements ± 1.

Proof: By the Delsarte-Goethals-Seidel theorem (see [7] or [22, p. 306]) we can ensure the conference matrix is symmetric and of the form

$$C = \begin{bmatrix} 0 & e_q^T \\ e_q & D \end{bmatrix}, \quad D^T = D,$$

where D has zero diagonal. We now simultaneously permute the rows and columns of D (so if row i and j are interchanged then column i and column j are also interchanged) to keep symmetry and obtain

$$E = \begin{bmatrix} 0 & 1 & e & e \\ 1 & 0 & e & -e \\ e_q^T & e_q^T & -C_1 & C_2 \\ e_q^T & -e_q^T & C_3 & C_4 \end{bmatrix}.$$

Since E is orthogonal $e - eC_1^T - eC_2^T = 0 = e - eC_1^T + eC_2^T$ so $eC_1^T = e$, $eC_2^T = 0$ and

$$C_1C_1^T + C_2C_2^T = C_3C_3^T + C_4C_4^T = qI - 2J,$$
$$eC_1^T = eC_4^T = e, \; eC_2^T = eC_3^T = 0,$$
$$C_1C_3^T - C_2C_4^T = 0, \quad C_1^T = C_1, \quad C_4^T = C_4, \quad C_3^T = C_2.$$

□

Lemma 10 *Suppose there exist two amicable $(0, +1, -1)$ matrices U, V of order u satisfying $UU^T + VV^T = (2u-1)I$. Further suppose U has zero diagonal and U, V have other elements $+1$ or -1. Then there exist matrices A, B of order $u-1$ satisfying*

$$AA^T + BB^T = (2u-1)I_{u-1} - 2J_{u-1},$$
$$eA^T = e, \quad eB^T = 0, \quad AB^T = BA^T,$$

where A has one zero element per row and column and the other entries of A and B are ± 1. Further if U and V are symmetric (or skew-type respectively) then A and B are symmetric (or skew-type respectively).

Furthermore if U and V satisfy $UU^T + VV^T = 2uI$ (U, V are $(1,-1)$ matrices), u even, then there exist matrices A, B of order $u-1$, with entries ± 1, satisfying

$$AA^T + BB^T = 2uI_{u-1} - 2J_{u-1},$$
$$eA^T = e, \quad eB^T = e, \quad AB^T = BA^T,$$

and if U and V are symmetric (or skew-type respectively) then A and B are symmetric (or skew-type respectively).

Proof: Without loss of generality assume V has its $(1,1)$ entry $+1$, otherwise replace it by $-V$. If U has no zeros and non zero $(1,1)$ entry assume it is -1 (the outcome is identical up to equivalence of the desired properties).

Assume U has zero diagonal. Define $D = U + iV$, then with $D\dagger$ written for the Hermitian conjugate (transpose and complex conjugate), we have

$$\begin{aligned} DD\dagger &= (U+iV)(U^T - iV^T) \\ &= UU^T + VV^T + i(UV^T - VU^T) \\ &= UU^T + VV^T \qquad \text{(by the amicability of } U \text{ and } V\text{)} \\ &= (2u-1)I_u, \end{aligned}$$

an orthogonal matrix with diagonal entries $\pm i$ and other entries $\pm 1 \pm i$. We wish to normalize the first row and column to

$$E = \begin{bmatrix} i & 1+i & 1+i & \ldots & 1+i \\ 1+i & & & & \\ 1+i & & & & \\ \vdots & & & F+iG & \\ 1+i & & & & \end{bmatrix}$$

$$\text{or } E_1 = \begin{bmatrix} i & 1+i & 1+i & \ldots & 1+i \\ -1-i & & & & \\ -1-i & & & & \\ \vdots & & & F+iG & \\ -1-i & & & & \end{bmatrix}$$

if U and V are skew-type. If the first element of row/column j of D is $1+i$, $1-i$, $-1+i$, $-1-i$ we multiply the row/column by 1, i, $-i$, -1 respectively, to form E. We only form E_1 if both U and V are skew type.

If U and V are symmetric (or skew-type respectively) the operation on row j is also carried out on column j preserving symmetry (skew-type respectively).

The operations performed have not affected the orthogonality so

$$EE\dagger = (2u-1)I_u.$$

We now write E or E_1 as

$$E = \begin{bmatrix} 0 & e \\ e^T & L \end{bmatrix} + i \begin{bmatrix} 1 & e \\ e^T & N \end{bmatrix}.$$

So

$$
\begin{aligned}
EE\dagger &= \begin{bmatrix} u-1 & eL^T \\ Le^T & J+LL^T \end{bmatrix} + \begin{bmatrix} u & e(1+N^T) \\ (1+N)e^T & J+NN^T \end{bmatrix} \\
&\quad -i\left(\begin{bmatrix} u-1 & eN^T \\ (1+L)e^T & J+LN^T \end{bmatrix} - \begin{bmatrix} u-1 & e(1+L^T) \\ Ne^T & J+NL^T \end{bmatrix}\right) \\
&= \begin{bmatrix} 2u-1 & e(L^T+N^T+1) \\ (1+L+N)e^T & 2J+LL^T+NN^T \end{bmatrix} \\
&\quad -i\begin{bmatrix} 0 & e(N^T-1-L^T) \\ (1+L-N)e^T & LN^T-NL^T \end{bmatrix} \\
&= (2u-1)I.
\end{aligned}
$$

Hence $LN^T = NL^T$, $(1+L+N)e^T = 0 = (1+L-N)e^T$, giving $eL^T = -e$, $eN^T = 0$ and $LL^T + NN^T = (2u-1)I - 2J$. Set $-L = M$ to get the result.

It remains to be shown that M has zero diagonal. Now $MM^T + NN^T = (2u-1)I - 2J$. So there is only one zero per row of $[M : N]$. Also u is odd so M and N have even order $u-1$. Hence $eN^T = 0$ tells us N has no zero entries and thus the one zero entry per row must be in M. Rearrange the columns of M (if necessary) to ensure M has zero diagonal.

If U and V were $(1,-1)$ matrices of even order then

$$
E = \begin{bmatrix} -1 & e \\ e^T & L \end{bmatrix} + i\begin{bmatrix} 1 & e \\ e^T & N \end{bmatrix} .
$$

and

$$
\begin{aligned}
EE\dagger &= \begin{bmatrix} 2u & e(L^T+N^T) \\ (L+N)e^T & 2J+LL^T+NN^T \end{bmatrix} \\
&\quad +i\begin{bmatrix} 0 & e(L^T-N^T+2) \\ (N-L-2)e^T & LN^T-NL^T \end{bmatrix} \\
&= 2uI.
\end{aligned}
$$

Hence $LN^T = NL^T$, $(L+N)e^T = 0 = (N-L-2)e^T$, giving $eL^T = -e$, $eN^T = e$ and $LL^T + NN^T = 2uI - 2J$. Set $-L = M$ to get the result. □

Remark 11 This lemma is very similar to the beautiful Lemma 1 of Miyamoto [12].

Remark 12 Let $I + W$ and V be normalized amicable Hadamard matrices of order h (see Jennifer Seberry [16] for a list of their orders). Then there exist two matrices A, B of order $h-1$ satisfying

$$\begin{array}{c} AA^T + BB^T = (2h-1)I_{h-1} - 2J_{h-1}, \\ eA^T = 0, \quad eB^T = e, \quad AB^T = BA^T, \quad A^T = -A, \quad B^T = B, \\ AA^T = (h-1)I - J, \quad BB^T = hI - J \ . \end{array}$$

where A has zero diagonal and the other entries of A and B are ± 1.

Remark 13 Let $I + W$ and V be amicable Hadamard matrices of order h (see Jennifer Seberry [16] for a list of their orders). Then there exist two matrices W, V of order h satisfying

$$WW^T + VV^T = (2h-1)I, \quad WV^T = VW^T, \quad W^T = -W, \quad V^T = V.$$

Remark 14 From Jennifer Seberry Wallis' restatement [22, p. 291] of a theorem of R.E.A.C. Paley we have

(i) If $q \equiv 3 \pmod 4$ is a prime power or there is a skew-Hadamard matrix of order $q+1$ then there is a skew symmetric matrix W of order q such that $WW^T = (q+1)I - J$, $W^T = -W$. Let R be a symmetric permutation matrix such that WR is symmetric (in the case of q a prime power the back diagonal matrix has this property) then

$$(WR)(WR)^T = (q+1)I - J, \quad (WR)^T = (WR),$$
$$\text{and} \quad (WR)I^T = I(WR)^T.$$

(ii) If $q \equiv 1 \pmod 4$ is a prime power or there is a symmetric conference matrix $C + I$ of order $q+1$ then there is a symmetric matrix Q of order q such that $QQ^T = qI - J$, $Q^T = Q$ and so that

$$(Q+I)(Q+I)^T + (Q-I)(Q-I)^T = 2(q+1)I - 2J.$$

Remark 15 From Geramita and Seberry's restatement [8, p. 92, Theorem 4.41] of a theorem of Goethals and Seidel we have

If $q \equiv 1 \pmod 4$ is a prime power there are two circulant symmetric, amicable matrices M and N of order $\frac{1}{2}(q+1)$ satisfying

$$MM^T + NN^T = qI_{\frac{1}{2}(q+1)}.$$

Remark 16 From Seberry-Wallis's restatement [22, p. 321, Theorem 4.6] of a theorem of Szekeres for $q \equiv 5 \pmod 8$ and by Yamada's theorem [45, Appendix] for $q = a^2 \equiv 1 \pmod 8$ we have

(i) If $q \equiv 5 \pmod 8$ is a prime power then there are two circulant or type 1 amicable matrices U, V of order q satisfying

$$UU^T + VV^T = 2qI - 2J,$$
$$eU^T = 0, \quad eV^T = 0, \quad UV^T = VU^T, \quad U^T = -U, \quad V^T = -V.$$

With R the appropriate permutation matrix (as mentioned in Remark 14(i) above) set $W = I + V$; then

$$UU^T + (WR)(WR)^T = (2q+1)I - 2J,$$
$$eU^T = 0, \quad e(WR)^T = e,$$
$$U(WR)^T = (WR)U^T, \quad U^T = -U, \quad (WR)^T = (WR).$$

(ii) If $q = a^2 \equiv 1 \pmod 8$ is a prime power then there are two circulant or type 1 amicable matrices U, V of order q satisfying

$$UU^T + VV^T = 2(q+1)I - 2J,$$
$$eU^T = e, \quad eV^T = e,$$
$$UV^T = VU^T, \quad U^T = U, \quad V^T = V.$$

Remark 17 From Seberry-Wallis's restatement [22, p. 323, Theorem 4.7] of a theorem found independently by Szekeres and Whiteman, we have

If $q = p^t \equiv 1 \pmod 8$ is a prime power, $p \equiv 5 \pmod 8$, then there are two circulant or type 1 amicable matrices U, V of order q satisfying

$$UU^T + VV^T = 2qI - 2J,$$
$$eU^T = 0, \quad eV^T = 0, \quad UV^T = VU^T, \quad U^T = -U, \quad V^T = -V.$$

With R the appropriate permutation matrix (as mentioned in Remark 14(i) above) set $W = I + V$ then

$$UU^T + (WR)(WR)^T = (2q+1)I - 2J,$$
$$eU^T = 0, \quad e(WR)^T = e,$$
$$U(WR)^T = (WR)U^T, \quad U^T = -U, \quad (WR)^T = (WR).$$

Remark 18 From Geramita and Seberry's restatement [8, p. 256, Theorem 5.80] of a theorem of Szekeres we have

If $q = 4m + 3 \equiv 3 \pmod 4$ is a prime power then there are two cyclic supplementary difference sets $2 - \{2m+1; m; m-1\}$, M and N, called Szekeres difference sets, such that $a \in M \Rightarrow -a \notin M$,

$B \in N \Rightarrow -b \in N$. Thus if $U - I$, V are the $(1,-1)$ incidence matrices of M, N respectively,

$$UU^T + VV^T = qI - 2J,$$
$$eU^T = 0, \quad eV^T = -e, \quad U^T = -U, \quad V^T = V.$$

Now let R be the back diagonal matrix (as above) and set $W = -VR$ then U and W are amicable matrices of order $\frac{1}{2}(q-1)$, U with zero diagonal and W symmetric such that

$$UU^T + WW^T = qI - 2J,$$
$$eU^T = 0, \quad eW^T = e, \quad U^T = -U, \quad W^T = W.$$

Indeed the process just described ensures that if there are Szekeres difference sets on an abelian group of order q then the matrices U and W, just mentioned, can be constructed of order q.

Remark 19 If $q \equiv 1 \pmod 4$ is a prime power, Yamada [42] showed that there exist two circulant matrices U, V of order $\frac{1}{2}(q-1)$ satisfying

$$UU^T + VV^T = qI - 2J,$$
$$eU^T = e, \quad eV^T = 0, \quad U^T = U,$$

where U has zero diagonal. With R the appropriate permutation matrix (as mentioned in Remark 14(i) above) set $W = VR$ then

$$UU^T + WW^T = qI - 2J,$$
$$eU^T = e, \quad eW^T = 0, \quad UW^T = WU^T, \quad U^T = U, \quad W^T = W.$$

Remark 20 If $q = s^2 + 4 \equiv 5 \pmod 8$ is a prime power then J. Wallis [29] and independently Yamada [45] showed that there are two.circulant or type 1 matrices U and V of order q where

$$UU^T + VV^T = (2q+1)I - 2J,$$
$$eU^T = 0, \quad eV^T = e, \quad U^T = -U, \quad V^T = V,$$

and where U has zero diagonal. Now let R be the back diagonal matrix (as above) and set $W = VR$ then U and W are amicable matrices of order q, U with zero diagonal and W symmetric with zero back diagonal such that

$$UU^T + WW^T = 2qI - 2J,$$
$$eU^T = 0, \quad eW^T = 0, \quad U^T = -U, \quad W^T = W, \quad UW^T = WU^T.$$

Note Yamada has observed that there are other suitable matrices for these orders.

4 A multiplication Theorem using M-structures

Theorem 21 *Let $N = (N_{ij})$, $i,j = 1,2,3,4$ be an Hadamard matrix of order $4n$ of M-structure. Further let T_{ij}, $i,j = 1,2,3,4$ be 16 $(0,+1,-1)$ type 1 or circulant matrices of order t which satisfy*

(i) $T_{ij} * T_{ik} = 0$, $T_{ji} * T_{ki} = 0$, $j \neq k$, *(* the Hadamard product)*

(ii) $\sum_{k=1}^{4} T_{ik}$ *is a* $(1,-1)$ *matrix,* *(8)*

(iii) $\sum_{k=1}^{4} T_{ik}T_{ik}^T = tI_t = \sum_{k=1}^{4} T_{ki}T_{ki}^T$,

(iv) $\sum_{k=1}^{4} T_{ik}T_{jk}^T = 0 = \sum_{k=1}^{4} T_{ki}T_{kj}^T$, $i \neq j$.

Then there is an M-structure Hadamard matrix of order $4nt$.

Proof: Define the matrix $X = (X_{ij})$ as follows

$$X_{ij} = \sum_{k=1}^{4} T_{ik} \times N_{jk}^T.$$

From the conditions of the T-matrices and from the M-structure, we have

$$\begin{aligned}
\sum_{j=1}^{4} X_{ij}X_{ij}^T &= \sum_{j=1}^{4} \left(\sum_{k=1}^{4} T_{ik} \times N_{jk}^T\right)\left(\sum_{m=1}^{4} T_{im} \times N_{jm}^T\right)^T \\
&= \sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{m=1}^{4} (T_{ik}T_{im}^T \times N_{jk}^T N_{jm}) \\
&= \sum_{k=1}^{4}\sum_{m=1}^{4} T_{ik}T_{im}^T \times \left(\sum_{j=1}^{4} N_{jk}^T N_{jm}\right).
\end{aligned}$$

If $k \neq m$, then $\sum_{j=1}^{4} N_{jk}^T N_{jm} = 0$. Hence the above equation becomes

$$\begin{aligned}
\sum_{j=1}^{4} X_{ij}X_{ij}^T &= \sum_{k=1}^{4} T_{ik}T_{ik}^T \times \sum_{j=1}^{4} N_{jk}^T N_{jk} \\
&= 4tnI_{tn}.
\end{aligned}$$

For $i \neq k$,

$$\sum_{j=1}^{4} X_{ij}X_{kj}^T = \sum_{j=1}^{4} \left(\sum_{g=1}^{4} T_{ig} \times N_{jg}^T\right)\left(\sum_{m=1}^{4} T_{km} \times N_{jm}^T\right)^T$$

$$
\begin{aligned}
&= \sum_{j=1}^{4}\sum_{g=1}^{4}\sum_{m=1}^{4} T_{ig}T_{km}^T \times N_{jg}^T N_{jm} \\
&= \sum_{g=1}^{4}\sum_{m=1}^{4} T_{ig}T_{km}^T \times \left(\sum_{j=1}^{4} N_{jg}^T N_{jm}\right) \\
&= \sum_{g=1}^{4} T_{ig}T_{kg}^T \times \sum_{j=1}^{4} N_{jg}^T N_{jg} \\
&= 0.
\end{aligned}
$$

Hence the matrix X is an Hadamard matrix of order $4nt$ of M-structure and the matrix $X' = (X_{ji})$ is also an Hadamard matrix of M-structure.

We further note that if $\sum_{k=1}^{4} T_{ki}$ is a $(1,-1)$ matrix and define the matrices $Y = (Y_{ij})$, $Z = (Z_{ij})$, and $W = (W_{ij})$ as follows:

$$
\begin{aligned}
Y_{ij} &= \textstyle\sum_{k=1}^{4} T_{ki} \times N_{kj}^T, \\
Z_{ij} &= \textstyle\sum_{k=1}^{4} T_{ki} \times N_{jk}^T, \quad \text{and} \\
W_{ij} &= \textstyle\sum_{k=1}^{4} T_{ik} \times N_{kj}^T.
\end{aligned}
$$

Then, as in the case for X, we see all three matrices Y, Z and W are Hadamard matrices of order $4nt$ of M-structure. Furthermore $Y' = (Y_{ji})$, $Z' = (Z_{ji})$, and $W' = (W_{ji})$ are also Hadamard matrices of M-structure. □

Corollary 22 *If there exists an Hadamard matrix of order $4h$ and an orthogonal design $OD(4u; u_1, u_2, u_3, u_4)$, then an $OD(8hu; 2hu_1, 2hu_2, 2hu_3, 2hu_4)$ exists.*

Proof: Let $H = (H_{ij})$, $i, j = 1,2,3,4$ be an Hadamard matrix of order $4h$. Put

$$P_i = \frac{1}{2}(H_{i1}+H_{i2}),\ Q_i = \frac{1}{2}(H_{i1}-H_{i2}),\ R_i = \frac{1}{2}(H_{i3}+H_{i4}),\ S_i = \frac{1}{2}(H_{i3}-H_{i4}),$$

and the required T-matrices of order $2h$ for the theorem are

$$T_{i1} = \begin{bmatrix} P_i & \\ & P_i \end{bmatrix},\ T_{i2} = \begin{bmatrix} Q_i & \\ & Q_i \end{bmatrix},\ T_{i3} = \begin{bmatrix} & R_i \\ R_i & \end{bmatrix},\ T_{i4} = \begin{bmatrix} & S_i \\ S_i & \end{bmatrix},$$

for $i = 1,2,3,4$. Since

$$\sum_{j=1}^{4} T_{ij}T_{ij}^T = \sum_{i=1}^{4} (P_iP_i^T + Q_iQ_i^T + R_iR_i^T + S_iS_i^T) \times I_2$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\sum_{j=1}^{4} H_{ij}H_{ij}^T\right) \times I_2 \\
&= 2hI_{2h},
\end{aligned}
$$

and

$$\sum_{k=1}^4 T_{ik}T_{jk}^T = 0, \quad \sum_{k=1}^4 T_{ki}T_{kj}^T = 0, \quad \text{for } i \neq j, \text{ and}$$
$$\sum_{k=1}^4 T_{ki}, \quad i = 1,2,3,4 \quad \text{is a } (1,-1) \text{ matrix.}$$

Now let the OD$(4u; u_1, u_2, u_3, u_4) = D = (D_{ij})$, $i, j = 1,2,3,4$ defined on the commuting variables x_1, x_2, x_3, x_4. Then we have

$$DD^T = (u_1x_1^2 + u_2x_2^2 + u_3x_3^2 + u_4x_4^2)I_{4u},$$

that is

$$
\begin{aligned}
\sum_{j=1}^{4} D_{ij}D_{ij}^T &= \sum_{j=1}^{4} D_{ij}^T D_{ij} \\
&= (u_1x_1^2 + u_2x_2^2 + u_3x_3^2 + u_4x_4^2)I_u,
\end{aligned}
$$

$$\sum_{k=1}^4 D_{ik}D_{jk}^T = 0, \qquad \sum_{k=1}^4 D_{ki}D_{kj}^T = 0, \quad i, j = 1,2,3,4, \quad i \neq j.$$

We now define the matrix $X = (X_{ij})$ as follows

$$X_{ij} = \sum_{k=1}^{4} T_{ik} \times D_{jk}^T.$$

Then, as in the theorem, we have

$$\sum_{j=1}^{4} X_{ij}X_{ij}^T = 2h(u_1x_1^2 + u_2x_2^2 + u_3x_3^2 + u_4x_4^2)I_{2hu},$$

and for $i \neq k$,

$$\sum_{k=1}^{4} X_{ij}X_{kj}^T = 0.$$

Thus $X = (X_{ij})$ and $X' = (X_{ji})$ are OD$(8hu; 2hu_1, 2hu_2, 2hu_3, 2hu_4)$ of M-structure and $Y = (Y_{ij}) = \left(\sum_{k=1}^4 T_{ki} \times D_{kj}^T\right)$, $Z = (Z_{ij}) = \left(\sum_{k=1}^4 T_{ki} \times D_{jk}^T\right)$ and $W = (W_{ij}) = \left(\sum_{k=1}^4 T_{ik} \times N_{kj}^T\right)$, $Y' = (Y_{ji})$, $Z' = (Z_{ji})$ and $W' = (W_{ji})$, are also OD$(8hu; 2hu_1, 2hu_2, 2hu_3, 2hu_4)$ of M-structure. □

Corollary 23 *If there exists an Hadamard matrix of order $4h$ and an orthogonal design $OD(4u; u, u, u, u)$, then there exists an $OD(8hu; 2hu, 2hu, 2hu, 2hu)$.*

This gives the theorem of Agayan and Sarukhanyan [2] as a corollary by setting all variables equal to one:

Corollary 24 *If there exists Hadamard matrices of orders $4h$ and $4u$ then there exists an Hadamard matrix of order $8hu$.*

We now give as a corollary a result, motivated by, and a little stronger than that of Agayan and Sarukhanyan [2]:

Corollary 25 *Suppose there are Williamson or Williamson type matrices of orders u and v. Then there are Williamson type matrices of order $2uv$.*

If the matrices of orders u and v are symmetric the matrices of order $2uv$ are also symmetric.

If the matrices of orders u and v are circulant and/or type 1 the matrices of order $2uv$ are type 1.

Proof: Suppose A, B, C, D are (symmetric) Williamson or Williamson type matrices of order u then they are pairwise amicable and satisfy

$$AA^T + BB^T + CC^T + DD^T = 4uI_u.$$

Define

$$E = \frac{1}{2}(A+B), \quad F = \frac{1}{2}(A-B), \quad G = \frac{1}{2}(C+D), \quad H = \frac{1}{2}(C-D),$$

then E, F, G, H are pairwise amicable (and symmetric) and satisfy

$$EE^T + FF^T + GG^T + HH^T = 2uI_u.$$

Now define

$$T_1 = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad T_2 = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & G \\ G & 0 \end{bmatrix}, \quad \text{and } T_4 = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix},$$

so that

$$\begin{aligned} T_1 &= T_1 1 = T_2 2 = T_3 3 = T_4 4, \\ T_2 &= T_1 2 = -T_2 1 = T_3 4 = -T_4 3, \\ T_3 &= T_1 3 = -T_3 1 = -T_2 4 = T_4 2 \quad \text{and} \\ T_4 &= T_1 4 = -T_4 1 = T_2 3 = -T_3 2, \end{aligned}$$

in the theorem. Note T_1, T_2, T_3, T_4 are pairwise amicable. If A, B, C, D were circulant (or type 1) they would be type 1 of order $2u$.

Let X, Y, Z, W be the Williamson or Williamson type (symmetric) matrices of order v. Then X, Y, Z, W are pairwise amicable and

$$XX^T + YY^T + ZZ^T + WW^T = 4vI_v.$$

Then

$$\begin{aligned} L &= T_1 \times X + T_2 \times Y + T_3 \times Z + T_4 \times W \\ M &= -T_1 \times Y + T_2 \times X + T_3 \times W - T_4 \times Z \\ N &= -T_1 \times Z - T_2 \times W + T_3 \times X + T_4 \times Y \\ P &= -T_1 \times W + T_2 \times Z - T_3 \times Y + T_4 \times X. \end{aligned}$$

are 4 Williamson type (symmetric) matrices of order $2uv$. If the matrices of orders u and v were circulant or type 1 these matrices are type 1. □

5 Miyamoto's Theorem and Corollaries via M-structures

We reformulate Miyamoto's results so that symmetric Williamson-type matrices can be obtained.

Lemma 26 (Miyamoto's Lemma Reformulated) *Let U_i, V_j, $i, j = 1, 2, 3, 4$ be $(0, +1, -1)$ matrices of order n which satisfy*

(i) U_i, U_j, $i \neq j$ *are pairwise amicable,*

(ii) V_i, V_j, $i \neq j$ *are pairwise amicable,*

(iii) $U_i \pm V_i$, $(+1, -1)$ *matrices,* $i = 1, 2, 3, 4$,

(iv) the row sum of U_1 *is 1, and the row sum of* U_j, $i = 2, 3, 4$ *is zero,*

(v) $\sum_{i=1}^{4} U_i U_i^T = (2n + 1)I - 2J$, $\sum_{i=1}^{4} V_i V_i^T = (2n + 1)I$.

Then there are 4 Williamson type matrices of order $2n + 1$. *If* U_i *and* V_i *are symmetric,* $i = 1, 2, 3, 4$ *then the Williamson-type matrices are symmetric. Hence there is a Williamson type Hadamard matrix of order* $4(2n + 1)$.

Proof: Let S_1, S_2, S_3, S_4 be 4 $(+1, -1)$-matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sum of $S_1 = 2$ and of $S_i = 0$, $i = 2,3,4$. Now define

$$X_1 = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_1 \end{bmatrix} \quad \text{and} \quad X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \quad i = 2,3,4.$$

First note that since U_i, U_j, $i \neq j$ and V_i, V_j, $i \neq j$ are pairwise amicable,

$$\begin{aligned} S_iS_j^T &= \left(U_i \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_i \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\left(U_j^T \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j^T \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \\ &= U_iU_j^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + V_iV_j^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= S_jS_i^T. \end{aligned}$$

(Note this relationship is valid if and only if conditions (i) and (ii) of the theorem are valid.)

$$\begin{aligned} \sum_{i=1}^{4} S_iS_i^T &= \sum_{i=1}^{4} U_iU_i^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \sum_{i=1}^{4} V_iV_i^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= 2\begin{bmatrix} 2(2n+1)I - 2J & -2J \\ -2J & 2(2n+1)I - 2J \end{bmatrix} \\ &= 4(2n+1)I_{2n} - 4J_{2n} \end{aligned}$$

Next we observe

$$X_1X_i^T = \begin{bmatrix} 1-2n & e_{2n} \\ e_{2n}^T & -J + S_1S_i^T \end{bmatrix} = X_iX_1^T \qquad i = 2,3,4,$$

and

$$X_iX_j^T = \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_iS_j^T \end{bmatrix} = X_jX_i^T \qquad i \neq j, \;\; i,j = 2,3,4.$$

Further

$$\begin{aligned} \sum_{i=1}^{4} X_iX_i^T &= \begin{bmatrix} 1+2n & -3e_{2n} \\ -3e_{2n}^T & J + S_1S_1^T \end{bmatrix} + \sum_{i=2}^{4} \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_iS_i^T \end{bmatrix} \\ &= \begin{bmatrix} 4(2n+1) & 0 \\ 0 & 4J + 4(2n+1)I - 4J \end{bmatrix}. \end{aligned}$$

Thus we have shown that X_1, X_2, X_3, X_4 are 4 Williamson type matrices of order $2n+1$.

Hence there is a Williamson type Hadamard matrix of order $4(2n+1)$. □

Corollary 27 *Let $q \equiv 1 \pmod 4$ be a prime power then there are symmetric Williamson type matrices of order $q+2$ whenever $\frac{1}{2}(q+1)$ is a prime power or $\frac{1}{2}(q+3)$ is the order of a symmetric conference matrix. Also there exists an Hadamard matrix of Williamson type of order $4(q+2)$.*

Proof: (i) Let B be the skew-symmetric core of order $\frac{1}{2}(q+1)$ formed via the quadratic residues (see Remark 14(i)) and R the back-diagonal matrix so that BR is back circulant or type2 and symmetric;

(ii) Let X be the symmetric core of order $\frac{1}{2}(q+1)$ of the conference matrix (see Remark 14(ii));

(iii) Let M, N be the two circulant symmetric matrices of order $\frac{1}{2}(q+1)$, M with zero diagonal satisfying $MM^T + NN^T = qI$ (see Remark 15).

Then in Lemma 26 use

(ia) $U_1 = I, U_2 = 0, U_3 = U_4 = BR,$

(iia) $V_1 = M, V_2 = N, V_3 = V_4 = R,$

(ib) $U_1 = I, U_2 = 0, U_3 = U_4 = X,$

(iib) $V_1 = M, V_2 = N, V_3 = V_4 = I,$

to obtain the result. □

Remark 28 Some of the results in Corollary 27 are also due to A.L. Whiteman [35]. This gives symmetric Williamson-type matrices of orders

7	11	15	19	27	39	51	55	63	75
83	91	99	123	159	195	243	279	315	339
363	399	423	451	459	543	579	615	627	663
675	735	759	843	879	883	999	1095	1155	1203
1215	1239	1251	1323	1383	1455	1623	1659	1683	1755
1875	1935	1995							

(since Mathon found conference matrices of orders 46 and 442). Almost all these, with symmetry, are new though Miyamoto [12] has found Williamson-type matrices for these orders and hence Hadamard matrices for four times these orders.

Koukouvinos and Kounias [10] have shown there are no circulant symmetric Williamson matrices of order 39 but here a symmetric but not circulant Williamson matrix of order 39 is given.

Corollary 29 *Let $q \equiv 1 \pmod 4$ be a prime power. Then*

(i) if there are Williamson type matrices of order $(q-1)/4$ or an Hadamard matrix of order $\frac{1}{2}(q-1)$ there exist Williamson type matrices of order q;

(ii) if there exist symmetric conference matrices of order $\frac{1}{2}(q-1)$ or a symmetric Hadamard matrix of order $\frac{1}{2}(q-1)$ then there exist symmetric Williamson type matrices of order q.

Hence there exists an Hadamard matrix of Williamson type of order $4q$.

Proof: (i) Use Yamada's matrices A and $C = BR$ of order $\frac{1}{2}(q-1)$ (see Remark 19) as

$$U_1 = A, \quad U_2 = C, \quad U_3 = U_4 = 0, \quad \text{and} \quad V_1 = I, \quad V_2 = 0,$$

and for

$$V_3 = \begin{bmatrix} W_1 & W_2 \\ W_2 & -W_1 \end{bmatrix}, \qquad V_4 = \begin{bmatrix} W_3 & W_4 \\ -W_4 & W_3 \end{bmatrix},$$

where W_i, $i = 1,2,3,4$ are Williamson-type matrices, or $V_3 = V_4 = H$, where H is an Hadamard matrix of order $\frac{1}{2}(q-1)$, and
(ii) with N the appropriate symmetric conference matrix and H the appropriate Hadamard matrix use

$$V_3 = N + I, \quad V_4 = N - I, \quad \text{or} \quad V_3 = V_4 = H,$$

as indicated in Lemma 26 to obtain Williamson-type matrices. □

Remark 30 Part (i) of Corollary 29 for Williamson matrices of order $(q-1)/4$ was found by Miyamoto [12]. Part (i) with Hadamard matrices of order $\frac{1}{2}(q-1)$ is new. Part (ii) with symmetry is new.

Corollary 29 part (ii) gives symmetric Williamson-type matrices of order q when $q \equiv 1 \pmod 4$ is a prime power and $\frac{1}{2}(q-1)$ is the order of a symmetric conference matrix. This gives symmetric Williamson-type matrices for the following orders:

13	29	37	53	61	101	109	125	149	181
197	229	277	317	349	389	397	461	541	557
677	701	709	797	821	1021	1061	1117	1229	1237
1549	1597	1621	1709	1861	1877	1997			

Corollary 29 will also give Williamson-type matrices of orders 293, 373, 613, 653, 733, 757, 853, 1013, 1069, 1213, 1277, 1373, 1381, 1453, 1493, 1669, 1693, 1733, 1901, 1933, or 1973 if symmetric conference matrices of orders 146, 186, 306, 326, 366, 378, 426, 506, 534, 606, 638, 686, 690, 726, 746, 834, 866, 950, 966 or 986 exist, respectively.

Corollary 29 part (ii) gives symmetric Williamson-type matrices of order q when $q \equiv 1 \pmod 4$ is a prime power and $\frac{1}{2}(q-1)$ is the order of a symmetric Hadamard matrix. Rembering that symmetric Hadamard matrices exist for orders $p+1$ when $p \equiv 3 \pmod 4$ is a prime power we have symmetric Williamson-type matrices for the following orders:

5	9	17	25	41	49	73	81	89	97
113	121	169	193	241	257	281	289	337	353
361	401	409	433	449	457	529	569	577	593
601	617	625	641	673	729	761	769	841	881
929	937	961	977	1009	1033	1049	1097	1129	1153
1201	1217	1249	1289	1297	1321	1361	1369	1409	1481
1489	1553	1601	1609	1657	1681	1697	1721	1777	1801
1849	1873								

Corollary 29 also gives symmetric Williamson-type matrices of orders 233, 313, 521, 809, 857, 953, 1193, 1433, 1753, 1889, 1913, and 1993 when symmetric Hadamard matrices of orders 4.29, 4.39, 4.65, 4.101, 4.107, 4.119, 4.149, 4.179, 4.219, 16.59, 4.239 and 4.249 are discovered.

Corollary 29 part (i) gives Williamson-type matrices of order q when $q \equiv 1 \pmod 4$ is a prime power and $\frac{1}{2}(q-1)$ is the order of an Hadamard matrix. This gives Williamson-type matrices for the following orders not given above:

137	233	313	521	809	953	1193	1753	1889	1993

Corollary 29 part (i) gives Williamson-type matrices of order q when $q \equiv 1 \pmod 4$ is a prime power and $(q-1)/4$ is the order of Williamson-type matrices. This result is also due to Miyamoto [12]. This gives Williamson-type matrices for the following orders:

157	173	293	373	613	757	757	773	1109	1301
1453	1493	1637	1693	1733	1741				

Corollary 29 will also gives Williamson-type matrices of orders 857, 1433 and 1913 when Hadamard matrices of orders 4.107, 4.179 and 4.239 are discovered. Further it will give Williamson-type matrices of orders

269	421	509	653	661	733	829	853	877	941
1069	1093	1181	1213	1277	1373	1381	1429	1613	1669
1789	1901	1933	1949	1973					

when Williamson-type matrices of orders

67	105	127	163	165	183	207	213	219	235
267	273	295	303	319	343	345	357	403	417
447	475	483	487	493					

are discovered.

Corollary 31 *Let* $q \equiv 1 \pmod 4$ *be a prime power or* $q+1$ *the order of a symmetric conference matrix. Let* $2q-1$ *be a prime power. Then there exist symmetric Williamson type matrices of order* $2q+1$ *and an Hadamard matrix of Williamson type of order* $4(2q+1)$.

Proof: Form the core Q as in Remark 14(i). Thus we choose a symmetric Q of order q satisfying $eQ = 0$, $QQ^T = qI - J$. From Remark 15 there exist symmetric matrices M and N of order q satisfying

$$MM^T + NN^T = (2q-1)I, \quad M \text{ with zero diagonal.}$$

Use

$$U_1 = I, \quad U_2 = U_3 = Q, \quad U_4 = 0,$$

and

$$V_1 = M, \quad V_2 = V_3 = I, \quad V_4 = N,$$

$$\sum_{i=1}^{4} U_i U_i^T = (2q+1)I - 2J, \qquad \sum_{i=1}^{4} V_i V_i^T = (2q+1)I.$$

Hence by Lemma 26 we have four symmetric Williamson type matrices of order $2q+1$ and a Williamson type Hadamard matrix of order $4(2q+1)$. □

Remark 32 Corollary 31 is satisfied for the appropriate primes or conference matrix orders to give symmetric Williamson-type matrices for the following orders:

11	19	27	51	75	83	91	99	123	195
243	315	339	363	451	459	579	627	675	843
883	1155	1203	1251	1323	1659	1683	1755	1875	1995
2019	2139	2403	2475	2595	2859	3043	3219	3315	3363
3483	3699	3723							

Note this last corollary is a modified version of Miyamoto's Corollary 5 (original manuscript). A new proof of Miyamoto's result, preserving symmetry, is:

Corollary 33 *Let $q \equiv 5 \pmod 8$ be a prime power. Further let $\frac{1}{2}(q-3)$ be a prime power or $\frac{1}{2}(q-1)$ be the order of a symmetric conference matrix then there exist symmetric Williamson type matrices of order q and an Hadamard matrix of Williamson type of order $4q$.*

Proof: Since $q \equiv 1 \pmod 4$ is a prime power, Yamada's matrices A and $C = BR$ of order $\frac{1}{2}(q-1)$ (see Remark 19) satisfy $A^T = A$, $eA = e$, $eB = 0$, $eC = 0$, A has zero diagonal, B and C have elements $+1$ and -1, and $AA^T + CC^T = qI - 2J$, where R is the back diagonal matrix which makes $C = BR$ symmetric.

From Remark 14, since $\frac{1}{2}(q-3)$ is a prime power $\equiv 1 \pmod 4$, there exists a symmetric conference matrix, N, of order $\frac{1}{2}(q-1)$. Let

$$X = N + I, \quad \text{and} \quad Y = N - I,$$

then X, Y are symmetric and amicable of order $\frac{1}{2}(q-1)$ satisfying

$$XX^T + YY^T = (q-1)I.$$

Let

$$\begin{aligned} &U_1 = A, \quad U_2 = C, \quad U_3 = U_4 = 0, \\ \text{and } &V_1 = I, \quad V_2 = 0, \quad V_3 = X, \quad V_4 = Y, \end{aligned}$$

then

$$\sum_{i=1}^{4} U_i U_i^T = qI - 2J, \quad \sum_{i=1}^{4} V_i V_i^T = qI.$$

So the lemma gives the result. □

Theorem 34 (Miyamoto's Theorem Reformulated) *Let U_{ij}, V_{ij}, $i,j = 1,2,3,4$ be $(0,+1,-1)$ matrices of order n which satisfy*

(i) U_{ki}, U_{kj}, $i \neq j$ are pairwise amicable, $k = 1,2,3,4$,

(ii) V_{ki}, V_{kj}, $i \neq j$ are pairwise amicable, $k = 1,2,3,4$,

(iii) $U_{ki} \pm V_{ki}$, $(+1,-1)$ matrices, $i,k = 1,2,3,4$,

(iv) the row sum of U_{ii} is 1, and the row sum of U_ij is zero, $i \neq j$, $i,j = 1,2,3,4$,

(v) $\sum_{i=1}^{4} U_{ji}U_{ji}^T = (2n+1)I - 2J$, $\sum_{i=1}^{4} V_{ji}V_{ji}^T = (2n+1)I$, $j = 1,2,3,4$,

(vi) $\sum_{i=1}^{4} U_{ji}U_{ki}^T = 0$, $\sum_{i=1}^{4} V_{ji}V_{ki}^T = 0$, $j \neq k$, $j,k = 1,2,3,4$.

If conditons (i) to (v) hold, there are four Williamson matrices type of order $2n+1$ and thus a Williamson type Hadamard matrix of order $4(2n+1)$. Furthermore if the matrices U_{ki} and V_{ki} are symmetric for all $i,j = 1,2,3,4$ the Williamson matrices obtained of order $2n+1$ are also symmetric.

If conditons (iii) to (vi) hold, there is an M-structure Hadamard matrix of order $4(2n+1)$.

Proof: Let S_{ij}, be 16 $(+1,-1)$-matrices of order $2n$ defined by

$$S_{ij} = U_{ij} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_{ij} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sum of $S_{ii} = 2$ and of $S_{ij} = 0$, $i \neq j$, $i, j = 1,2,3,4$. Now define

$$X_{11} = \begin{bmatrix} -1 & -e \\ -e^T & S_{11} \end{bmatrix} \quad X_{12} = \begin{bmatrix} 1 & e \\ e^T & S_{12} \end{bmatrix} \quad X_{13} = \begin{bmatrix} 1 & e \\ e^T & S_{13} \end{bmatrix} \quad X_{14} = \begin{bmatrix} -1 & e \\ e^T & S_{14} \end{bmatrix}$$

$$X_{21} = \begin{bmatrix} 1 & e \\ e^T & S_{21} \end{bmatrix} \quad X_{22} = \begin{bmatrix} -1 & -e \\ -e^T & S_{22} \end{bmatrix} \quad X_{23} = \begin{bmatrix} 1 & e \\ e^T & S_{23} \end{bmatrix} \quad X_{24} = \begin{bmatrix} -1 & e \\ e^T & S_{24} \end{bmatrix}$$

$$X_{31} = \begin{bmatrix} 1 & e \\ e^T & S_{31} \end{bmatrix} \quad X_{32} = \begin{bmatrix} 1 & e \\ e^T & S_{32} \end{bmatrix} \quad X_{33} = \begin{bmatrix} -1 & -e \\ -e^T & S_{33} \end{bmatrix} \quad X_{34} = \begin{bmatrix} -1 & e \\ e^T & S_{34} \end{bmatrix}$$

$$X_{41} = \begin{bmatrix} -1 & e \\ e^T & -S_{41} \end{bmatrix} \quad X_{42} = \begin{bmatrix} 1 & e \\ e^T & -S_{42} \end{bmatrix} \quad X_{43} = \begin{bmatrix} -1 & e \\ e^T & -S_{43} \end{bmatrix} \quad X_{44} = \begin{bmatrix} -1 & -e \\ -e^T & -S_{44} \end{bmatrix}$$

We note that the following always holds as it is just a case of Miyamoto's Lemma Reformulated:

$$\sum_{i=1}^{4} S_{ji} S_{ji}^T = 4(2n+1)I_{2n} - 4J_{2n}. \tag{9}$$

In all cases though assumption (vi) assures us that

$$\sum_{i=1}^{4} S_{ki} S_{ji}^T = 0, \quad j \neq k. \tag{10}$$

We separate the remainder of the proof into two parts: Case A where conditions (i) to (v) of the enunciation hold and Case 2 where conditions (iii) to (vi) of the enunciation hold.

Case A. We now note that, as in Miyamoto's Lemma:

$$S_{ki} S_{ji}^T = S_{ji} S_{ki}^T \tag{11}$$

if and only if U_{ki}, U_{kj}, $i \neq j$ are pairwise amicable, $k = 1,2,3,4$, and V_{ki}, V_{kj}, $i \neq j$ are pairwise amicable, $k = 1,2,3,4$. Thus

$$X_{44} X_{4j}^T = \begin{bmatrix} 1-2n & -e_{2n} \\ -e_{2n}^T & -J + S_{44} S_{4j}^T \end{bmatrix} = X_{4j} X_{44}^T \quad j = 1,2,3$$

and

$$X_{4k} X_{4j}^T = \begin{bmatrix} 1+2n & -e_{2n} \\ -e_{2n}^T & J + S_{4k} S_{4j}^T \end{bmatrix} = X_{4j} X_{4k}^T \quad k \neq j, \quad j,k = 1,2,3.$$

Further we note

$$\begin{aligned}
\sum_{i=1}^{4} X_{4i}X_{4i}^T &= \begin{bmatrix} 1+2n & 3e_{2n} \\ 3e_{2n}^T & J+S_{44}S_{44}^T \end{bmatrix} + \sum_{i=1}^{3} \begin{bmatrix} 1+2n & -e_{2n} \\ -e_{2n}^T & J+S_{4i}S_{4i}^T \end{bmatrix} \\
&= \begin{bmatrix} 4(2n+1) & 0 \\ 0 & 4J+4(2n+1)I-4J \end{bmatrix} \\
&= 4(2n+1)I_{2n+1}
\end{aligned}$$

Hence X_{41}, X_{42}, X_{43}, X_{44} are 4 Williamson type matrices of order $2n+1$ and thus a Williamson type Hadamard matrix of order $4(2n+1)$ exists.

Case B. We now assume conditions (i) and (ii) do not hold but that condition (vi) does hold. By straightforward checking we can assert that

$$\sum_{i=1}^{4} X_{ji}X_{ki}^T = 0 \quad j \neq k, \quad \text{if and only if (10) holds.}$$

$$\sum_{i=1}^{4} X_{ji}X_{ji}^T = 4(2n+1)I_{2n+1} \quad j = 1,2,3,4 \quad \text{as (9) holds.}$$

Hence there is an M-structure Hadamard matrix of order $4(2n+1)$. □

Note that if we write our M-structure from the theorem as

$$\begin{matrix}
-1 & 1 & 1 & -1 & -e & e & e & e \\
1 & -1 & 1 & -1 & e & -e & e & e \\
1 & 1 & -1 & -1 & e & e & -e & e \\
1 & 1 & 1 & 1 & -e & -e & -e & e \\
-e^T & e^T & e^T & e^T & S_{11} & S_{12} & S_{13} & S_{14} \\
e^T & -e^T & e^T & e^T & S_{21} & S_{22} & S_{23} & S_{24} \\
e^T & e^T & -e^T & e^T & S_{31} & S_{32} & S_{33} & S_{34} \\
-e^T & -e^T & -e^T & e^T & S_{41} & S_{42} & S_{43} & S_{44}
\end{matrix}$$

and we can see Yamada's matrix with trimming [46] or the J. Wallis-Whiteman [30] matrix with a border embodied in the construction.

Corollary 35 *Suppose there exists a symmetric conference matrix of order $q+1 = 4t+2$ and an Hadamard matrix of order $4t = q-1$. Then there is an Hadamard matrix with M-structure of order $4(4t+1) = 4q$. Further if the Hadamard matrix is symmetric the Hadamard matrix of order $4q$ is of the form*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix},$$

where X, Y are amicable and symmetric.

Proof: Use Lemma 9 to obtain four matrices C_1, C_2, C_3, C_4, of order $\frac{1}{2}(q-1)$ satisfying

$$\begin{aligned} C_1C_1^T + C_2C_2^T &= C_3C_3^T + C_4C_4^T \\ &= qI - J \end{aligned}$$

$$eC_1^T = eC_4^T = e, \quad eC_2^T = eC_3^T = 0, \quad C_1^TC_3^T - C_3^TC_4^T = 0,$$
$$C_1^T = C_1, \quad C_4^T = C_4, \quad C_3^T = C_2.$$

Write the Hadamard matrix with four blocks of size $\frac{1}{2}(q-1)$ as

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}.$$

If this matrix is symmetric then $H_1^TH_3^T + H_3^TH_4^T = 0$, $H_1^T = H_1$, $H_4^T = H_4$, $H_3^T = H_2$.

Now write $U = (U_{ij})$ and $V = (V_{ij})$ with 16 blocks of size $\frac{1}{2}(q-1) \times \frac{1}{2}(q-1)$

$$U = \begin{bmatrix} C_1 & C_2 & 0 & 0 \\ -C_3 & C_4 & 0 & 0 \\ 0 & 0 & C_1 & C_2 \\ 0 & 0 & -C_3 & C_4 \end{bmatrix}, \quad \text{and } V = \begin{bmatrix} I & 0 & H_1 & H_2 \\ 0 & I & H_3 & H_4 \\ -H_1^T & -H_3^T & I & 0 \\ -H_2^T & -H_4^T & 0 & I \end{bmatrix},$$

and straightforward use of Miyamoto's theorem gives the result. □

We note that complex Hadamard matrices of order $n \equiv 2 \pmod 4$ do exist when symmetric conference matrices cannot exist (see [22, Chapter VI]). These complex Hadamard matrices may be written as $K = X + iY$ where $KK* = kI_k$ ($*$ the Hermitian conjugate).

Hence we have

Corollary 36 *Let $q \equiv 4f + 1$ be a prime power. Suppose there is a complex Hadamard matrix of order $2f$. Then there is an Hadamard matrix of order $4(4f+1)$.*

Proof: Use Yamada's construction (see the method of Remark 19) to make A with zero diagonal and ± 1 elsewhere, $A^T = A$, and back-circulant B with elements ± 1 of order $\frac{1}{2}(q-1) = 2f$ satisfying $AA^T + BB^T = qI - 2J$.

Let $C = X + iY$ be a complex Hadamard matrix of order $2f$. Choose

$$U = \begin{bmatrix} A & B & 0 & 0 \\ -B & A & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -B & A \end{bmatrix} \quad \text{and}$$

$$V = \begin{bmatrix} I & 0 & X+Y & X-Y \\ 0 & I & -X+Y & X+Y \\ -X^T-Y^T & X^T-Y^T & I & 0 \\ -X^T+Y^T & -X^T-Y^T & 0 & I \end{bmatrix}.$$

Then the theorem gives us an Hadamard matrix of order $4(4f+1)$. □

Note complex Hadamard matrices exist for orders 22, 34, 58, 86, 306, 650, 870, 1046, 2450, 3782, ..., for which either a symmetric conference matrix cannot exist or is not known. None of these orders give new Hadamard matrices.

6 Using 64 Block M-structures

In a similar fashion, we consider the following lemma so symmetric 8-Williamson-type matrices can be obtained.

Lemma 37 *Let U_i, V_j, $i,j = 1,\ldots,8$ be $(0,+1,-1)$ matrices of order n which satisfy*

(i) U_i, U_j, $i \neq j$ are pairwise amicable,

(ii) V_i, V_j, $i \neq j$ are pairwise amicable,

(iii) $U_i \pm V_i$, $(+1,-1)$ matrices, $i = 1,\ldots,8$,

(iv) the row(column) sums of U_1 and U_2 are both 1, and the row sum of U_j, $i = 3,\ldots,8$ is zero,

(v) $\sum_{i=1}^{8} U_iU_i^T = 2(2n+1)I - 4J$, $\sum_{i=1}^{8} V_iV_i^T = 2(2n+1)I$.

Then there are 8-Williamson type matrices of order $2n+1$. Furthermore, if the U_i and V_i are symmetric, $i = 1,\ldots,8$, then the 8-Williamson type matrices are symmetric. Hence there is a block type Hadamard matrix of order $8(2n+1)$.

Proof: Let $S_1, \ldots, S_8$ be 8 $(+1,-1)$-matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sums of S_1 and S_2 are both 2 and of $S_i = 0$, $i = 3, \ldots, 8$. Now define

$$X_j = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_j \end{bmatrix}, \quad j = 1,2 \text{ and } X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \quad i = 3, \ldots, 8.$$

First note that since U_i, U_j, $i \neq j$ and V_i, V_j, $i \neq j$ are pairwise amicable,

$$\begin{aligned} S_i S_j^T &= \left(U_i \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_i \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\left(U_j^T \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j^T \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \\ &= U_i U_j^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + V_i V_j^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= S_j S_i^T. \end{aligned}$$

(Note this relationship is valid if and only if conditions (i) and (ii) of the theorem are valid.)

$$\begin{aligned} \sum_{i=1}^{8} S_i S_i^T &= \sum_{i=1}^{8} U_i U_i^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \sum_{i=1}^{8} V_i V_i^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= 2 \begin{bmatrix} 4(2n+1)I - 4J & -4J \\ -4J & 4(2n+1)I - 4J \end{bmatrix} \\ &= 8(2n+1)I_{2n} - 8J_{2n}. \end{aligned}$$

Next we observe

$$X_1 X_2^T = \begin{bmatrix} 1+2n & -3e_{2n} \\ -3e_{2n}^T & J + S_1 S_2^T \end{bmatrix} = X_2 X_1^T,$$

$$X_k X_i^T = \begin{bmatrix} 1-2n & e_{2n} \\ e_{2n}^T & -J + S_k S_i^T \end{bmatrix} = X_i X_k^T, \quad k = 1,2, \text{ and } i = 3, \ldots, 8,$$

and

$$X_i X_j^T = \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_i S_j^T \end{bmatrix} = X_j X_i^T, \quad i \neq j \quad i,j = 3, \ldots, 8.$$

Further

$$\sum_{i=1}^{8} X_i X_i^T = 2\begin{bmatrix} 1+2n & -3e_{2n} \\ -3e_{2n}^T & J+S_1S_1^T \end{bmatrix} + \sum_{i=3}^{8} \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J+S_iS_i^T \end{bmatrix}$$

$$= 2\begin{bmatrix} 8(2n+1) & 0 \\ 0 & 8J+8(2n+1)I-8J \end{bmatrix}.$$

Thus we have shown that $X_1, \ldots, X_8$ are 8-Williamson type matrices of order $2n+1$.

Hence there is a block type Hadamard matrix of order $8(2n+1)$ obtained by replacing the variables of an orthogonal design OD$(8; 1,1,1,1,1,1,1,1)$ by the 8-Williamson type matrices. □

Corollary 38 *Let $q+1$ be the order of amicable Hadamard matrices $I+S$ and P. Suppose there exist 4 Williamson type matrices of order q. Then there exist Williamson type matrices of order $2q+1$. Furthermore there exists an Hadamard matrix of block type of order $8(2q+1)$.*

Proof: Now $(I+S)P^T = P(I+S)^T$ and write e for the $1 \times q$ matrix of ones. From Remark 12 we have matrices A, B of order q satisfying:

$$AB^T = BA^T, \quad B^T = -B, \quad A^T = -A, \quad eA = -e, \quad eB = 0,$$

$$AA^T = (q+1)I - J, \quad BB^T = qI - J.$$

Thus we choose

$$U_1 = U_2 = -A, \quad U_3 = U_4 = B, \quad U_5 = U_6 = U_7 = U_8 = 0,$$

$$\text{and } V_1 = V_2 = 0, \quad V_3 = V_4 = I, \quad V_i + 4 = W_i,$$

where W_i are Williamson type matrices. Hence

$$\sum_{i=1}^{8} U_iU_i^T = 2(2q+1)I - 4J, \quad \sum_{i=1}^{8} V_iV_i^T = 2(2q+1)I.$$

These are then used in the Lemma 37 to obtain the result. □

Using the amicable Hadamard matrices given in [22] and [16, Table 1] we get 8 Williamson type matrices for the following orders for which 4 Williamson matrices are not known:
47, 111, 127, 167, 319, 487, 655, 831, ...

This gives new constructions for Hadamard matrices of orders 8.167 and 8.487.

Corollary 39 *Let q be a prime power and $(q-1)/2$ be the order of four (symmetric) Williamson type matrices. Then there exist (symmetric) 8-Williamson type matrices of order q and an Hadamard matrix of block structure of order $8q$.*

Proof: If $q \equiv 1 \pmod 4$, by Remark 19, Yamada has found circulant matrices A, B of order $(q-1)/2$ where

$$AA^T + BB^T = qI - 2J, \quad eA = e, \quad eB = 0,$$

where A has zero diagonal. Let R be the back-diagonal matrix so $C = BR$ is symmetric; then A and C are amicable. Choose

$$U_1 = U_2 = A, \quad U_3 = U_4 = C, \quad U_5 = U_6 = U_7 = U_8 = 0,$$

$$V_1 = V_2 = I, \quad V_3 = V_4 = 0, \quad V_i + 4 = W_i,$$

$i = 1, 2, 3, 4$, where

$$\sum_{i=1}^{8} U_i U_i^T = 2qI - 4J, \quad \sum_{i=1}^{8} V_i V_i^T = 2qI,$$

and W_i are (symmetric) Williamson type matrices. The result now follows from Lemma 37.

If $q \equiv 3 \pmod 4$, by Remark 18, Szekeres has found circulant matrices A, B of order $\frac{1}{2}(q-1)$ where

$$AA^T + BB^T = qI - 2J, \quad eA = 0, \quad eB = -e,$$

and A has zero diagonal. Let R be the back-diagonal matrix so $C = -BR$ is symmetric; then A and C are amicable and $eC = e$. Choose

$$U_1 = U_2 = C, \quad U_3 = U_4 = A, \quad U_5 = U_6 = U_7 = U_8 = 0,$$

so the U_i are pairwise amicable of order $\frac{1}{2}(q-1)$ and

$$V_1 = V_2 = 0, \quad V_3 = V_4 = I, \quad V_i + 4 = W_i, \quad i = 1, 2, 3, 4,$$

where

$$\sum_{i=1}^{8} U_i U_i^T = 2qI - 4J, \quad \sum_{i=1}^{8} V_i V_i^T = 2qI,$$

and W_i are (symmetric) Williamson type matrices. Since Williamson type matrices are by definition amicable, the V_i are all pairwise amicable (and symmetric) and thus we have the conditions of the lemma satisfied and hence the corollary follows. □

In particular we have 8-Williamson matrices for the following orders for which no Williamson type matrices are known:
59, 67, 103, 107, 151, 163, 179, 227, 251, 283, 347, 463, 467, 523, 563, 571, 587, 631, 643, 823, 859, 919, 947, ...

This gives new Hadamard matrices or new constructions for Hadamard matrices of orders 8.107, 8.163, 8.179, 8.251, 8.283, 8.347, 8.463, 8.523, 8.571, 8.631, 8.643, 8.823, 8.859, 8.919, 8.947, ...

Corollary 40 *Let $q \equiv 1 \pmod 4$ be a prime power or $q+1$ the order of a symmetric conference matrix. Suppose there exist four (symmetric) Williamson type matrices of order q. Then there exist (symmetric) 8-Williamson type matrices of order $2q+1$ and an Hadamard matrix of block structure of order $8(2q+1)$.*

Proof: Form the core Q as in Remark 14(ii). Thus we choose

$$U_1 = I + Q, \quad U_2 = I - Q, \quad U_3 = U_4 = Q, \quad U_5 = U_6 = U_7 = U_8 = 0$$

$$\text{and } V_1 = V_2 = 0, \quad V_3 = V_4 = I, \quad V_{i+4} = W_i,$$

$i = 1, 2, 3, 4$, where W_i are (symmetric) Williamson type matrices. Then

$$\sum_{i=1}^{8} U_i U_i^T = 2(2q+1)I - 4J, \quad \sum_{i=1}^{8} V_i V_i^T = 2(2q+1)I.$$

These U_i and V_i are then used in Lemma 37 to obtain the (symmetric) 8-Williamson type matrices. □

This corollary gives 8 Williamson type matrices for the following new orders: 219, 275, 299, 395, 483, 515, 579, 635, 699, 707, 723, 779, 795, 803, 899, 915, 923, ...

It does not give new Hadamard matrices for these orders.

Corollary 41 *Let $q = 9^t$, $t > 0$. Now there exist four (symmetric) Williamson type matrices of order 9^t, $t > 0$. Hence there exist (symmetric) 8-Williamson type matrices of order $2 \cdot 9^t + 1$, $t > 0$, and an Hadamard matrix of block structure of order $8(2 \cdot 9^t + 1)$.*

This gives symmetric 8-Williamson type matrices for the new order 163, 13123, ...

Also we have the following theorem:

Theorem 43 *Let U_{ij}, V_{ij}, $i,j = 1,\ldots,8$ be $(0,+1,-1)$ matrices of order n which satisfy*

(i) *U_{ki}, U_{kj}, $i \neq j$ are pairwise amicable, $k = 1,\ldots,8$,*

(ii) *V_{ki}, V_{kj}, $i \neq j$ are pairwise amicable, $k = 1,\ldots,8$,*

(iii) *$U_{ki} \pm V_{ki}$, $(+1,-1)$ matrices, $i,k = 1,\ldots,8$,*

(iv) *the row(column) sum of U_{ab} is 1 for $(a,b)\epsilon\{(i,i),(i,i+1),(i+1,i)\}$, $i = 1,3,5,7$, the row(column) sum of U_{aa} is -1 for $(a,a) = 2,4,6,8$ and otherwise, and the row(column) sum of U_{ij}, $i \neq j$ is zero,*

(v) *$\sum_{i=1}^{8} U_{ji}U_{ji}^T = 2(2n+1)I - 4J$, $\sum_{i=1}^{8} V_{ji}V_{ji}^T = 2(2n+1)I$, $j = 1,\ldots,8$,*

(vi) *$\sum_{i=1}^{8} U_{ji}U_{ki}^T = 0$, $\sum_{i=1}^{8} V_{ji}V_{ki}^T = 0$, $j \neq k$, $j,k = 1,\ldots,8$.*

If (iii) to (vi) hold, there is a 64 block M-structure Hadamard matrix of order $8(2n+1)$.

Proof: Let S_{ij} be 64 $(+1,-1)$-matrices of order $2n$ defined by

$$S_{ij} = U_{ij} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_{ij} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row(column) sum of S_{ii}, $S_{i,i+1}$, $S_{i+1,i}$ $i = 1,3,5,7$ is 2, the row(column) sum of S_{ii} is -2 for (i,i), $i = 2,4,6,8$ and otherwise, the row(column) sum of $S_{ij} = 0$, $i \neq j$. Now define

$$X_{11} = \begin{bmatrix} -1 & -e \\ -e^T & S_{11} \end{bmatrix}, \quad X_{12} = \begin{bmatrix} -1 & -e \\ -e^T & S_{12} \end{bmatrix}, \quad X_{13} = \begin{bmatrix} 1 & e \\ e^T & S_{13} \end{bmatrix}, \quad X_{14} = \begin{bmatrix} 1 & e \\ e^T & S_{14} \end{bmatrix},$$

$$X_{15} = \begin{bmatrix} 1 & e \\ e^T & S_{15} \end{bmatrix}, \quad X_{16} = \begin{bmatrix} 1 & e \\ e^T & S_{16} \end{bmatrix}, \quad X_{17} = \begin{bmatrix} -1 & e \\ e^T & S_{17} \end{bmatrix}, \quad X_{18} = \begin{bmatrix} -1 & e \\ e^T & S_{18} \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} -1 & -e \\ -e^T & S_{21} \end{bmatrix}, \quad X_{22} = \begin{bmatrix} 1 & e \\ e^T & S_{22} \end{bmatrix}, \quad X_{23} = \begin{bmatrix} 1 & e \\ e^T & S_{23} \end{bmatrix}, \quad X_{24} = \begin{bmatrix} -1 & -e \\ -e^T & S_{24} \end{bmatrix},$$

$$X_{25} = \begin{bmatrix} 1 & e \\ e^T & S_{25} \end{bmatrix}, \quad X_{26} = \begin{bmatrix} -1 & -e \\ -e^T & S_{26} \end{bmatrix}, \quad X_{27} = \begin{bmatrix} -1 & e \\ e^T & S_{27} \end{bmatrix}, \quad X_{28} = \begin{bmatrix} 1 & -e \\ -e^T & S_{28} \end{bmatrix},$$

$$X_{31} = \begin{bmatrix} 1 & e \\ e^T & S_{31} \end{bmatrix}, \quad X_{32} = \begin{bmatrix} 1 & e \\ e^T & S_{32} \end{bmatrix}, \quad X_{33} = \begin{bmatrix} -1 & -e \\ -e^T & S_{33} \end{bmatrix}, \quad X_{34} = \begin{bmatrix} -1 & -e \\ -e^T & S_{34} \end{bmatrix},$$

$$X_{35} = \begin{bmatrix} 1 & e \\ e^T & S_{35} \end{bmatrix}, \quad X_{36} = \begin{bmatrix} 1 & e \\ e^T & S_{36} \end{bmatrix}, \quad X_{37} = \begin{bmatrix} -1 & e \\ e^T & S_{37} \end{bmatrix}, \quad X_{38} = \begin{bmatrix} -1 & e \\ e^T & S_{38} \end{bmatrix},$$

$$X_{41}=\begin{bmatrix}1 & e\\ e^T & S_{41}\end{bmatrix},\quad X_{42}=\begin{bmatrix}-1 & -e\\ -e^T & S_{42}\end{bmatrix},\quad X_{43}=\begin{bmatrix}-1 & -e\\ -e^T & S_{43}\end{bmatrix},\quad X_{44}=\begin{bmatrix}1 & e\\ e^T & S_{44}\end{bmatrix},$$

$$X_{45}=\begin{bmatrix}1 & e\\ e^T & S_{45}\end{bmatrix},\quad X_{46}=\begin{bmatrix}-1 & -e\\ -e^T & S_{46}\end{bmatrix},\quad X_{47}=\begin{bmatrix}-1 & e\\ e^T & S_{47}\end{bmatrix},\quad X_{48}=\begin{bmatrix}1 & -e\\ -e^T & S_{48}\end{bmatrix},$$

$$X_{51}=\begin{bmatrix}1 & e\\ e^T & S_{51}\end{bmatrix},\quad X_{52}=\begin{bmatrix}1 & e\\ e^T & S_{52}\end{bmatrix},\quad X_{53}=\begin{bmatrix}1 & e\\ e^T & S_{53}\end{bmatrix},\quad X_{54}=\begin{bmatrix}1 & e\\ e^T & S_{54}\end{bmatrix},$$

$$X_{55}=\begin{bmatrix}-1 & -e\\ -e^T & S_{55}\end{bmatrix},\quad X_{56}=\begin{bmatrix}-1 & -e\\ -e^T & S_{56}\end{bmatrix},\quad X_{57}=\begin{bmatrix}-1 & e\\ e^T & S_{57}\end{bmatrix},\quad X_{58}=\begin{bmatrix}-1 & e\\ e^T & S_{58}\end{bmatrix},$$

$$X_{61}=\begin{bmatrix}1 & e\\ e^T & S_{61}\end{bmatrix},\quad X_{62}=\begin{bmatrix}-1 & -e\\ -e^T & S_{62}\end{bmatrix},\quad X_{63}=\begin{bmatrix}1 & e\\ e^T & S_{63}\end{bmatrix},\quad X_{64}=\begin{bmatrix}-1 & -e\\ -e^T & S_{64}\end{bmatrix},$$

$$X_{65}=\begin{bmatrix}-1 & -e\\ -e^T & S_{65}\end{bmatrix},\quad X_{66}=\begin{bmatrix}1 & e\\ e^T & S_{66}\end{bmatrix},\quad X_{67}=\begin{bmatrix}-1 & e\\ e^T & S_{67}\end{bmatrix},\quad X_{68}=\begin{bmatrix}1 & -e\\ -e^T & S_{68}\end{bmatrix},$$

$$X_{71}=\begin{bmatrix}1 & -e\\ -e^T & S_{71}\end{bmatrix},\quad X_{72}=\begin{bmatrix}1 & -e\\ -e^T & S_{72}\end{bmatrix},\quad X_{73}=\begin{bmatrix}1 & -e\\ -e^T & S_{73}\end{bmatrix},\quad X_{74}=\begin{bmatrix}-1 & -e\\ -e^T & S_{74}\end{bmatrix},$$

$$X_{75}=\begin{bmatrix}1 & -e\\ -e^T & S_{75}\end{bmatrix},\quad X_{76}=\begin{bmatrix}1 & -e\\ -e^T & S_{76}\end{bmatrix},\quad X_{77}=\begin{bmatrix}1 & e\\ e^T & S_{77}\end{bmatrix},\quad X_{78}=\begin{bmatrix}1 & e\\ e^T & S_{78}\end{bmatrix},$$

$$X_{81}=\begin{bmatrix}1 & -e\\ -e^T & S_{81}\end{bmatrix},\quad X_{82}=\begin{bmatrix}-1 & e\\ e^T & S_{82}\end{bmatrix},\quad X_{83}=\begin{bmatrix}1 & -e\\ -e^T & S_{83}\end{bmatrix},\quad X_{84}=\begin{bmatrix}-1 & e\\ e^T & S_{84}\end{bmatrix},$$

$$X_{85}=\begin{bmatrix}1 & -e\\ -e^T & S_{85}\end{bmatrix},\quad X_{86}=\begin{bmatrix}-1 & e\\ e^T & S_{86}\end{bmatrix},\quad X_{87}=\begin{bmatrix}1 & e\\ e^T & S_{87}\end{bmatrix},\quad X_{88}=\begin{bmatrix}-1 & -e\\ -e^T & S_{88}\end{bmatrix},$$

Then provided conditions (i) to (v) hold and $S_{7i}^T = S_{7i}$, $i = 1,\ldots,8$ are symmetric, X_{7i}, $i = 1,\ldots,8$ are symmetric 8-Williamson type matrices. Otherwise X_{7i}, $i = 1,\ldots,8$ are 8-Williamson type matrices. This can be verified by straightforward checking. Hence there is an Hadamard matrix of block structure of order $8(2n+1)$.

If conditions (iii) to (vi) hold then straightforward verification shows the 64 block M-structure X_{ij} is an Hadamard matrix of order $8(2n+1)$. □

Corollary 43 *Let q be an odd prime power and suppose there exist Williamson-type matrices of order $\frac{1}{2}(q-1)$. Then there exists an M-structure Hadamard matrix of order $8q$.*

Proof: Let $U = (U_{ij})$ and $V = (V_{ij})$ be defined by the following M-structures and write O for the matrix of zeros of order $\frac{1}{2}(q-1)$. Let

$$U = \begin{bmatrix} C & C & A & A & 0 & 0 & 0 & 0 \\ C & -C & A & -A & 0 & 0 & 0 & 0 \\ A & A & C & C & 0 & 0 & 0 & 0 \\ A & -A & C & -C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & C & A & A \\ 0 & 0 & 0 & 0 & C & -C & A & -A \\ 0 & 0 & 0 & 0 & A & A & C & C \\ 0 & 0 & 0 & 0 & A & -A & C & -C \end{bmatrix} \quad \text{and}$$

$$V = \begin{bmatrix} 0 & 0 & I & I & W_1 & W_2 & W_3 & W_4 \\ 0 & 0 & I & -I & -W_2 & W_1 & -W_4 & W_3 \\ -I & -I & 0 & 0 & -W_3 & W_4 & W_1 & -W_2 \\ -I & I & 0 & 0 & -W_4 & -W_3 & W_2 & W_1 \\ -W_1^T & W_2^T & W_3^T & W_4^T & 0 & 0 & -I & -I \\ -W_2^T & -W_1^T & -W_4^T & W_3^T & 0 & 0 & -I & I \\ -W_3^T & W_4^T & -W_1^T & -W_2^T & I & I & 0 & 0 \\ -W_4^T & -W_3^T & W_2^T & -W_1^T & I & -I & 0 & 0 \end{bmatrix},$$

where A, C are defined in the proof of Corollary 39 and W_1, W_2, W_3, and W_4 are Williamson-type matrices. Then by Theorem 41 we have the result. □

Remark 44 This corollary gives new Hadamard matrices of order $8q$ for $q = 179, 1087, 1283, 1327, 1619, 1907, 2099, 2459, 2579, 2647, \ldots$.

Corollary 45 *Let* $q = 2m+1 \equiv 9 \pmod{16}$ *be a prime power. Suppose there are Williamson-type matrices of order* q. *Then there is a M-structure Hadamard matrix of order* $8(2q+1)$.

Proof: J. Wallis and A.L. Whiteman [22, Theorem 4.17, pp. 334–336] showed there are four circulant or type 1 matrices with entries ± 1, and row and column sum ± 1 at will.

We construct, using cyclotomy, the type 1 $4 - \{2m+1; m; 2(m-1)\}$ supplementary difference sets X_1, X_2, X_3 and X_4, where $y \in X_i \Rightarrow -y \notin X_i$, $i = 1, 2, 3, 4$.

Let A be the back-circulant or type 2 matrix given by

$$A = (J - 2X_1)R \text{ so } A \text{ has row sum } +1.$$

Let B, C and D be the circulant or type 1 matrices given by

$B \; = J - 2X_2$ so B has row sum $+1$,

$C \; = J - 2X_3 - I$ so C has row sum 0 and zero diagonal, and

$D \; = J - 2X_4 - I$ so D has row sum 0 and zero diagonal.

Now we modify the Wallis-Whiteman core, noting that

$$AA^T + BB^T + CC^T + DD^T = 2(q+1)I - 4J.$$

We use V as in Corollary 43 and the following matrix for U to obtain the result

$$U = \begin{bmatrix} A & B & C & D & 0 & 0 & 0 & 0 \\ B & -A & -D^T & C^T & 0 & 0 & 0 & 0 \\ -C & -D^T & A & B^T & 0 & 0 & 0 & 0 \\ D & -C^T & B^T & -A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & B & C & D \\ 0 & 0 & 0 & 0 & B & -A & -D^T & C^T \\ 0 & 0 & 0 & 0 & -C & -D^T & A & B^T \\ 0 & 0 & 0 & 0 & D & -C^T & B^T & -A \end{bmatrix}.$$

□

The analogous Yamada-J. Wallis-Whiteman structure to Theorem 42 is:

$$\begin{matrix} -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -e & -e & e & e & e & e & e & e \\ -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -e & e & e & -e & e & -e & e & -e \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & e & e & -e & -e & e & e & e & e \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & e & -e & -e & e & e & -e & e & -e \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & e & e & e & e & -e & -e & e & e \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & e & -e & e & -e & -e & e & e & -e \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -e & -e & -e & -e & -e & -e & e & e \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -e & e & -e & -e & e & e & -e & e \\ -e^T & -e^T & e^T & e^T & e^T & e^T & e^T & e^T & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\ -e^T & e^T & e^T & -e^T & e^T & -e^T & e^T & -e^T & S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\ e^T & e^T & -e^T & -e^T & e^T & e^T & e^T & e^T & S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} \\ e^T & -e^T & -e^T & e^T & e^T & -e^T & e^T & -e^T & S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\ e^T & e^T & e^T & e^T & -e^T & -e^T & e^T & e^T & S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} \\ e^T & -e^T & e^T & -e^T & -e^T & e^T & e^T & -e^T & S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} \\ -e^T & -e^T & -e^T & -e^T & -e^T & -e^T & e^T & e^T & S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} \\ -e^T & e^T & -e^T & e^T & -e^T & e^T & e^T & -e^T & S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} \end{matrix}$$

We can see Yamada's matrix with trimming [46] or the J. Wallis-Whiteman [30] matrix with a border embodied in the construction.

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* Department of Computer Science
University College
University of New South Wales
ADFA
Canberra, 2600
AUSTRALIA

† Department of Mathematics
Tokyo Woman's Christian University
Zempukuji
Suginami-Ku
Tokyo, 167
JAPAN