Chromatic Polynomials of $C_4 \times P_n$ and $C_5 \times P_n$

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Abstract

The chromatic polynomial of a graph G, $P(G;\lambda)$, is the polynomial in λ which counts the number of distinct proper vertex λ-colorings of G, given λ colors. We compute $P(C_4 \times P_n;\lambda)$ and $P(C_5 \times P_n;\lambda)$ in matrix form and will find the generating function for each of these sequences.

1 Chromatic Polynomial of $C_4 \times P_n$

For more information on chromatic polynomials see [1] and [2]. Let Θ_n denote the graph $C_4 \times P_n$. For $n \geq 1$, by applying deletion-contraction method to Θ_n – more precisely, deleting and contracting the edges linking Θ_{n-1} to last copy of C_4 – one may write chromatic polynomial of this graph in terms of $P(\Theta_{n-1};\lambda)$ and $P(\Theta'_{n-1};\lambda)$ where Θ'_n is obtained from Θ_n by linking two non-adjacent vertices in the last copy of C_4 (see Figure 1). Similarly, $P(\Theta'_n;\lambda)$ can be written in terms of chromatic polynomials of Θ_{n-1} and Θ'_{n-1}. We have

$$P(\Theta_n;\lambda) = (\lambda^4 - 8\lambda^3 + 28\lambda^2 - 47\lambda + 31)P(\Theta_{n-1};\lambda) - (4\lambda - 10)P(\Theta'_{n-1};\lambda),$$

$$P(\Theta'_n;\lambda) = (\lambda-2)(\lambda^3 - 7\lambda^2 + 19\lambda - 19)P(\Theta_{n-1};\lambda) + (\lambda-5)(\lambda-3)P(\Theta'_{n-1};\lambda).$$

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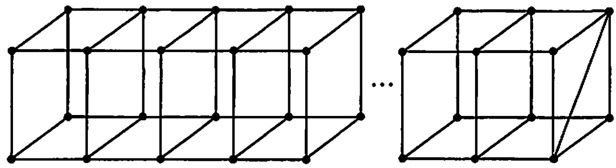

Figure 1: Θ'_n

Let $A = (a_{ij})$ be the matrix of the coefficients of the above recursive relations. Clearly, we have

$$\begin{bmatrix} P(\Theta_n;\lambda) \\ P(\Theta'_n;\lambda) \end{bmatrix} = A^n \begin{bmatrix} P(\Theta_0;\lambda) \\ P(\Theta'_0;\lambda) \end{bmatrix} \tag{1}$$

By substituting $P(C_4;\lambda) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)$ and $P(K_4-e;\lambda) = \lambda(\lambda-1)(\lambda-2)^2$ for $P(\Theta_0;\lambda)$ and $P(\Theta'_0;\lambda)$, respectively, we will have $P(\Theta_n;\lambda)$ and $P(\Theta'_n;\lambda)$, for all n.

Now let $\mathfrak{f}_1(x)$ and $\mathfrak{f}_2(x)$ be the generating functions for chromatic polynomials of Θ_n and Θ'_n, respectively. Using (1), we have

$$\begin{bmatrix} \mathfrak{f}_1(x) \\ \mathfrak{f}_2(x) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} P(\Theta_n;\lambda) \\ P(\Theta'_n;\lambda) \end{bmatrix} x^n$$

$$= \sum_{n=0}^{\infty} (xA)^n \begin{bmatrix} P(\Theta_0;\lambda) \\ P(\Theta'_0;\lambda) \end{bmatrix} = (I-xA)^{-1} \begin{bmatrix} P(\Theta_0;\lambda) \\ P(\Theta'_0;\lambda) \end{bmatrix}$$

and after inverting $(I-xA)^{-1}$, we derive

$$\mathfrak{f}_1(x) = \frac{P(\Theta_0;\lambda) + \left(a_{12}P(\Theta'_0;\lambda) - a_{22}P(\Theta_0;\lambda)\right)x}{1-(a_{11}+a_{22})x + (a_{11}a_{22}-a_{12}a_{21})x^2},$$

$$\mathfrak{f}_2(x) = \frac{P(\Theta'_0;\lambda) + \left(a_{21}P(\Theta_0;\lambda) - a_{11}P(\Theta'_0;\lambda)\right)x}{1-(a_{11}+a_{22})x + (a_{11}a_{22}-a_{12}a_{21})x^2}.$$

2 Chromatic Polynomial of $C_5 \times P_n$

Now let Φ_n denote the graph $C_5 \times P_n$. Similar to the previous example, chromatic polynomial of Φ_n, for $n \geq 1$, can be computed by deleting and contracting the edges linking Φ_{n-1} to last copy of C_5. With some effort, one may check that chromatic polynomial of this graph can be written in terms of $P(\Phi_{n-1};\lambda)$, $P(\Phi'_{n-1};\lambda)$, and $P(\Phi''_{n-1};\lambda)$ where Φ'_n and Φ''_n are graphs obtained from Φ_n by linking a vertex in the last copy of C_5 to, respectively, one and two non-adjacent vertices in the same copy of C_5 (see Figure 2). Similarly, $P(\Phi'_n;\lambda)$ can be written in terms of chromatic polynomials of Φ_{n-1}, Φ'_{n-1}, and Φ''_{n-1}. It follows that we have

$$P(\Phi_n;\lambda) = (\lambda-2)^3(\lambda^2-4\lambda+9)\,P(\Phi_{n-1};\lambda)$$

$$-(5\lambda^2-28\lambda+38)\,P(\Phi'_{n-1};\lambda)+(\lambda-6)\,P(\Phi''_{n-1};\lambda),$$

$$P(\Phi'_n;\lambda) = (\lambda^5-11\lambda^4+53\lambda^3-135\lambda^2+177\lambda-95)\,P(\Phi_{n-1};\lambda)$$

$$+(\lambda-2)(\lambda-3)(\lambda-9)\,P(\Phi'_{n-1};\lambda)+(4\lambda-14)\,P(\Phi''_{n-1};\lambda).$$

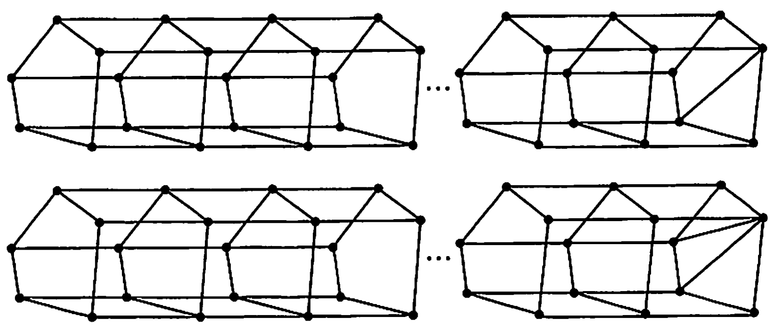

Figure 2: Φ'_n and Φ''_n

On the other hand, one may easily check that $P(\Phi_0;\lambda)$ is equal to $P(\Phi'_0;\lambda)+P(K_3+e;\lambda)$ and $P(\Phi''_0;\lambda)-P(K_3+e;\lambda)$ by respectively applying addition-contraction and deletion-contraction. Hence, $P(\Phi''_0;\lambda) = 2P(\Phi'_0;\lambda)-P(\Phi_0;\lambda)$. In a similar fashion, we have $P(\Phi''_n;\lambda) = 2P(\Phi'_n;\lambda)-P(\Phi_n;\lambda)$ using induction. By substituting this back in above recursions, we derive

$$P(\Phi_n;\lambda) = (\lambda^5-10\lambda^4+45\lambda^3-110\lambda^2+139\lambda-66)P(\Phi_{n-1};\lambda)$$

$$-(5\lambda^2-30\lambda+50)P(\Phi'_{n-1};\lambda),$$

$$P(\Phi'_n;\lambda) = (\lambda^5-11\lambda^4+53\lambda^3-135\lambda^2+173\lambda-81)\,P(\Phi_{n-1};\lambda)$$

$$+(\lambda^3-14\lambda^2+59\lambda-82)\,P(\Phi'_{n-1};\lambda).$$

By letting $B=(b_{ij})$ to be the matrix of the coefficients of the above recursive relations, we write

$$\begin{bmatrix} P(\Phi_n;\lambda) \\ P(\Phi'_n;\lambda) \end{bmatrix} = B^n \begin{bmatrix} P(\Phi_0;\lambda) \\ P(\Phi'_0;\lambda) \end{bmatrix}$$

where $P(\Phi_0; \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)$ and $P(\Phi_0'; \lambda) = \lambda(\lambda-1)$ $(\lambda-2)(\lambda^2-3\lambda+3)$. This allows us to compute chromatic polynomial of Φ_n and Φ_n' and consecutively, that of Φ_n''.

Finally, let $\mathfrak{g}_1(x)$, $\mathfrak{g}_2(x)$, and $\mathfrak{g}_3(x)$ denote the generating functions for chromatic polynomials of Φ_n, Φ_n', and Φ_n'', respectively. Clearly, $\mathfrak{g}_3(x) = 2\mathfrak{g}_2(x) - \mathfrak{g}_1(x)$ while

$$\mathfrak{g}_1(x) = \frac{P(\Phi_0;\lambda) + \left(b_{12}P(\Phi_0';\lambda) - b_{22}P(\Phi_0;\lambda)\right)x}{1-(b_{11}+b_{22})\,x + (b_{11}b_{22}-b_{12}b_{21})\,x^2},$$

$$\mathfrak{g}_2(x) = \frac{P(\Phi_0';\lambda) + \left(b_{21}P(\Phi_0;\lambda) - b_{11}P(\Phi_0';\lambda)\right)x}{1-(b_{11}+b_{22})\,x + (b_{11}b_{22}-b_{12}b_{21})\,x^2}.$$

References

[1] N. L. Biggs, *Algebraic Graph Theory* (Cambridge University Press, Cambridge, Second Edition 1993).

[2] F.M. Dong, K.M. Koh, K.L. Teo, *Chromatic Polynomials and Chromaticity of Graphs* (World Scientific, Singapore, 2005).