Properties of 1-Tough Graphs[1]

Robin W. Dawes *and* Marion G. Rodrigues

Department of Computing and Information Science
Queen's University
Kingston, Ontario, Canada

Abstract. A graph G is defined by Chvátal [4] to be *n-tough* if, given any set of vertices $S, S \subseteq G, c(G-S) \leq \frac{|S|}{n}$. We present several results relating to the recognition and construction of 1-tough graphs, including the demonstration that all n-regular, n-connected graphs are 1-tough. We introduce the notion of minimal 1-tough graphs, and tough graph augmentation, and present results relating to these topics.

1. Introduction

All graphs are assumed to be undirected, finite and connected, with no loops or multiple edges.

Let G be a graph on v vertices and n be a real number. Let $|S|$ denote the cardinality of S, and $c(G)$ denote the number of components of G. Then G is said to be *n-tough*, if, given any set of vertices S where $S \subseteq G, c(G-S) \leq \frac{|S|}{n}$ [4].

Graph toughness was originally introduced by Chvátal [4] as an invariant property important for hamiltonicity. Chvátal indicated that every hamiltonian graph is 1-tough, and made several conjectures relating toughness and hamiltonicity. In keeping with the emphasis set by Chvátal, much of the subsequent research involving the property of toughness has continued to elucidate properties of graphs contingent upon toughness. Enomoto, et al [6], have proven Chvátal's conjecture that every k-tough graph has a k-factor, where k is a positive integer, with the restrictions that $k|G|$ must be even and $|G| \geq k+1$. Nishizeki [8] has shown that 1-toughness is not a sufficient condition for a maximal planar graph to be hamiltonian. Other results relating 1-toughness and hamiltonicity, and 1-toughness and k-factors may be found in Ainouche and Christofides [1], and Katerinis [7], respectively. Relationships have also been established between degree of toughness and degree of matching extendability [9], and 1-toughness and Delaunay triangulations [5].

Toughness is a measure of how well the vertices of a graph are bound together. Our interest in toughness stems from its natural application to problems of network design and analysis. For example, with regard to flow reliability, it is quite clear that a network with one or more bottlenecks is likely to have a graph topology with a low degree of toughness. On the other hand, a network with no bottlenecks is

[1] This paper was presented at the Third Carbondale Combinatorics Conference.

likely to have a topology with a high degree of toughness (for example, a complete graph is infinitely tough).

Graph toughness also appears to be a desirable property for fault-tolerant networks. The traditional measure of fault-tolerance in networks has been graph connectivity. Connectivity effectively measures the smallest number of node failures required to disconnect a network. One problem with this measure is that it provides no information as to the size of the resulting network components. (Note that a distributed processing network with one or more failed nodes where a sufficiently large connected component remains active may still be able to function adequately.) Skillicorn and Kocay [10] have proposed a more informative measure, the connectivity function, $c(G,t)$, which denotes the minimum number of vertices that must be deleted from G in order to produce a graph in which the smallest component has size t. However, because this measure does not indicate the number of components, it does not necessarily provide any information regarding the size of the largest component. As an alternative measure, 1-toughness guarantees that if k nodes fail, at least one component will have $\lceil \frac{n-k}{k} \rceil$ vertices. Comparisons of several such "vulnerability" measures, including connectivity and toughness, may be found in [2].

To the best of our knowledge, very little research has sought to uncover exactly what structural characteristics render a graph 1-tough. The aim of this research is to begin to enumerate various classes of 1-tough graphs. In this paper, we restrict our attention to the recognition and construction of 1-tough graphs, as defined by Chvatal, particularly non-hamiltonian 1-tough graphs.

2. Recognition and Construction of 1-Tough Graphs

2.1 Recognition of 1-Toughness

Given a graph G, how difficult is it to determine whether G is 1-tough? Clearly, 1-tough recognition is in co-NP. Very recently, Bauer, Hakimi, and Schmeichel [3] have shown that recognizing 1-toughness is an NP-hard problem. This suggests that this problem is probably not in NP, since the equality of NP and co-NP would be an immediate consequence.

Under these circumstances, structural characterizations of 1-tough graphs become essential for recognition. Hamiltonicity is sufficient, but not necessary for 1-toughness. The following result utilizes more easily-recognized structural properties:

Theorem 1. *Let G be a k-regular, k-connected graph on n vertices (where $k \geq 2$ and $n \geq 3$); then G is 1-tough.*

Proof: Let S be any set of vertices of G, $|S| = t$. Let $C_1, C_2, \cdots, C_r$ be the components of $G - S$ (see Figure 1). Suppose $r > t$ (i.e. G is not 1-tough).

Let x_i be a vertex in C_i, $1 \leq i \leq r$. By a well-known theorem of Whitney [11], each x_i must be connected to each x_j, $i \neq j$, by k disjoint paths. Since x_i and x_j $(i \neq j)$ are in different components of $G - S$, all of these paths must pass through S. Thus there must exist at least k edges joining each C_i to vertices in S. Therefore there are at least $k \cdot r$ edges incident to vertices in S. However, the sum of all degrees in S is exactly $k \cdot t$, which is less than $k \cdot r$.

Therefore, all k-regular, k-connected graphs are 1-tough. ∎

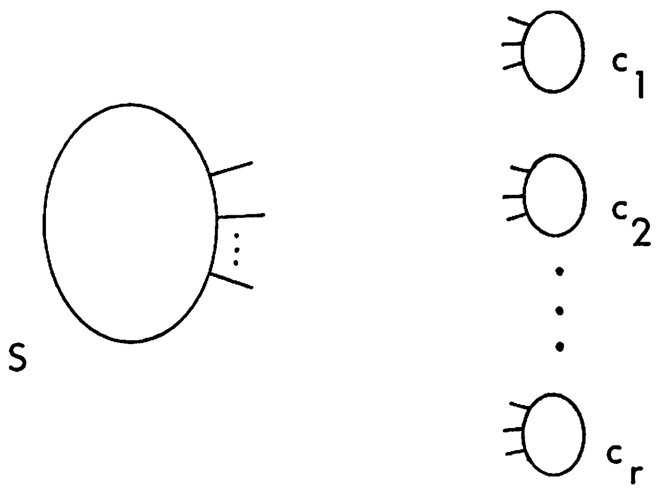

Figure 1.

2.2 Construction of 1-Tough Graphs

Consider the following problem: Given n, how can we construct a 1-tough graph on n vertices?

Constructing hamiltonian 1-tough graphs is trivial. Any cycle on n vertices, C_n, is 1-tough; one can begin with C_n, then add edges to achieve any desired property (e.g. high connectivity, small diameter, etc.). However, as hamiltonicity is not a necessary condition for 1-toughness, we turn our attention to the construction of non-hamiltonian tough graphs.

Nishizeki [8] demonstrates a construction of a highly structured family of 1-tough, non-hamiltonian maximal planar graphs, essentially by embedding graphs in others. We now apply a similar technique to construct a diverse family of non-hamiltonian 1-tough graphs.

Theorem 2. *Let G be a 1-tough graph, with a vertex x of degree 2. Replace x with the graph H shown in Figure 2. The resulting graph G' is 1-tough.*

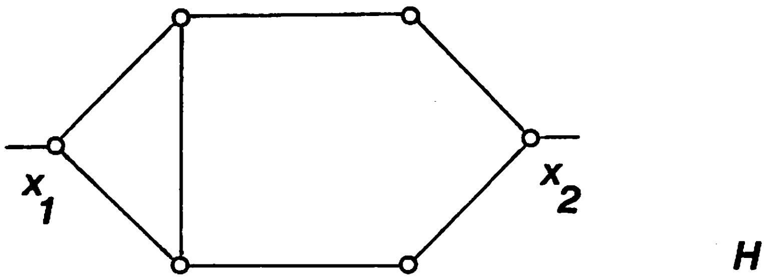

Figure 2.

Proof: Suppose G' is not 1-tough. Let V' be the vertex set of G', and let $S \subseteq V'$ be such that $c(G' - S) > |S|$.

Let $T = S \cap H$. Clearly $|T| \neq 0$.

Let $S' = S - T$.

Clearly, $c(G' - S') \leq |S'|$ (else G is not 1-tough).

Consider the component M of $G' - S'$ containing H. Simple examination reveals that $\forall T \in H, c(M - T) \leq |T| + 1$.

Thus $c(G' - S) = c(G' - S') + c(M - T) - 1 \leq |S'| + |T| = |S|$.

Furthermore, we observe that the graph G' constructed in the theorem just stated is non-hamiltonian.

In fact, the graph H used in the previous theorem may be replaced by any cycle of length ≥ 6 with one added edge, provided that at least one of x_1 and x_2 is adjacent to two vertices of degree 2 in the augmented cycle, and that the added edge is incident to neither x_1 nor x_2. The graph H shown is the smallest such, but whatever graph is used, the resulting graph will be 1-tough and non-hamiltonian.

This technique permits us to observe that the cardinality of the set of non-hamiltonian 1-tough graphs is at least as great as that of the hamiltonian 1-tough graphs.

Let G be any hamiltonian graph. If G contains a vertex of degree 2, the construction step outlined above can be applied, giving a non-hamiltonian 1-tough graph containing vertices of degree 2. If G contains no vertex of degree 2, let e be any edge on a hamiltonian cycle of G. Replacing e by a path of length 3 (i.e. subdividing e) gives a hamiltonian graph with a vertex of degree 2, to which the construction step may be applied.

Thus each hamiltonian graph generates a family of non-hamiltonian 1-tough graphs. ∎

2.2.2 Construction of Non-Hamiltonian Minimal 1-Tough Graphs

Let $G(V, E)$ be a graph. Then G is a *minimal* 1-tough graph if G is 1-tough, but $\forall e \in E, G - e$ is not 1-tough.

Clearly, every 1-tough graph can be constructed from some minimal 1-tough graph by adding 0 or more edges. For example, all hamiltonian minimal 1-tough graphs are cycles. As with non-hamiltonian 1-tough graphs in general, the structural requirements for non-hamiltonian minimal 1-tough graphs are unknown. However, we can use the following construction method to generate one relatively dense class of these graphs:

Theorem 3. *Given a complete graph K_t, construct a tree by identifying one end-vertex of t paths, each path of length ≥ 3. Let v be the resulting vertex of degree t. Attach one leaf of the tree to each vertex of K_t, as depicted in Figure 3. The resulting graph is a minimally 1-tough graph. If $t = 2$, then G is a cycle and thus minimally 1-tough; if $t \geq 3$, G is a non-hamiltonian minimal 1-tough graph.*

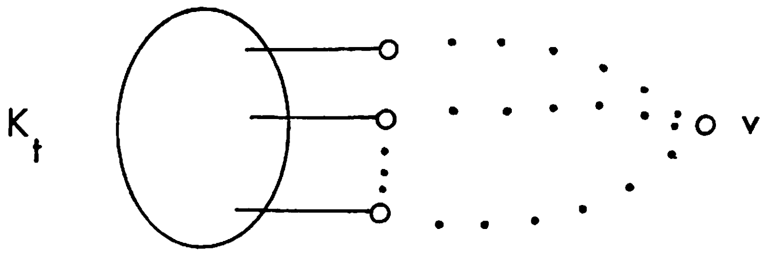

Figure 3.

Proof: For $t = 2$, the proof is immediate. Assume $t \geq 3$, and let G be constructed as above. Clearly G is non-hamiltonian. To see that G is 1-tough, consider the following. Suppose $S \subseteq V$ is such that $c(G - S) > |S|$, and let S be the smallest such set of vertices. Clearly $v \in S$ (else $c(G - S) \leq 2$, and $|S| \geq 3$). No neighbour of v can be in S, so $S = v + \{r \text{ vertices of } K_t\}$ for $0 \leq r \leq t$. But deleting v and any r vertices of K_t produces precisely $r + 1$ components. Thus G is 1-tough.

We now demonstrate that G is minimally 1-tough. Let $e = (x, y)$ be any edge of G. If e is incident to a vertex of degree 2, then clearly $G - e$ is not 1-tough ($G - e$ has a vertex of degree 1). If e is not incident to a vertex of degree 2, then e is an edge of K_t. Consider $G' = G - e$.

Let $S = K_t - x - y + v$. $|S| = t - 1$, but $c(G' - S) = t$. G' is not 1-tough, and it follows that G is minimally 1-tough. ∎

The density of these graphs approaches 1/4 as t approaches infinity. We conjecture that 1/4 is an upper bound on the density of minimally 1-tough graphs.

2.2.3 1-Tough Graph Augmentation

We now address the following problem:

Given a graph G, find the minimum set of edges S such that $G + S$ is 1-tough. Clearly, the complexity of this problem is at least equivalent to that of 1-toughness recognition. However, if the *minimum* requirement is relaxed, the problem becomes more tractable. For example, consider the following rather simple-minded algorithm:

Given a graph G on n vertices:

1. Find any path P in G.
2. Add edges as required to extend P into a hamiltonian cycle.

This method will require the addition of no more than $n - 1$ edges. Since hamiltonicity is not necessary for 1-toughness, we cannot expect this algorithm to find optimal solutions, even if we could identify a maximum length path in G.

We observe that any 1-connected graph on $n \geq 3$ vertices can be augmented to 1-toughness with no more than $n - 2$ edges, and that this bound is achieved for the graph $K_{1,n-1}$.

Similarly, $K_{t,n-t}$, $n > 2t$, can be augmented to 1-toughness with an edge-set of size $n - 2t$. We conjecture that this is the upper bound for augmenting sets for t-connected graphs.

3. Concluding Remarks

We have introduced several new directions for research into the property of graph toughness, particularly 1-toughness. We have shown that all k-regular, k-connected graphs are 1-tough. This is an interesting result in view of the fact that many such graphs are non-hamiltonian. We have also shown that it is possible to construct infinitely many non-hamiltonian tough graphs and infinitely many minimal tough graphs, and have proposed a tractable though simplistic algorithm for the tough graph augmentation problem.

Certainly, much work remains to be done on the subject of the structural characteristics of tough graphs. We are presently investigating critically-1-tough graphs (1-tough graphs on n vertices which have no 1-tough subgraphs on $n-1$ vertices), and extensions to the results presented here.

Acknowledgement

This work was supported by an operating grant from the Natural Sciences and Engineering Research Council of Canada. We thank the referee for suggesting a simplification in the proof of Theorem 1.

References

1. Ainouche, A., and Christofides, N., *Conditions for the existence of hamiltonian circuits in graphs based on vertex degrees*, Journal of the London Mathematical Society 32 (1985), 385–391.
2. Barefoot, C.A., Entringer, R., and Swart, H., *Vulnerability in graphs – a comparative survey*, Journal of Combinatorial Mathematics and Combinatorial Computing 1 (1987), 13–22.
3. Bauer, D., Hakimi, S., Schmeichel, E. communicated by M. Dillencourt, (1989).
4. Chvátal, V., *Tough graphs and hamiltonian circuits*, Discrete Mathematics 5 (1973), 215–228.
5. Dillencourt, M.B., *Toughness and Delaunay triangulations*, Discrete and Computational Geometry (1989) (to appear).
6. Enomoto, H., Jackson, B., Katerinis, P., and Saito, A., *Toughness and the existence of k-factors*, Journal of Graph Theory 9 (1985), 87–95.
7. Katerinis, P., *Two sufficient conditions for a 2-factor in a bipartite graph*, Journal of Graph Theory 11 (1987), 1–6.
8. Nishizeki, T., *A 1-tough nonhamiltonian maximal planar graph*, Discrete Mathematics 30 (1980), 305–307.
9. Plummer, M.D., *Toughness and matching extension in graphs*, Discrete Mathematics 72 (1988), 311–320.
10. Skillicorn, D.B., and Kocay, B., *A global measure of network connectivity*, Journal of Parallel and Distributed Computing 7 (1989), 165–177.
11. Whitney, H., *Congruent graphs and the connectivity of graphs*, American Journal of Mathematics 54 (1932), 330–332.