On ρ-labeling up to ten vertex-disjoint C_{4x+1}

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Abstract

Let G be a graph of size n with vertex set $V(G)$ and edge set $E(G)$. A *ρ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, \ldots, 2n\}$ such that $\{|f(u) - f(v)| : \{u, v\} \in E(G)\} = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in \{1, 2, \ldots, n\}$ either $x_i = i$ or $x_i = 2n+1-i$. Such a labeling of G yields a cyclic G-decomposition of K_{2n+1}. It is conjectured by El-Zanati and Vanden Eynden that every 2-regular graph G admits a ρ-labeling. We show that the union of up to ten vertex-disjoint C_{4x+1} admits a ρ-labeling.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i + 1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set

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$\Gamma = \{G_1, G_2, \ldots, G_t\}$ of pairwise disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If H is K_k, a G-decomposition Γ of H is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex-disjoint union of r copies of G.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [20], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one endvertex in A and the other in B) such that

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a *ρ-labeling*;

(ℓ1), (ℓ4) is called a *σ-labeling*;

(ℓ2), (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [20].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [20] and [13], respectively.

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

If G with n edges is not bipartite, then the best that could be obtained until recently (see [12]) from a Rosa-type labeling was a cyclic G-decomposition of K_{2n+1}. A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [6], Blinco et al. introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nt+1}. They called this labeling a γ-labeling.

Let G be a graph with n edges and h a labeling of the vertices of G. We call h a *γ-labeling* of G if the following conditions hold.

g1 The function h is a ρ-labeling of G.

g2 The graph G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\hat{b} \in B$ such that $\{\hat{b}, c\}$ is the unique edge joining an element of B to c.

g3 If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

g4 We have $|h(c) - h(\hat{b})| = n$.

Note that if a graph G has a γ-labeling, then it is almost-bipartite as defined earlier. In this case, removing the edge $\{c, \hat{b}\}$ from G produces a bipartite graph. See Figure 1 for γ-labelings of C_5 and of C_9.

Let G be a graph with n edges and Eulerian components and let h be a σ-labeling of G. It is well-known (see [3]) that we must have $n \equiv 0$ or 3 (mod 4). Moreover, if such a G is bipartite, then $n \equiv 0 \pmod 4$. We shall refer to this restriction as the *parity condition.* There are no such restrictions on $|E(G)|$ if h is a ρ-labeling.

Theorem 3 (Parity Condition) *If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$. If such a G is bipartite, then $n \equiv 0 \pmod 4$.*

In [20], Rosa presented α- and β-labelings of C_{4m} and of C_{4m+3}, respectively. It is also known that both C_{4m+1} and C_{4m+2} admit ρ-labelings. It was shown in [13] that there exists a ρ^+-labeling of C_{4m+2}, for all positive integers m. It can be easily checked that this labeling is actually a ρ^{++}-labeling.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). If a 2-regular graph G is bipartite, then it is known that G admits a σ^+-labeling if the parity condition is satisfied (see [13]) and a ρ^{++}-labeling otherwise (see [5]). A 2-regular graph G need not admit a β-labeling even if the parity condition is satisfied. For example, it is shown in [17] that rC_3 does not admit a β-labeling for all $r > 1$ and rC_5 does not admit a β-labeling for all $r \geq 1$. Moreover, it is known that $C_3 \cup C_3 \cup C_5$ is the smallest 2-regular graph that satisfies the parity condition, yet fails to have a β-labeling (see [2]). It is thus reasonable to focus on labelings that are less restrictive than β-labelings when studying 2-regular graphs.

Here, we shall show that for $r \leq 10$ and $x \geq 1$, the 2-regular graph rC_{4x+1} has a ρ-labeling. These results are already known if $r \leq 4$ (see [10]). Our results provide further evidence in support of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a ρ-labeling.

2 Summary of Some of the Known Results

As stated in the previous section, the following is known for cycles (see [19], [20] and [13]).

Theorem 4 *Let $m \geq 3$ be an integer. Then C_m admits an α-labeling if $m \equiv 0 \pmod 4$, a ρ-labeling if $m \equiv 1 \pmod 4$, a ρ^{++}-labeling if $m \equiv 2 \pmod 4$, and a β-labeling if $m \equiv 3 \pmod 4$.*

For 2-regular graphs with two components, we have the following important result from Abrham and Kotzig [2].

Theorem 5 *Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \cup C_n$ has a β-labeling if and only if $m + n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m + n \equiv 0 \pmod 4$.*

If the parity condition is not satisfied, then we have the following from [5] and [11].

Theorem 6 *Let $m \geq 3$ and $n \geq 3$ be integers such that $m + n \equiv 1$ or $2 \pmod 4$. Then $C_m \cup C_n$ has a ρ^{++}-labeling if both m and n are even and a ρ-labeling otherwise.*

For 2-regular graphs with more than two components, the following is known. In [17], Kotzig shows that if $r > 1$, then rC_3 does not admit a β-labeling. Similarly, he shows that rC_5 does not admit a β-labeling for any r. In [18], Kotzig shows that $3C_{4k+1}$ admits a β-labeling for all $k \geq 2$. From results in [9], it can be shown that rC_3 admits a ρ-labeling for all $r \geq 1$. The ρ-labeling in [9] can be modified to produce a σ-labeling of rC_3 when the parity condition is satisfied. In [14], Eshghi shows that $C_{2m} \cup C_{2n} \cup C_{2k}$ has an α-labeling for all m, n, and $k \geq 2$ with $m+n+k \equiv 0 \pmod 2$ except when $m = n = k = 2$. In [1], Abrham and Kotzig show that rC_4 has an α-labeling for all positive integers $r \neq 3$. In [10], it is shown that $3C_m$ and $4C_m$ admit σ-labelings if the parity condition is satisfied and ρ-labelings otherwise. An additional result follows by combining results from [13] and from [5].

Theorem 7 *Let G be a 2-regular bipartite graph of order n. Then G has a σ^+-labeling if $n \equiv 0 \pmod 4$ and a ρ^{++}-labeling if $n \equiv 2 \pmod 4$.*

Finally we note that in [8], it is shown that every 2-regular almost-bipartite graph $G \neq C_3 \cup (kC_4)$, $k \in \{0, 1\}$, has a γ-labeling.

3 Main results

Let $V(C_{4x+1}) = \{v_1, v_2, \cdots, v_{4x+1}\}$ and $E(C_{4x+1}) = \{\{v_i, v_{i+1}\} : 1 \leq i \leq 4x\} \cup \{v_{4x+1}, v_1\}$. Let $A = \{v_{2i-1} : 1 \leq i \leq 2x\}$, $B = \{v_{2i} : 1 \leq i \leq 2x\}$, and $C = \{v_{4x+1}\}$. Let $\hat{b} = v_{4x}$ and let $c = v_{4x+1}$. Thus A, B, C is a vertex tripartition of C_{4x+1} with a unique edge e joining $\hat{b} = v_{4x} \in B$ to the sole vertex $c = v_{4x+1} \in C$. In [6], it is shown (though described differently) that the following function f defines a γ-labeling of C_{4x+1}.

$$f(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \leq 4x-1 \text{ is odd} \\ 4x - \frac{i-2}{2} & \text{if } i \leq 2x-2 \text{ is even} \\ 4x - \frac{i}{2} & \text{if } i \geq 2x \text{ is even} \\ 6x+1 & \text{if } i = 4x+1 \end{cases}$$

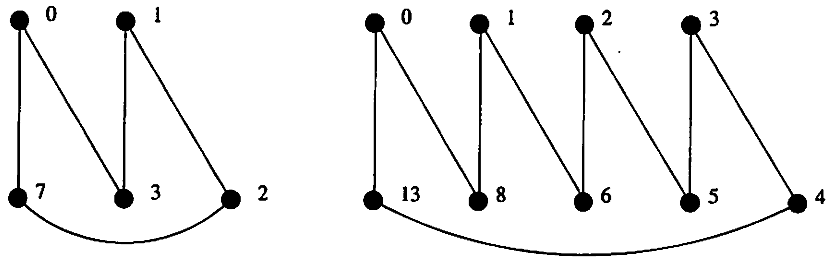

Figure 1: Examples of γ-labelings of C_5 and C_9.

See Figure 1 for γ-labelings of C_5 and of C_9.

As stated earlier, the following is proved in [6].

Theorem 8 *Let G be a graph with n edges having a γ-labeling. Then G divides K_{2nt+1} cyclically for all positive integers t.*

We illustrate how Theorem 8 works. For the actual proof, we direct the reader to [6]. Let G have n edges and let h be a γ-labeling for G, with A, B, C, c, and $\hat{b}$ as in the above definition. Let $B_1, B_2, \ldots, B_t$ be t vertex-disjoint copies of B, and let $c_1, c_2, \ldots, c_t$ be t new vertices. The vertex in B_i corresponding to $b \in B$ will be called b_i. Let $B^* = \bigcup_1^t B_i$ and $C^* = \{c_1, c_2, \ldots, c_t\}$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \le i \le t$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and $\{\hat{b}_i, c_i\}$, $1 \le i \le t$. Clearly G^* has nt edges and G divides G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)2n & v = b_i \in B_i, \\ h(c) + (t-i)2n & v = c_i. \end{cases}$$

It can be shown that the labeling h^* is a ρ-labeling of G^* and thus the result follows by Theorem 1.

Figure 2 shows a γ-labeling of C_5 and the C_5-decomposition of G^* which can be used to obtain a cyclic C_5-decomposition of K_{31}.

For $1 \le i \le t$, let G_i be the subgraph of G^* induced by the vertex set $V(G_i) = A \bigcup B_i \bigcup \{c_i\}$. Thus $\Delta = \{G_1, G_2, \ldots, G_t\}$ is a G-decomposition of G^*. If G is C_{4x+1} and if $t \le 10$, we will obtain a ρ-labeling of tC_{4x+1} by modifying the labelings of $G_1, G_2, \ldots, G_t$ as to make the new G_i's vertex-disjoint.

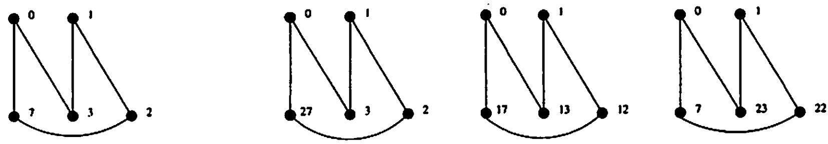

Figure 2: (a) A γ-labeling of C_5. (b) Three C_5's used for a cyclic decomposition of K_{31}.

Theorem 9 *If $t \leq 10$, then tC_{4x+1} admits a ρ-labeling.*

Proof. We will modify the labelings of $G_1, G_2, \ldots, G_t$ from the γ-labeling of C_{4x+1} to obtain the desired results.

Case 1 $t = 1$.
By definition, the γ-labeling of C_{4x+1} is a ρ-labeling.

Case 2 $t = 2$.
From the γ-labeling of C_{4x+1}, the labels for G_1 and G_2 are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$14x+3$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$6x+1$

We will keep the labels in G_1 as is and add $4x+2$ to every label in G_2. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$14x+3$
G_2	$[4x+2, 6x+1]$	$\subset [14x+4, 16x+4]$	$10x+3$

It is easy to see that the labels of G_1 and G_2 are disjoint.

Case 3 $t = 3$.
From the γ-labeling of C_{4x+1}, the labels for G_1, G_2, and G_3 are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$22x+5$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$14x+3$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_3, and add $6x+2$ to every label in G_2. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$22x+5$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$20x+5$
G_3	$[4x+2, 6x+1]$	$\subset [22x+6, 24x+6]$	$10x+3$

It is easy to see that the labels of G_1, G_2 and G_3 are disjoint.

Case 4 $t = 4$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_4$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$30x+7$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$22x+5$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$14x+3$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_4, add $6x+2$ to every label in G_2, and add $8x+2$ to every label in G_3. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$30x+7$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$28x+7$
G_3	$[8x+2, 10x+1]$	$\subset [26x+6, 28x+6]$	$22x+5$
G_4	$[4x+2, 6x+1]$	$\subset [30x+8, 32x+8]$	$10x+3$

It is easy to see that the labels of $G_1, \ldots, G_4$ are disjoint.

Case 5 $t = 5$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_5$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$38x+9$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$30x+7$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$22x+5$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$14x+3$
G_5	$[0, 2x-1]$	$\subset [34x+8, 36x+8]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_5, add $6x+2$ to every label in G_2, and add $8x+2$ to every label in G_4, and add $10x+4$ to every label in G_3. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$38x+9$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$36x+9$
G_3	$[10x+4, 12x+3]$	$\subset [28x+8, 30x+8]$	$32x+9$
G_4	$[8x+2, 10x+1]$	$\subset [34x+8, 36x+8]$	$22x+5$
G_5	$[4x+2, 6x+1]$	$\subset [38x+10, 40x+10]$	$10x+3$

It is easy to see that the labels of $G_1, \ldots, G_5$ are disjoint.

Case 6 $t = 6$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_6$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$46x+11$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$38x+9$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$30x+7$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$22x+5$
G_5	$[0, 2x-1]$	$\subset [34x+8, 36x+8]$	$14x+3$
G_6	$[0, 2x-1]$	$\subset [42x+10, 44x+10]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_6, add $6x+2$ to every label in G_2, and add $8x+2$ to every label in G_5, and add $10x+4$ to every label in G_3, and add $12x+4$ to every label in G_4. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$46x+11$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$44x+11$
G_3	$[10x+4, 12x+3]$	$\subset [28x+8, 30x+8]$	$40x+11$
G_4	$[12x+4, 14x+3]$	$\subset [38x+10, 40x+10]$	$34x+9$
G_5	$[8x+2, 10x+1]$	$\subset [42x+10, 44x+10]$	$22x+5$
G_6	$[4x+2, 6x+1]$	$\subset [46x+12, 48x+12]$	$10x+3$

It is easy to see that the labels of $G_1, \ldots, G_6$ are disjoint.

Case 7 $t = 7$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_7$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$54x+13$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$46x+11$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$38x+9$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$30x+7$
G_5	$[0, 2x-1]$	$\subset [34x+8, 36x+8]$	$22x+5$
G_6	$[0, 2x-1]$	$\subset [42x+10, 44x+10]$	$14x+3$
G_7	$[0, 2x-1]$	$\subset [50x+12, 52x+12]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_7, add $6x+2$ to every label in G_2, and add $8x+2$ to every label in G_6, and add $10x+4$ to every label in G_3, and add $12x+4$ to every label in G_5, and add $14x+4$ to every label in G_4. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$54x+13$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$52x+13$
G_3	$[10x+4, 12x+3]$	$\subset [28x+8, 30x+8]$	$48x+13$
G_4	$[14x+4, 16x+3]$	$\subset [40x+10, 42x+10]$	$44x+11$
G_5	$[12x+4, 14x+3]$	$\subset [46x+12, 48x+12]$	$34x+9$
G_6	$[8x+2, 10x+1]$	$\subset [50x+12, 52x+12]$	$22x+5$
G_7	$[4x+2, 6x+1]$	$\subset [54x+14, 56x+14]$	$10x+3$

It is easy to see that the labels of $G_1, \ldots, G_7$ are disjoint.

Case 8 $t = 8$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_8$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$62x+15$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$54x+13$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$46x+11$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$38x+9$
G_5	$[0, 2x-1]$	$\subset [34x+8, 36x+8]$	$30x+7$
G_6	$[0, 2x-1]$	$\subset [42x+10, 44x+10]$	$22x+5$
G_7	$[0, 2x-1]$	$\subset [50x+12, 52x+12]$	$14x+3$
G_8	$[0, 2x-1]$	$\subset [58x+14, 60x+14]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_8, add $6x+2$ to every label in G_2, add $8x+2$ to every label in G_7, add $10x+4$ to every label in G_3, add $12x+4$ to every label in G_6, add $14x+4$ to every label in G_4, and add $18x+6$ to every label in G_5. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$62x+15$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$60x+15$
G_3	$[10x+4, 12x+3]$	$\subset [28x+8, 30x+8]$	$56x+15$
G_4	$[14x+4, 16x+3]$	$\subset [40x+10, 42x+10]$	$52x+13$
G_5	$[18x+6, 20x+5]$	$\subset [52x+14, 54x+14]$	$48x+13$
G_6	$[12x+4, 14x+3]$	$\subset [54x+14, 56x+14]$	$34x+9$
G_7	$[8x+2, 10x+1]$	$\subset [58x+14, 60x+14]$	$22x+5$
G_8	$[4x+2, 6x+1]$	$\subset [62x+16, 64x+16]$	$10x+3$

In this case, if $x > 1$, then the label $54x+14$ appears on two vertices. It is the label of the vertex corresponding to v_2 in G_5 and it is the label of the vertex corresponding to v_{4x} in G_6. In G_5, the two vertices adjacent to v_2 are v_1 with label $18x+6$ and v_3 with label $18x+7$. Thus the edge $\{v_1, v_2\}$ has label $36x+8$, which corresponds to length $28x+9$ (in K_{64x+17}), while the edge $\{v_2, v_3\}$ has label $36x+7$, which in turn corresponds to length $28x+10$. Thus if we re-label v_2 in G_5 with $46x+16$, we preserve the two edge lengths and use a label that has not been used on any vertices. This would make the labels of $G_1, \ldots, G_8$ disjoint.

Case 9 $t = 9$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_9$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$70x+17$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$62x+15$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$54x+13$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$46x+11$
G_5	$[0, 2x-1]$	$\subset [34x+8, 36x+8]$	$38x+9$
G_6	$[0, 2x-1]$	$\subset [42x+10, 44x+10]$	$30x+7$
G_7	$[0, 2x-1]$	$\subset [50x+12, 52x+12]$	$22x+5$
G_8	$[0, 2x-1]$	$\subset [58x+14, 60x+14]$	$14x+3$
G_9	$[0, 2x-1]$	$\subset [66x+16, 68x+16]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_9, add $6x+2$ to every label in G_2, add $8x+2$ to every label in G_8, add $10x+4$ to every label in G_3, add $12x+4$ to every label in G_7, add $14x+4$ to every label in G_4, add $18x+6$ to every label in G_6, and add $20x+6$ to every label in G_5. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$70x+17$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$68x+17$
G_3	$[10x+4, 12x+3]$	$\subset [28x+8, 30x+8]$	$64x+17$
G_4	$[14x+4, 16x+3]$	$\subset [40x+10, 42x+10]$	$60x+15$
G_5	$[20x+6, 22x+5]$	$\subset [54x+14, 56x+14]$	$58x+15$
G_6	$[18x+6, 20x+5]$	$\subset [60x+16, 62x+16]$	$48x+13$
G_7	$[12x+4, 14x+3]$	$\subset [62x+16, 64x+16]$	$34x+9$
G_8	$[8x+2, 10x+1]$	$\subset [66x+16, 68x+16]$	$22x+5$
G_9	$[4x+2, 6x+1]$	$\subset [70x+18, 72x+18]$	$10x+3$

In this case, two vertex labels can appear twice. First, $22x+5$ will be the label of v_{4x-1} in G_5 and it will also be the label of v_{4x+1} in G_8. Similarly, if $x > 1$, then $62x+16$ is the label of v_2 in G_6 and it is also the label of v_{4x} in G_7. First we note that the two neighbors of v_2 in G_6 are v_1 with label $18x+6$ and v_3 with label $18x+7$. Thus $\{v_1, v_2\}$ has label $44x+10$ and hence length $28x+9$, and $\{v_2, v_3\}$ has label $44x+9$ and hence length $28x+10$. If we re-label v_2 with $46x+16$, which is a label that has not been used, the $\{v_1, v_2\}$ would have length $28x+10$ and $\{v_2, v_3\}$ would have length $28x+9$. This would eliminate the repeated vertex label $62x+16$ from G_6 while preserving the edge-lengths.

Similarly, we note that the two neighbors of v_{4x+1} in G_8 are v_1 with label $8x+2$ and v_{4x} with label $66x+16$. Thus $\{v_{4x+1}, v_1\}$ has label $14x+3$ which is also its length, and $\{v_{4x+1}, v_{4x}\}$ has label $44x+11$ and hence length $28x+8$. If we re-label v_{4x+1} with $52x+13$, which is a label that has not been used, then $\{v_{4x+1}, v_1\}$ would have length $28x+8$ and $\{v_{4x+1}, v_{4x}\}$ would have length $14x+3$. It is easy to see that the labels of $G_1, \ldots, G_9$ are now disjoint.

Case 10 $t = 10$.
From the γ-labeling of C_{4x+1}, the labels for $G_1, \ldots, G_{10}$ are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$78x+19$
G_2	$[0, 2x-1]$	$\subset [10x+2, 12x+2]$	$70x+17$
G_3	$[0, 2x-1]$	$\subset [18x+4, 20x+4]$	$62x+15$
G_4	$[0, 2x-1]$	$\subset [26x+6, 28x+6]$	$54x+13$
G_5	$[0, 2x-1]$	$\subset [34x+8, 36x+8]$	$46x+11$
G_6	$[0, 2x-1]$	$\subset [42x+10, 44x+10]$	$38x+9$
G_7	$[0, 2x-1]$	$\subset [50x+12, 52x+12]$	$30x+7$
G_8	$[0, 2x-1]$	$\subset [58x+14, 60x+14]$	$22x+5$
G_9	$[0, 2x-1]$	$\subset [66x+16, 68x+16]$	$14x+3$
G_{10}	$[0, 2x-1]$	$\subset [74x+18, 76x+18]$	$6x+1$

We will keep the labels in G_1 as is, add $4x+2$ to every label in G_{10}, add $6x+2$ to every label in G_2, add $8x+2$ to every label in G_9, add $10x+4$ to every label in G_3, add $12x+4$ to every label in G_8, add $14x+4$ to every label in G_4, add $18x+6$ to every label in G_7, add $20x+6$ to every label in G_5, and add $22x+6$ to every label in G_6. The new labels are as follows.

Graph	Labels of A-set	Labels of B-set	Label of c-vertex
G_1	$[0, 2x-1]$	$\subset [2x, 4x]$	$78x+19$
G_2	$[6x+2, 8x+1]$	$\subset [16x+4, 18x+4]$	$76x+19$
G_3	$[10x+4, 12x+3]$	$\subset [28x+8, 30x+8]$	$72x+19$
G_4	$[14x+4, 16x+3]$	$\subset [40x+10, 42x+10]$	$68x+17$
G_5	$[20x+6, 22x+5]$	$\subset [54x+14, 56x+14]$	$66x+17$
G_6	$[22x+6, 24x+5]$	$\subset [64x+16, 66x+16]$	$60x+15$
G_7	$[18x+6, 20x+5]$	$\subset [68x+18, 70x+18]$	$48x+13$
G_8	$[12x+4, 14x+3]$	$\subset [70x+18, 72x+18]$	$34x+9$
G_9	$[8x+2, 10x+1]$	$\subset [74x+18, 76x+18]$	$22x+5$
G_{10}	$[4x+2, 6x+1]$	$\subset [78x+20, 80x+20]$	$10x+3$

As in the previous case, two vertex labels can appear twice. First, $22x+5$ will be the label of v_{4x-1} in G_5 and it will also be the label of v_{4x+1} in G_9. Similarly, if $x > 1$, then $70x+18$ is the label of v_2 in G_7 and it is also the label of v_{4x} in G_8. First we note that the two neighbors of v_2 in G_7 are v_1 with label $18x+6$ and v_3 with label $18x+7$. Thus $\{v_1, v_2\}$ has label $52x+12$ and hence length $28x+9$, and $\{v_2, v_3\}$ has label $52x+11$ and hence length $28x+10$. If we re-label v_2 (in G_7) with $46x+16$, which is a label that has not been used, then $\{v_1, v_2\}$ would have length $28x+10$ and $\{v_2, v_3\}$ would have length $28x+9$. This would eliminate the repeated vertex label $70x+18$ from G_7 while preserving the edge-lengths.

Eliminating the repeated label $22x+5$ will be slightly more challenging. Note that the two neighbors of v_{4x+1} in G_9 are v_1 with label $8x+2$ and

v_{4x} with label $74x + 18$. Thus $\{v_{4x+1}, v_1\}$ has label $14x + 3$ which is also its length, and $\{v_{4x+1}, v_{4x}\}$ has label $52x + 13$ which corresponds to length $28x + 8$. First, we re-label v_{4x+1} in G_9 with $60x + 15$. Thus $\{v_{4x+1}, v_1\}$ would now have label $52x + 13$, which is length $28x + 8$, and $\{v_{4x+1}, v_{4x}\}$ would have label $14x + 3$, which is its new length. Unfortunately, $60x + 15$ is also the label of v_{4x+1} in G_6. The two neighbors of v_{4x+1} in G_6 are v_1 with the label $22x + 6$ and v_{4x} with the label $64x + 16$. Here we re-label v_{4x+1} with $26x + 7$ (which has not been used). This induces the label $4x + 1$ on the edge $\{v_{4x+1}, v_1\}$ and the label $38x + 9$ on $\{v_{4x+1}, v_{4x}\}$. Thus the labels on the two edges are simply switched. The labels of $G_1, \ldots, G_{10}$ are now disjoint. □

Using Theorems 1 and 9, we get the following.

Corollary 10 *Let $G = tC_{4x+1}$ where $t \leq 10$ and let $n = (4x + 1)t$. Then there exists a cyclic G-decomposition of K_{2n+1}.*

4 Concluding Remarks

Unfortunately, the approach used in the proof of Theorem 9 will not yield additional ρ-labelings of tC_{4x+1} for $t > 10$ (as can be seen in the complications in the proof of case 10 above). Moreover, this approach does not work with any of the currently known γ-labelings of cycles of the length $4x+3$ to produce ρ-labelings of tC_{4x+3}. It is clear that a different approach will be necessary to produce additional results of this type. Nonetheless, we note that our results here, along with results from [3], [9] and [16] among others, provide further evidence in support of the following conjecture of El-Zanati and Vanden Eynden.

Conjecture 11 *Every 2-regular graph has a ρ-labeling.*

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