A Result on Chromatic Uniqueness of Edge-Gluing of Graphs

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Abstract

In this paper, it is shown that the graph obtained by overlapping the cycle C_m ($m \geq 3$) and the complete bipartite graph $K_{3,3}$ at an edge is uniquely determined by its chromatic polynomial.

Let G be a finite graph with neither loops nor multiple edges and let $P(G;\lambda)$ denote its chromatic polynomial. Then G is said to be *chromatically unique* if it is uniquely determined by its chromatic polynomial. A graph is *vertex-transitive* (respectively *edge-transitive*) if its group of automorphisms acts transitively on the vertex-set (respectively edge-set).

Let K_n denote a complete graph on n vertices. Suppose G_1 and G_2 are two graphs where each contains a complete subgraph K_n. Let $G_1 \cup_n G_2$ denote any graph obtained by overlapping G_1 and G_2 at K_n. The graph $G_1 \cup_2 G_2$ is sometimes referred to as the *edge-gluing* of G_1 and G_2. Notice that different choices of K_n might give non-isomorphic graphs. Two non-isomorphic graphs for $G_1 \cup_2 C_4$ are shown in Figure 1(b) and (c). More about the properties of $G_1 \cup_n G_2$ can be found in [3].

A graph G is *quasi-separable* if it contains a complete subgraph K_n such that $G - K_n$ is disconnected (see [1], p. 61). A *quasi-block* Q of G is a maximal subgraph of G that is not quasi-separable.

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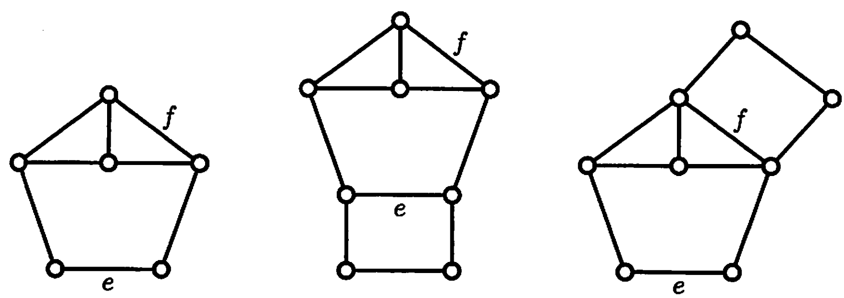
Theorem 1 *([3], [10])*
Let G be a graph consisting of two quasi-blocks Q_1 and Q_2 with $Q_1 \cap Q_2 = K_2$. Suppose G is chromatically unique. Then
(i) at least one of Q_1 or Q_2 is triangle-free;
(ii) Q_1 and Q_2 are chromatically unique;
(iii) Q_1 and Q_2 are edge-transitive. Further, at least one of Q_1 or Q_2 is vertex-transitive.

Question 5 in [4] asks whether the converse of Theorem 1 is true. In this paper, we show that $K_{3,3} \cup_2 C_m$ is chromatically unique for all $m \geq 3$. This result forms part of the continuing effort in supporting the fact that the answer to Question 5 asked in [4] could be true. Earlier, in [5], it was shown that $K_{2,s} \cup_2 C_m$ is chromatically unique for all $s \geq 1$ and all $m \geq 3$, while in [6], it was shown that $K_{2,2,2} \cup_2 C_m$ is chromatically unique for all $m \geq 3$. Here, $K_{r_1,r_2,\ldots,r_t}$ denote the complete t-partite graph whose t partite sets have $r_1, r_2, \ldots, r_t$ vertices.



(a) G_1 (b) $G_1 \cup_2 C_4$ (c) $G_1 \cup_2 C_4$

Figure 1: The graph G_1 and two non-isomorphic graphs for $G_1 \cup_2 C_4$

On occasion we will make use of the following theorem. By a K_4-*homeomorph* we mean a subdivision of K_4.

Theorem 2 *([2]) Let G be a connected graph and let $P(G;\lambda) = (\lambda - 1)T(G;\lambda)$. Then*
(i) $|T(G;1)| = 1$ if and only if G is a 2-connected graph and contains no K_4-homeomorph as a subgraph, and

(ii) $|T(G;1)| \geq 2$ *if and only if G is a 2-connected graph and contains at least one* K_4*-homeomorph as a subgraph.*

Let G be a graph and let A be a subgraph in G. Let $n(A,G)$ denote the number of subgraphs A in G. Let C_m^* denote a chordless cycle on m vertices ($m \geq 3$). The following lemma is a consequence of Theorems 1 and 2 of [7].

Lemma 1 *Let G and Y be two graphs such that* $P(G;\lambda) = P(Y;\lambda)$. *Then G and Y have the same number of vertices, edges and triangles. Moreover, in the event that G has at most one triangle, then* $n(C_4^*,G) = n(C_4^*,Y)$ *and*

$$-n(C_5^*,G) + n(K_{2,3},G) = -n(C_5^*,Y) + n(K_{2,3},Y).$$

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Then the *cyclomatic number* of G is $|E(G)| - |V(G)| + 1$.

Lemma 2 *([5]) Let G be a connected graph with cyclomatic number c. Then the number of* $K_{2,3}$ *in G is at most* $\binom{c+1}{3}$.

Let G be a graph and let e be an edge of G. Let $G - e$ denote the graph obtained by deleting the edge e from G. Also, let $G \cdot e$ be the graph obtained from G by identifying the two end-vertices of this edge e and removing any loop and all but one of the multiple edges, if they arise. Then, the chromatic polynomial of G can be obtained by using the following reduction formula

$$P(G;\lambda) = P(G-e;\lambda) - P(G \cdot e;\lambda) \tag{1}$$

Let $b_k(G)$ denote the coefficient of ω^k in $Q(G;\omega) = P(G;\lambda)$ where $\omega = \lambda - 1$. By Theorem 1 of [11], G contains a cut-vertex if and only if $b_1(G) = 0$.

Let H be a non-complete graph and let R (respectively T) be any graph obtained by identifying the end-vertices of a path P_m ($m \geq 3$) with two adjacent (respectively non-adjacent) vertices x and y of H. That is, $R = H \cup_2 C_m$. Also, let $H \cdot (x,y)$ be the graph obtained from H by identifying the two end-vertices x and y (regardless of whether they are adjacent or not).

Lemma 3 *([9])*
$Q(T;\omega) = Q(R;\omega) + (-1)^{m-1}Q(H \cdot (x,y);\omega)$.

Let $S(s,a,b)$ denote the set of all graphs obtained in the following way. Identify the end-vertices of a paths (not necessary of the same length) with any two non-adjacent degree-2 vertices (not necessary of the same pair) of $K_{2,s}$, $s \geq 3$. Call each of the a paths an *α-path*. We require that each α-path to have length at least 2. Also, overlap b cycles (not necessary of the same length) at any edge (not necessary of the same edge) of $K_{2,s}$. Figure 2 depicts an example of a graph in $\mathcal{S}(s,3,1)$. Notice that the subgraph $K_{2,s}$ is shown by the dark vertices and thicker lines and that the two paths with end vertices x_1 and x_2 (so is the path with end vertices x_2 and x_3) are α-paths.

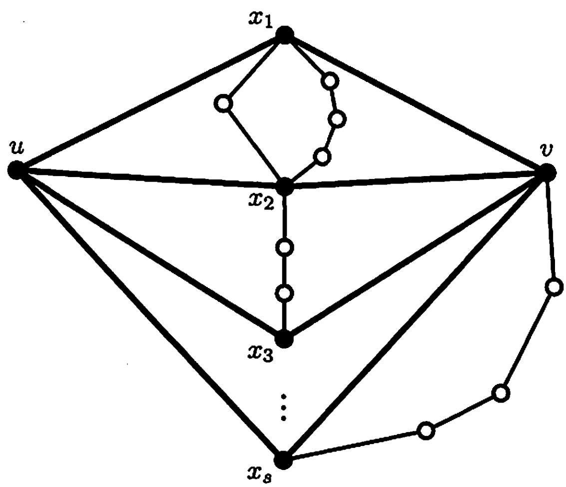


Figure 2: A graph in $\mathcal{S}(s,3,1)$

Lemma 4 *Let $G \in \mathcal{S}(s,a,b)$. Then*
(i) $|b_1(G)| \leq a+1$ if $s=3$ and $0 \leq a \leq 2$,
(ii) $|b_1(G)| \leq 4$ if $s=3$ and $a \geq 3$,
(iii) $|b_1(G)| \leq 2^a$ if $s \geq 4$.

Proof: It follows from Theorem 1 that $|b_1(G)| = 1$ if $G \in \mathcal{S}(3,0,b)$.

Suppose $a \geq 1$. We shall apply Lemma 3. Here T is the graph G, $H \in \mathcal{S}(s,a-1,b)$ is obtained from G by removing all edges and vertices (except the end-vertices) of an α-path P whose end-vertices are x and y, $R \in \mathcal{S}(s,a-1,b+1)$ is obtained from H by identifying the end-vertices of P with any two adjacent vertices of $K_{2,s}$.

Notice that, in this case, $H \cdot (x, y)$ contains a cut-vertex if and only if there is another α-path in G with x and y as end-vertices. In the event that $H \cdot (x, y)$ does not contain a cut-vertex, then $H \cdot (x, y) \in \mathcal{S}(s-1, a-1, b)$. In any case, we see that $|b_1(H \cdot (x, y))| \leq 1$ if $s = 3$.

By Lemma 3, we have

$$|b_1(G)| \leq |b_1(R) \pm b_1(H \cdot (x, y))| .$$

The proof for (i) then follow by induction on a.

For (ii), we observe that if $a \geq 4$, then there exist two α-paths with same end-vertices in $K_{2,s}$ (because $s = 3$), in which case, $H \cdot (x, y)$ contains a cut-vertex and $R \in \mathcal{S}(s, a-1, b+1)$ when Lemma 3 is applied to G with respect to one of these α-paths. Consequently, $|b_1(G)| = |b_1(R)|$.

Repeat the same argument to the graph R and in a finite number of steps, we have $|b_1(G)| = |b_1(R_1)|$ where $R_1 \in \mathcal{S}(s, d, f)$ for some $d \leq 3$ and some $f \geq b+1$.

If $0 \leq d \leq 2$, then $|b_1(R_1)| \leq 4$ by the result in (i). So assume that $d = 3$ and that no two α-paths have the same end-vertices in $K_{2,s}$.

Apply Lemma 3 to R_1 with respect to any of the α-paths, we have

$$|b_1(R_1)| \leq |b_1(R_2) \pm b_1(H \cdot (x, y))|$$

where $R_2 \in \mathcal{S}(3, 2, f+1)$ and $H \cdot (x, y) \in \mathcal{S}(2, 2, f)$ contains no K_4-homeomorphs. By the result in (i), $|b_1(R_2)| \leq 3$ and $|b_1(H \cdot (x, y))| = 1$. This finishes the proof for (ii).

Note that (i) and (ii) imply that $|b_1(G)| \leq 2^a$ where $G \in \mathcal{S}(3, a, b)$ and $a \geq 0$. We can then use this fact to prove (iii) by using the same inequality above and by induction on a. □

Lemma 5 *([2]) Let G be a 2-connected graph. If H is a 2-connected subgraph of G, then $|b_1(G)| \geq |b_1(H)|$.*

Let $S_m(s)$ denote the graph obtained by identifying the end-vertices of a path P_m with two non-adjacent degree-3 vertices of $K_{3,s}$ where $s \geq 3$ and $m \geq 3$.

Lemma 6 *Let G denote the graph $K_{3,s} \cup_2 C_m$ where $s, m \geq 3$. Then $|b_1(G)| \neq |b_1(S_m(s))|$ and hence $Q(G; \omega) \neq Q(S_m(s); \omega)$.*

Proof: By applying Lemma 3 to the graph $S_m(s)$, we have $R = G$, $H = K_{3,s}$ and

$$Q(S_m(s); \omega) = Q(G; \omega) + (-1)^{m-1} Q(K_{3,s-1}; \omega)$$

and the lemma follows by noting that $b_1(K_{3,s-1}) \neq 0$ (because $K_{3,s-1}$ contains no cut-vertices). □

Lemma 7 *Suppose G^* is a homeomorph of the graph G. Then $|b_1(G^*)| = |b_1(G)|$.*

Proof: Let P be a path in G^* which is the result of subdividing an edge of G. Let e be an edge on P. Apply the reduction formula (1) to G^* on the edge e, we see that $G^* - e$ contains a cut-vertex so that $|b_1(G^* - e)| = 0$ and hence $|b_1(G^*)| = |b_1(G^* \cdot e)|$.

Repeat the same argument to the graph $G^* \cdot e$ for every edge on P and for every such path until we reach the graph G. Therefore $|b_1(G^*)| = |b_1(G)|$. □

Let X (respectively Y) be a graph containing a subgraph of the form $K_{2,l}$ (respectively $K_{3,l}$) for some $l \geq 2$. Let x be a vertex in $X - K_{2,l}$ (respectively $Y - K_{3,l}$). Then x is called a *t-vertex* to $K_{2,l}$ (respectively a *t^*-vertex* to $K_{3,l}$) if x is adjacent to precisely two vertices of $K_{2,l}$ (respectively three vertices of $K_{3,l}$) so that the resulting graph $K_{2,l} \cup \{x\}$ (respectively $K_{3,l} \cup \{x\}$) is isomorphic to $K_{2,l+1}$ (respectively $K_{3,l+1}$), $l \geq 2$.

Suppose J is a tree. Let $\mathcal{G}$ denote the set of all 2-connected graphs G obtained by joining four new edges from J to the graph $K_{2,3}$ with the following properties:

(i) there are no t^*-vertices from J to $K_{2,3}$,

(ii) G has at most one triangle, and

(iii) $n(K_{2,3}, G) \geq 5$.

Lemma 8 *Suppose $G \in \mathcal{G}$. Then either G is a homeomorph of the graph $S_3(3)$ or else $|b_1(G)| \leq 4$.*

Proof: Let u_1, u_2, u_3 denote the vertices of degree 2 in $K = K_{2,3}$ and v_1, v_2 those of degree 3. There are two cases to consider.

Case(1): J contains no t-vertex to K.

Since $n(K_{2,3}, G) \geq 5$, there exist two vertices x and y in J that are adjacent to the same two degree-2 vertices in K, say u_1 and u_2 (so that u_1, u_2, v_1, v_2, x, y form a $K_{2,4}$). Since J is joined to K by exactly 4 edges, J is a path (because G is 2-connected) with end-vertices x and y. Evidently $G \in S(4,2,0)$ and we have $|b_1(G)| \leq 4$ by Lemma 4.

Case(2): J contains a t-vertex to K.

Let $z_0 \in J$ denote a t-vertex to K. Let z_1 be a vertex in $J - z_0$ such that z_1 is adjacent to some vertices in K.

Suppose z_1 is adjacent to two vertices of K. Then J is a path with z_0 and z_1 as its end-vertices (because G is 2-connected). If z_1 is a t-vertex to K, then $G \in \mathcal{S}(5,1,0)$ and we have $|b_1(G)| \leq 4$ by Lemma 4. If z_1 is adjacent to any two distinct vertices u_i and u_j, then G is a homeomorph of the graph $S_3(3)$. If z_1 is adjacent to v_r and u_s for some r and s, then $n(K_{2,3}, G) < 5$, in which case, $G \notin \mathcal{G}$.

Hence we assume that z_1 is adjacent to only one vertex in K. Then there exists a vertex z_2 in $J - \{z_0, z_1\}$ such that z_2 is also adjacent to only one vertex in K. Since $n(K_{2,3}, G) \geq 5$, there is an $1 \leq i \leq 2$ such that $d(z_0, z_i) = 1$ and z_i is adjacent to a degree-2 vertex of K. Without loss of generality, we may assume that z_1 is adjacent to z_0 and u_1. Since G is 2-connected, J is a path. Since z_2 is in $J - \{z_0, z_1\}$, there is a path P connecting z_0 and z_2.

Subcase (2.1): P contains z_1

(In this case z_0 and z_2 are end-vertices of P.) If z_2 is adjacent to v_1 or v_2, then $G \in \mathcal{S}(3,3,0)$. If z_2 is adjacent to u_1, then $G \in \mathcal{S}(3,2,1)$. In either case, we have $|b_1(G)| \leq 4$ by Lemma 4. Hence z_2 is adjacent to either u_2 or u_3. But then G is a homeomorph of the graph $S_3(3)$.

Subcase (2.2): P does not contain z_1

(In this case z_1 and z_2 are end-vertices of P.) If z_2 is adjacent to v_i, then $G \in \mathcal{S}(4,1,1)$. If z_2 is adjacent to u_j, then $G \in \mathcal{S}(4,2,0)$. In either case, we have $|b_1(G)| \leq 4$ by Lemma 4.

This completes the proof. □

Theorem 3 *For any $m \geq 3$, the graph $K_{3,3} \cup_2 C_m$ is chromatically unique.*

Proof: Let $G = K_{3,3} \cup_2 C_m$. Suppose Y is such that $P(Y;\lambda) = P(G;\lambda)$. Then Y is a 2-connected graph on $m+4$ vertices and $m+8$ edges and contains at most one triangle. Note that $n(K_{2,3}, G) = 6$. By Lemma 1, $n(K_{2,3}, Y) \geq 6$ if $m \neq 5$ and $n(K_{2,3}, Y) \geq 5$ if $m = 5$. In either case, we see that Y contains a subgraph $K_{2,3}$. Let K denote this subgraph.

Let J be the graph $Y - K$ and assume that there are α edges joining K to J. Now note that J has $m-1$ vertices and $m+2-\alpha$ edges and so

$$|E(J)| - |V(J)| = 3 - \alpha$$

Let $J_1, \ldots, J_k$ be the connected components of J, $k \geq 1$. Suppose there are α_i edges joining K and J_i, $i = 1, \ldots, k$. Let c_i denote the cyclomatic number of J_i, $i = 1, \ldots, k$. Then $\sum_{i=1}^{k} c_i = 3 - \alpha + k \geq 0$. Consequently, $\alpha \leq 3 + k$. Since $\alpha \geq 2k$, it follows that $1 \leq k \leq 3$.

Note that $|b_1(Y)| = |b_1(G)| = 5$. The cyclomatic numbers of G and Y are both equal to 5. We make the following observations.

$O(1)$: Y is neither the graph $S_m(3)$ nor its homeomorph, where $m \geq 3$. This follows from Lemmas 6 and 7.

$O(2)$: Y is neither the graph in $\mathcal{S}(s, a, b)$ nor its homeomorph, where $s + a + b \leq 6$ and $s \geq 3$.

This follows from Lemmas 4 and 7 by noting that otherwise $|b_1(Y)| \leq 4$.

Case(1): J contains no t^*-vertex to K.

Suppose $\alpha_i = 2$ for all $i = 1, \ldots, k$. Then $\sum_{i=1}^{k} c_i = 3 - 2k + k = 3 - k \leq 2$. If $\sum_{i=1}^{k} c_i = 0$, then each J_i is a tree (and hence a path because Y is 2-connected) and Y is a graph in $\mathcal{S}(3 + x_1, x_2, x_3)$ or its homeomorph where $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 \leq 3$ (because $k \leq 3$). Therefore $(3 + x_1) + x_2 + x_3 \leq 6$. However this is impossible by $O(2)$. If $\sum_{i=1}^{k} c_i = 1$, then $k = 2$. Further, one of the components of J is a path and the other a unicyclic graph. It is not difficult to check that $n(K_{2,3}, Y) < 5$ in this case. If $\sum_{i=1}^{k} c_i = 2$, then $k = 1$ and we have $n(K_{2,3}, Y) \leq 1 + \binom{c_1+1}{3} < 5$ by Lemma 2.

Now suppose $\alpha_i \geq 3$ for some $1 \leq i \leq k$. Then $2k + 1 \leq \sum_{i=1}^{k} \alpha_i = \alpha \leq k + 3$ and this implies that $k \leq 2$. We claim that no J_j contains a $K_{2,3}$ as subgraph, $j = 1, \ldots, k$. This is because otherwise $2 \leq \sum_{j=1}^{k} c_j = 3 - \alpha + k$ and this implies that $2k + 1 \leq \alpha \leq k + 1$, a contradiction.

If $k = 1$, then $3 \leq \alpha \leq 4$. Suppose $\alpha = 3$. Then J is unicyclic. Since $n(K_{2,3}, Y) \geq 5$, J contains a t-vertex v to K. Moreover there exists a vertex u in J such that u is adjacent to v and a degree-2 vertex w of K (so that u, v, w and the two degree-3 vertices of K form another $K_{2,3}$ of Y). But then, in this case, J is a cycle (because Y is 2-connected) and we have $Y \in \mathcal{S}(3, 2, 1)$ in which case, $|b_1(Y)| \leq 4$ by Lemma 4. Hence $\alpha = 4$ and J is a tree. By Lemma 8, either $|b_1(Y)| \leq 4$ or else Y is homeomorph of the graph $S_3(3)$. The latter case contradicts $O(1)$.

If $k = 2$, then $\alpha = 5$ and $\sum_{i=1}^{k} c_i = 0$, and hence the components of J are trees. Without loss of generality, assume that $\alpha_1 = 2$ and $\alpha_2 = 3$. Let w_1, w_2, w_3 denote the degree-2 vertices of K.

Subcase(1.1): J contains no t-vertex to K.

Since $n(K_{2,3}, Y) \geq 5$, each J_i contains one vertex that is adjacent to the same two degree-2 vertices (say w_2 and w_3) of K. This implies that J_1 is an isolated vertex and J_2 is a path (because Y is 2-connected). If the other end-vertex of J_2 is adjacent to a degree-3 vertex of K, then $Y \in \mathcal{S}(4, 2, 0)$. If the other end-vertex of J_2 is adjacent to w_2 or w_3, then $Y \in \mathcal{S}(4, 1, 1)$. In either case, we have $|b_1(Y)| \leq 4$ by Lemma 4. Hence the other end-vertex of J_2 is adjacent to w_1. But then Y is a homeomorph of $S_3(3)$, a contradiction to $O(1)$.

Subcase(1.2): J contains a t-vertex to K.

Because $n(K_{2,3}, Y) \geq 5$, we see that J contains either another t-vertex to K or a vertex that is adjacent to two degree-2 vertices of K. Whatever the case is, J_2 is a path because Y is 2-connected. Let z denote an end-vertex of J_2 that is neither a t-vertex to K nor a vertex that is adjacent to two degree-2 vertices of K.

Subcase (1.2.1): J_1 contains a t-vertex to K.

Then J_1 is an isolated vertex.

Suppose J_2 contains a t-vertex to K. If z is adjacent to a degree-3 vertex of K, then $Y \in \mathcal{S}(5,0,1)$. If z is adjacent to a degree-2 vertex of K, then $Y \in \mathcal{S}(5,1,0)$. In either case, we have $|b_1(Y)| \leq 4$ by Lemma 4.

Now suppose J_2 contains a vertex that is adjacent to w_2 and w_3. If z is adjacent to a degree-3 vertex of K, then $Y \in \mathcal{S}(3,3,0)$. If z is adjacent to w_2 or w_3, then $Y \in \mathcal{S}(3,2,1)$. In either case, we have $|b_1(Y)| \leq 4$ by Lemma 4. Hence z is adjacent to w_1. But then Y is a homeomorph of $S_3(3)$, a contradiction to $O(1)$.

Subcase (1.2.2): J_1 contains no t-vertex to K.

Then J_2 must contain a t-vertex to K. As remarked earlier, J_1 is then an isolated vertex and is adjacent to two degree-2 vertices of K. If z is adjacent to a degree-3 vertex of K, then $Y \in \mathcal{S}(4,1,1)$. If z is adjacent to a degree-2 vertex of K, then $Y \in \mathcal{S}(4,2,0)$. In either case, we have $|b_1(Y)| \leq 4$ by Lemma 4.

Case(2): J contains a t^*-vertex to K.

Note that J contains exactly one t^*-vertex to K. This is because otherwise the cyclomatic number of Y would be more than 5. Since J contains a t^*-vertex to K, we have $\alpha_i \geq 3$ for some i and this implies that $2k+1 \leq \alpha \leq 3+k$ and that $k \leq 2$. If $k=1$, then $\alpha = 4$ (otherwise $\alpha = 3$ and this results in Y having a cut-vertex). Consequently, $\sum_{i=1}^{k} c_i = 0$ and J is a path with one end-vertex being a t^*-vertex. By $O(1)$, the other end-vertex u of J is not adjacent to a degree-3 vertex of K. Hence u is adjacent to a degree-2 vertex of K and the resulting graph is isomorphic to G.

If $k=2$, then $\alpha = 5$. Here, J must contain an isolated vertex which is a t^*-vertex to K. The other component of J is a path whose end-vertices cannot be adjacent to two non-adjacent vertices of K by $O(1)$. Hence the resulting graph is isomorphic to G. □

Combining Theorem 3 with those results known previously, we have the following.

Corollary 1 *Let H be an edge-transitive graph on at most 6 vertices. If H is chromatically unique, then so is the graph $H \cup_2 C_m$, for any $m \geq 3$.*

References

[1] N.L. Biggs, *Algebraic Graph Theory,* Cambridge University Press, London, New York (1994).

[2] C.Y. Chao and L.-C. Zhao, Chromatic polynomials of connected graphs, *Arch. Math. (Basel)* **43** (1984) 187-192.

[3] G.L. Chia, Some remarks on the chromatic uniqueness of graphs, *Ars Combinat.*, Vol. 26A (1988), 65–72.

[4] G.L. Chia, Some problems on chromatic polynomials, *Discrete Math.* **172** (1997) 39-44.

[5] G.L. Chia and C.-K. Ho, On the chromatic uniqueness of edge-gluing of complete bipartite graphs and cycles, *Ars Combinat.* **60** (2001) 193–199.

[6] G.L. Chia and C.-K. Ho, On the chromatic uniqueness of edge-gluing of complete tripartite graphs and cycles, *Bull. Malaysian Math. Sc. Soc.* (Second Series) **26** (2003) 87–92.

[7] E.J. Farrell, On chromatic coefficients, *Discrete Math.* **29** (1980) 257-264.

[8] K.M. Koh and K.L. Teo, The search for chromatically unique graphs, *Graphs and Combin.* **6** (1990) 259-285.

[9] R.C. Read, Broken wheels are *SLC*, *Ars Combinat.* **21A** (1986) 123-128.

[10] R.C. Read, Connectivity and chromatic uniqueness, *Ars Combinat.* **23** (1987) 209-218.

[11] E.G. Whitehead Jr. and L.-C. Zhao, Cutpoints and the chromatic polynomial, *J. Graph Theory* **8** (1984) 371-377.