The effect of vertex and edge deletion on the number of sizes of maximal independent sets

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Abstract. A graph G is said to be in the collection M_t if there are precisely t different sizes of maximal independent sets of vertices in G. For $G \in M_t$ and $v \in G$, we determine the extreme values that x can assume where $G \setminus \{v\}$ belongs to M_x. For both the minimum and maximum values graphs are given that achieve them, showing that the bounds are sharp. The effect of deleting an edge from G on the number of sizes of maximal independent sets is also considered.

1 Introduction

A graph G is said to be in the collection M_t (introduced by A. Finbow, B. Hartnell and C.A. Whitehead [2]) if there are precisely t different sizes of maximal independent sets of vertices in G. Further properties of graphs in this class are given in [1]. Thus the M_1 graphs are the well-covered ones (introduced by M. Plummer [3]) where all the maximal independent sets are of one size. A very useful construction in building well-covered graphs is based on extendable vertices. An *extendable* vertex of a well-covered graph is one whose removal results in another well-covered graph. For instance, the 5-cycle is well-covered and removing a vertex results in the path on 4 vertices which is also well-covered. On the other hand, the 4-cycle is a well-covered graph but removing a vertex results in a graph which is not well-covered. If one has two well-covered graphs, say G_1 and G_2, each with an extendable vertex, say v_1 and v_2, then the resulting graph formed when v_1 and v_2 are joined by an edge is also well-covered. Partly motivated by the importance of extendable vertices in the well-covered setting, we examine $G \setminus \{v\}$ where $G \in M_t$.

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2 Some Bounds

First we consider for an arbitrary graph G and any vertex v of G how large x can be when $G \in M_t$ and $G\backslash\{v\} \in M_x$.

Lemma 1 *For any given graph G in M_t there is no vertex in $V(G)$ such that $G\backslash\{v\} \in M_x$ where $x \geq (2t+1)$.*

Proof: Let $v \in V(G)$. Take the graph $G\backslash\{v\}$, and let I be a maximal independent set in $G\backslash\{v\}$. Then I can be of two types:

1)$I \cap N(v) = \emptyset$, or

2)$I \cap N(v) \neq \emptyset$

We can have at most t sizes of maximal independent sets of type 2, since in this case, I is also a maximal independent set in G, and $G \in M_t$.

If I is of type 1, then $I \cup \{v\}$ is a maximal independent set in G, and since $G \in M_t$, we can have at most only t different sizes for such sets.

In the situation that there is no overlap in the sets of Type 1 with those of Type 2, we have the maximum of $2t$ different sizes. □

We observe that the bound given in Lemma 1 is sharp. In particular the graph in Figure 1 belongs to M_t but $G\backslash\{v\} \in M_{2t}$.

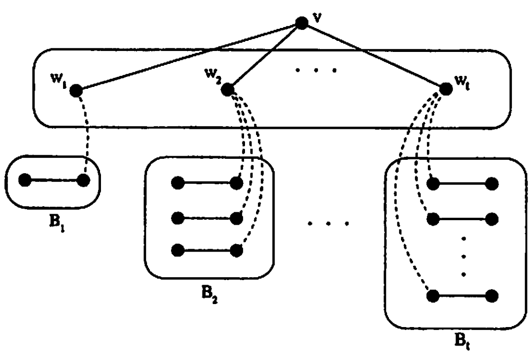

Figure 1: $G \in M_t$ with $G\backslash\{v\} \in M_{2t}$

There are $2i-1$ edges in B_i $(i \leq i \leq t)$. Each w_i is joined to all vertices of B_i, for $i \neq j$, and to exactly one endpoint of each edge of B_i (the *missing* edges are indicated in the diagram). The vertices $w_1, w_2, ..., w_t$

are completely joined and each vertex of B_i is joined to all vertices of B_j, for $i \neq j$.

Note that G has maximal independent sets of sizes $2, 4, 6, ..., 2t$ whereas $G\backslash\{v\}$ has $1, 2, 3, ..., 2t-1, 2t$. Observe that in the case $t = 1$, the graph is simply a path on 4 vertices.

Furthermore it is straight forward to modify the construction so that $G\backslash\{v\} \in M_r$ for any r, where $t+1 \leq r \leq 2t$. Let b_i represent the number of edges in B_i and observe that G has maximal independent sets of sizes $b_1+1, b_2+1, ..., b_t+1$ whereas $G\backslash\{v\}$ has $b_1+1, b_2+1, ..., b_t+1$ and $b_1, b_2, ..., b_t$. So if the b_i values are consecutive we have $G\backslash\{v\} \in M_{t+1}$ and if there are s non-consecutive pairs, $G\backslash\{v\} \in M_{t+1+s}$. Thus Figure 1 gives the upper bound situation when $s = t-1$.

Next we observe that it is possible for x to be small as 1. That is, it is possible for G to belong to M_t but $G\backslash\{v\} \in M_1$. As an illustration, consider the graph indicated in Figure 2.

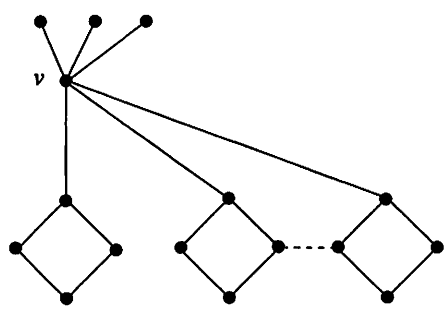

Figure 2: A graph $G \in M_k$, such that $G\backslash\{v\} \in M_1$.

We also note that there can be more than one vertex such that its removal results in a graph that has twice the number of maximal independent sets. For example, see Figure 3 (every vertex of the copy of the graph on the left side is joined to every vertex of the copy of the graph on the right side).

As mentioned in the introduction, extendable vertices have played an important role in the study of well-covered graphs. Those graphs that have the property that any vertex can be deleted and the resulting graph is still well-covered (called 1-well-covered or in W_2) are an important sub-class. Here we observe that an analogous collection exists for graphs in M_t. It can be easily verified that the graph in Figure 4 has the property that it belongs to M_{k+1} and removing any vertex still leaves a graph in M_{k+1}.

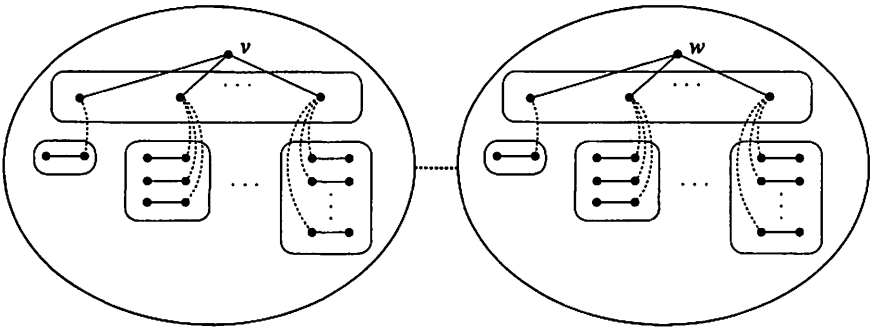

Figure 3: A graph with 2 vertices v and w such that its removal results in a graph that has twice the number of maximal independent sets.

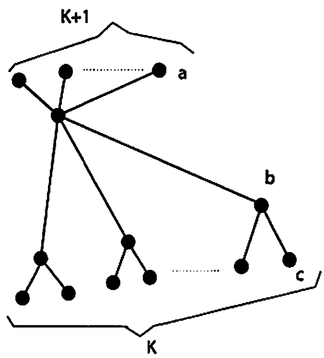

Figure 4: A $1 - M_{k+1}$ graph.

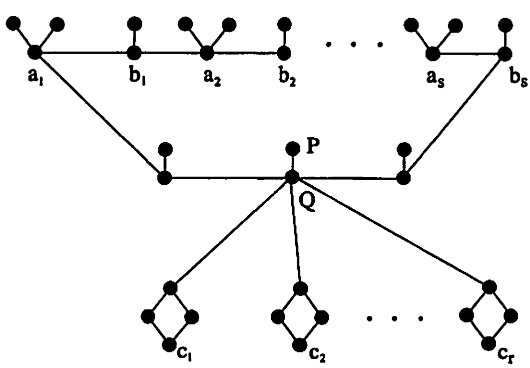

Figure 5: Graph for Proposition 1

A more general construction (see Figure 5) establishes the following proposition:

Proposition 1 *For any t and any $h \leq t$, there is a graph $G \in M_t$ with a vertex v such that $G\backslash\{v\} \in M_h$.*

Proof:

Let G be the graph indicated in Figure 5.

If P part of a maximal independent set, then sizes are:

$(1 + 2r) + (2 + 2s)$

$(1 + 2r) + (3 + 2s)$

....

$(1 + 2r) + (2 + 3s)$

If Q part of a maximal independent set, then sizes are:

$(1 + r) + (2 + 2s)$

$(2 + r) + (2 + 2s)$

....

$(1 + 2r) + (2 + 3s)$

Then, $G \in M_{[(1+2r)+(2+3s)]-[(1+r)+(2+2s)-1]} = M_{r+s+1}$

and $G\backslash\{Q\} \in M_{[(1+2r)+(2+3s)]-[(1+2r)+(2+2s)-1]} = M_{s+1}$.

Thus for any t and any $h \leq t$, there is a graph $G \in M_t$ with a vertex v such that $G\backslash\{v\} \in M_h$. □

We summarize Lemma 1 and the more general construction given after it along with Proposition 1 in the following Theorem.

Theorem 1 *For any t and any h where $1 \leq h \leq 2t$, $\exists$ a graph $G \in M_t$ with a vertex v such that $G\backslash\{v\} \in M_h$.*

We conclude by observing that if $G \in M_t$, then it is possible for $G\backslash\{e\}$ to have as many as $2t$ different maximal independent set sizes or as few as half the original number.

Consider the graph shown in Figure 6. It belongs to M_t but if the edge joining v and v' is removed, the resulting graph belongs to M_{2t}. On the other hand, the graph shown in Figure 7 belongs to M_{4t-2}, but deleting the edge between v and v' gives a graph belonging to M_{2t-1}.

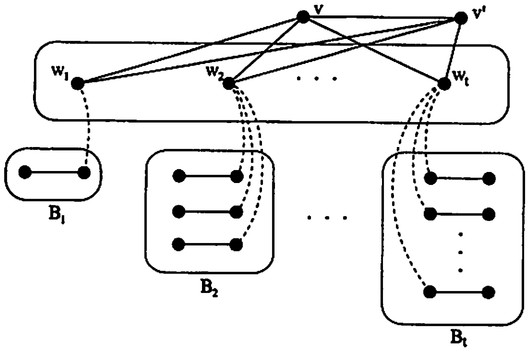

Figure 6: $G \in M_t$ and $G \backslash vv' \in M_{2t}$

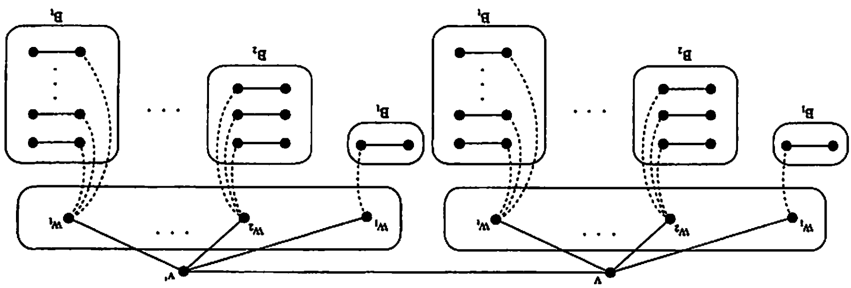

Figure 7: $G \in M_{4t-2}$ and $G \backslash vv' \in M_{2t-1}$

References

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[2] A. Finbow, B. Hartnell and C. A. Whitehead. *A Characterization of graphs of girth eight or more with exactly two sizes of maximal independent sets. Discrete Math. 125 (1994), 153-167.*

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