

The Hamilton-Waterloo problem with triangle-factors and Hamilton cycles: The case $n \equiv 3 \pmod{18}$

Alan C.H. Ling
Dept. of Computer Science
University of Vermont
Burlington, Vermont

J.H. Dinitz
Dept. of Mathematics and Statistics
University of Vermont
Burlington, Vermont

February 13, 2008

Abstract

The Hamilton-Waterloo problem in the case of triangle-factors and Hamilton cycles asks for a 2-factorization of K_n in which each 2-factor is either a Hamilton cycle or a triangle-factor. Necessarily $n \equiv 3 \pmod{6}$. The case of $n \equiv 9 \pmod{18}$ was completely solved in 2004 by Horak, Nedela and Rosa. In this note we solve the problem when $n \equiv 3 \pmod{18}$ and there are at least two Hamilton cycles. A companion paper treats the case when there is exactly one Hamilton cycle and $n \equiv 3 \pmod{6}$.

1 Definitions and Background

This paper is based on the paper by Horak, Nedela and Rosa [4]. We refer the reader to that paper and to [3] for background on the

Hamilton-Waterloo problem and also to [1] or [2] for general reference to terms which may not be defined. We begin by giving the necessary definitions from [4]. Let $HW^*(6k+3)$ be the set of all integers r such that there exists a 2-factorization of K_{6k+3} in which r of the 2-factors are Hamilton cycles, and the remaining $s = 3k + 1 - r$ two-factors are triangle-factors (spanning sets of $2k + 1$ triangles). Clearly then $HW^*(6k+3) \subseteq \{0, 1, \dots, 3k+1\}$. Define $I(6k+3) = \{0, 1, \dots, 3k+1\}$. It is well-known that Hamilton factorizations of K_{6k+3} exist as do triangle-factorizations (Kirkman triple systems), hence $\{0, 3k+1\} \subset HW^*(6k+3)$.

Let the vertex set of K_{6k+3} be the set $\mathbb{Z}_{2k+1} \times \{0, 1, 2\}$, where $k \geq 1$. As was done in [4] the notation is simplified by letting $\mathbb{Z}_{2k+1} \times \{0\} = A = \{a_0, a_1, \dots, a_{2k}\}$, $\mathbb{Z}_{2k+1} \times \{1\} = B = \{b_0, b_1, \dots, b_{2k}\}$ and $\mathbb{Z}_{2k+1} \times \{2\} = C = \{c_0, c_1, \dots, c_{2k}\}$. All indices will be taken modulo $2k+1$. For $0 \leq d \leq 2k$, define the set of edges

$$\begin{aligned} (AB)_d &= \{\{a_i, b_{i+d} : i = 0, 1, \dots, 2k\}, \\ (BC)_d &= \{\{b_i, c_{i+d} : i = 0, 1, \dots, 2k\}, \text{ and} \\ (CA)_d &= \{\{c_i, a_{i+d} : i = 0, 1, \dots, 2k\}. \end{aligned}$$

Hence the edge set E of the complete graph K_{6k+3} can be written as

$$E = E([A]) \cup E([B]) \cup E([C]) \cup \bigcup_{d=0}^{2k} \{(AB)_d \cup (BC)_d \cup (CA)_d\}.$$

where $[X]$ is the complete graph induced by the set of vertices X .

For each $0 \leq d \leq 2k$, let F_d denote the subgraph induced by the set of edges $(AB)_d \cup (BC)_d \cup (CA)_{-2d}$. Then clearly, for each $0 \leq d \leq 2k$, F_d is a triangle-factor of K_{6k+3} where the edges of F_d form the triangles (a_i, b_{i+d}, c_{i+2d}) for $i = 0, 1, \dots, 2k$.

The following three results are all from [4].

Lemma 1.1 [4] *Let $-2k \leq p, q, r \leq 2k$ be integers such that $p+q+r$ and $2k+1$ are relatively prime. Then the set of edges $(AB)_p \cup (BC)_q \cup (CA)_r$ induce a Hamilton cycle of K_{6k+3} .*

Corollary 1.2 [4] *For each $d = 0, 1, \dots, 2k-1$, the edges of $F_d \cup F_{d+1}$ can be decomposed into two Hamilton cycles.*

The following gives a partial result for $HW^*(n)$ in the case when $n \equiv 3 \pmod{18}$.

Theorem 1.3 [4] *Let $n = 6k + 3$ with $k \equiv 0$ or $2 \pmod{3}$. Then $\{(n+3)/6, (n+3)/6+2, (n+3)/6+3, \dots, (n-1)/2\} \subset HW^*(n)$.*

2 Main Result

We begin with a corollary to Lemma 1.1.

Corollary 2.1 *The edges of $F_1 \cup F_2 \cup F_3$ can be decomposed into three Hamilton cycles.*

Proof. View the edge set of $F_1 \cup F_2 \cup F_3$ as the union of sets $G_1 \cup G_2 \cup G_3$ where $G_1 = (AB)_1 \cup (BC)_1 \cup (CA)_{-4}$, $G_2 = (AB)_2 \cup (BC)_2 \cup (CA)_{-6}$, and $G_3 = (AB)_3 \cup (BC)_3 \cup (CA)_{-2}$. As all three numbers $1+1-4, 2+2-6$ and $3+3-2$ are relatively prime to $2k+1$, by Lemma 1.1 each of the three graphs G_1, G_2 and G_3 is a Hamilton cycle. \square

An important ingredient in the main proof is a Hanani triple system. Assume that $v \equiv 1 \pmod{6}$, a *Hanani triple system*, $\text{HTS}(v)$, is a Steiner triple system on v points with a partition of its blocks into $(v-1)/2$ maximum parallel classes, and a single partial parallel class with $(v-1)/6$ blocks. If we let $v = 2k+1$, then we see that there are k classes each missing exactly one point and there is one class which misses $k+1$ points. It follows since the replication number of an $\text{STS}(2k+1)$ is k , that each point is missing from exactly one class of blocks. The following theorem gives the necessary and sufficient conditions for the existence of a Hanani triple system of order v .

Theorem 2.2 [5] *A Hanani triple system of order v exists if and only if $v \equiv 1 \pmod{6}$ and $v \notin \{7, 13\}$.*

We are now ready to prove our main result.

Theorem 2.3 *Let $n = 6k+3$ with $k \equiv 0 \pmod{3}$ and $k \neq 3, 6$. Then $\{2, 3, \dots, (n-1)/2\} \subset \text{HW}^*(n)$.*

Proof. Since $n = 6k+3$ with $k \equiv 0 \pmod{3}$ we have that $|A| = |B| = |C| \equiv 1 \pmod{6}$. Hence from Theorem 2.2 there exists a Hanani triple system of order $|A| = 2k+1$. Put the blocks of a $\text{HTS}(2k+1)$ on the points of A such that for $0 \leq i \leq k-1$ the point a_i is missing from the i th maximum parallel class (call this A_i) and the points $a_k, a_{k+1}, \dots, a_{2k}$ are missing from the last class, A_k . Do the same on the points of B and C . Now for $0 \leq i \leq k-1$, let $T_i = A_i \cup B_i \cup C_i \cup \{a_i, b_i, c_i\}$ and let $T_k = A_k \cup B_k \cup C_k \cup \bigcup_{j=k}^{2k} \{a_j, b_j, c_j\}$. There are $k+1$ of these T_i 's each of which is a triangle-factor of K_n and $\bigcup_{i=0}^k T_i = E([A]) \cup E([B]) \cup E([C]) \cup F_0$.

To show that $2m \in HW^*(n)$ for $2m \leq 2k$ apply Corollary 1.2 to $\{F_{2i-1}, F_{2i}\}$ for each $i = 1, 2, \dots, m$. The triangle-factors are formed by F_j , $j = 2m + 1, 2m + 2, \dots, 2k$ and by T_j , $j = 0, 1, \dots, k$. To show that $2m + 1 \in HW^*(n)$ for $3 \leq 2m + 1 \leq 2k - 1$ first apply Corollary 2.1 to $\{F_1, F_2, F_3\}$ to obtain 3 Hamilton cycles. If $2m + 1 \geq 5$ apply Corollary 1.2 to $\{F_{2i}, F_{2i+1}\}$ for $i = 2, 3, \dots, m$. The T_i 's and the remaining F_i 's form the triangle-factors in the 2-factorization.

Hence we have that $\{2, 3, \dots, 2k - 1, 2k\} = \{2, 3, \dots, (n - 3)/3\} \subset HW^*(n)$. Combining these values with those from Theorem 1.3 completes the proof. \square

Combining Theorem 2.3 with some other known results give us the (almost) complete spectrum for $HW(n)$ in the case where $n \equiv 3 \pmod{18}$. We state this as our final theorem.

Theorem 2.4 *Let $n \equiv 3 \pmod{18}$, then $HW^*(n) = I(n) = \{0, 1, \dots, (n - 1)/2\}$ except possibly that $1 \notin HW^*(n)$ if $n = 93, 111, 129, 183$, or 201 .*

Proof. When $n = 21$ and $n = 39$ the result is proven in [4]. The existence of Hamilton cycle decompositions for all K_v , with v odd proves that $0 \in HW^*(n)$ for $n \equiv 3 \pmod{18}$ and hence in conjunction with Theorem 2.3 we have that $I(n) \setminus \{1\} \subset HW^*(n)$ for all $n \equiv 3 \pmod{18}$. In [3] it is shown that $1 \in HW^*(n)$ for all but 13 values of n . These exceptional cases when $n \equiv 3 \pmod{18}$ are $n = 93, 111, 129, 183$, and 201 .

3 Conclusion

We conclude by summarizing what is known about the Hamilton-Waterloo problem on n vertices with triangle-factors and Hamilton cycles. The necessary condition is that $n \equiv 3 \pmod{6}$. The problem is completely solved when $n \equiv 3$ or $9 \pmod{18}$ and the number of Hamilton cycles is at least 2 (from [4] and this paper). The problem is also solved when $n \equiv 15 \pmod{18}$ and the number of Hamilton cycles is in the set $\{(n + 3)/6, (n + 3)/6 + 2, (n + 3)/6 + 3, \dots, (n - 1)/2\}$ [4] (this is the top 2/3 of the possible spectrum).

In the case where there is exactly one Hamilton cycle and all the other cycles are triangle factors, there is a solution to the Hamilton-Waterloo problem for all $n \equiv 3 \pmod{6}$ except possibly for 14 cases, namely when $n \in \{93, 111, 123, 129, 141, 153, 159, 177, 183, 201, 207, 213, 237, 249\}$ [3].

References

- [1] C.J. Colbourn and J.H. Dinitz, *Handbook of Combinatorial Designs*, Second Edition, Chapman & Hall/CRC, Boca Raton, Fl., 2007.
- [2] C.J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, Oxford, 1999.
- [3] J.H. Dinitz and A.C.H. Ling, The Hamilton-Waterloo problem: The case of triangle-factors and one Hamilton cycle, submitted.
- [4] P. Horak, R. Nedela, and A. Rosa, The Hamilton-Waterloo problem: the case of Hamilton cycles and triangle-factors, *Discrete Math.* **284** (2004), pp. 181–188.
- [5] S.A. Vanstone, D.R. Stinson, P.J. Schellenberg, A. Rosa, R.S. Rees, C.J. Colbourn, M.W. Carter, and J. Carter, Hanani triple systems, *Israel J. Math.* **83** (1993), pp.305–319.