New weighing matrices of order $2n$ and weight $2n-5$

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Abstract

In this paper we find ten new weighing matrices of order $2n$ and weight $2n-5$ constructed from two circulants, by forming a conjecture on the locations of the five zeros in a potential solution. Establishing patterns for the locations of zeros in sequences that can be used to construct weighing matrices seems to be a worthwhile path to explore, as it reduces significantly the computational complexity of the problem.

Keywords: Weighing matrices, algorithm, patterns, locations of zeros.

MSC classification: 05B20, 62K05.

1 Introduction

A weighing matrix $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore W satisfies $WW^t = kI_n$. The number k is called the weight of W. Weighing matrices have been studied extensively, see [6] and references therein. A well-known necessary condition for the existence of $W(2n,k)$ matrices states that if there exists a $W(2n,k)$ matrix with n odd, then $k < 2n$ and k is the sum of two squares. In this paper we are focusing on $W(2n,k)$ constructed from two circulants. The two circulants construction for weighing matrices is described in the theorem below, taken from [5].

Theorem 1 *If there exist two circulant matrices A_1, A_2 of order n, with $0, \pm 1$ elements, satisfying $A_1 A_1^t + A_2 A_2^t = f I_n$ and f is an integer, then there exists a $W(2n,f)$, given as*

$$W(2n,f) = \begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_1^t \end{pmatrix} \quad or \quad W(2n,f) = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

2 Exhaustive searches for $n \leq 11$

We wrote a bash shell script metaprogram to generate via the Maple `CodeGeneration` package the C programs to perform exhaustive searches for $W(2n,k)$ constructed from two circulants whose first rows are given by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. We denote by a and b the sums of these two sequences, i.e. $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_n$. An exhaustive search in the $2n$ triadic $\{-1, 0, +1\}$ variables $a_1, \ldots, a_n, b_1, \ldots, b_n$ corresponds roughly to an exhaustive search on $3.17n$ binary $\{-1, +1\}$ variables, as can be seen by solving the equation $3^{2n} = 2^x$ for x.

The results of the exhaustive searches for all $n \leq 11$ and for all k such that $1 \leq k \leq 2n - 1$ are summarized in the following table. The values of $k \leq 22$ for which the Diophantine equation $a^2 + b^2 = k$ does not have a solution, (i.e. $k = 3, 6, 7, 11, 12, 14, 15, 19, 21, 22$) do not appear in the table.

The symbol – signifies that $k > 2n - 1$ and so $W(2n,k)$ is undefined. The symbol 0 signifies that there are no $W(2n,k)$ constructed from two circulants even though the diophantine equation $a^2 + b^2 = k$ has solutions.

$n \downarrow$ $k \rightarrow$	1	2	4	5	8	9	10	13	16	17	18	20
2	8	16	–	–	–	–	–	–	–	–	–	–
3	12	36	72	72	–	–	–	–	–	–	–	–
4	16	64	176	256	–	–	–	–	–	–	–	
5	20	100	400	800	800	200	–	–	–	–	–	–
6	24	144	696	1440	1728	0	2592	–	–	–	–	–
7	28	196	1232	3136	5488	3136	8624	1176	–	–	–	–
8	32	256	1760	4096	17152	8192	30720	0	–	–	–	–
9	36	324	2592	4536	27216	0	89424	23328	15552	1944	–	–
10	40	400	3560	9600	38400	23200	217600	102400	54400	6400	10400	–
11	44	484	4840	9680	67760	106480	232320	387200	214896	81312	48400	38720

The computational results of these exhaustive searches, in conjunction with the results appearing in [5, 6] support the following conjecture.

Conjecture 1 For any permissible odd value of n, there is a weighing matrix $W(2n, 2n-5)$ constructed from two circulants.

A permissible value of n is a value such that the Diophantine equation $a^2 + b^2 = 2n - 5$ has solutions.

A number of other relevant conjectures on weighing matrices appears in [6].

3 New weighing matrices of order $2n$ and weight $2n-5$

The paper [6] contains a comprehensive list of open cases for $W(2n, k)$ constructed from two circulants. Here we focus on $W(2n, 2n-5)$ i.e. weighing matrices of weight $2n-5$, constructed from two circulants. Results on the structure of weighing matrices $W(n, n-2)$, $W(n, n-3)$, $W(n, n-4)$ (not necessarily constructed from two circulants) are given in [2]. It is well-known that if the Diophantine equation $a^2 + b^2 = k$ has no solutions, then there do not exist $W(2n, k)$ constructed from two circulants. Therefore we focus our attention to the permissible odd values of n.

The basic premise of our approach is that by analyzing the exhaustive search results for $W(2n, 2n-5)$ for all initial permissible odd values of n, we should be able to make accurate predictions for the location of the 5 zeros in a solution. This would transform a problem with $2n$ triadic $\{-1, 0+1\}$ variables to a problem with $2n-5$ binary $\{-1, +1\}$ variables, which is much easier to solve.

Conjecture 1 will be refined below (in conjecture 2) in order to find some new weighing matrices $W(2n, 2n-5)$ constructed from two circulants.

The initial permissible odd values of n for which exhaustive search for weighing matrices $W(2n, 2n-5)$ are feasible within reasonable time with serial C programs are $n = 5, 7, 9, 11$. We performed these exhaustive searches and displayed the result following the format $[a_1, \ldots, a_n, b_1, \ldots, b_n]$. By observing carefully these 4 exhaustive search results, we see that solutions with 4 consecutive zeros occur frequently and moreover these 4 consecutive zeros are located as shown below

$$\begin{array}{ccccccccccc} \star & \ldots & \star & 0 & 0 & 0 & 0 & \star & \ldots & \star \\ a_1 & \ldots & a_{n-2} & a_{n-1} & a_n & b_1 & b_2 & b_3 & \ldots & b_n \end{array} \tag{1}$$

where $\star$ denotes a binary $\{-1, +1\}$ variable, with one exception corresponding to the fifth zero. To predict accurately the location of the fifth zero in a solution string with 4 consecutive zeros as shown above, we remark that the fifth zero can be located in any of the $n-2$ places in the vector of the a variables or in any of the $n-2$ places in the vector of the b variables. Therefore the fifth zero can only be located in any one of $2n-4$ places in a potential solution vector.

To predict accurately the location of the fifth zero, we wrote a bash shell script metaprogram to generate via the Maple `CodeGeneration` package the $2n-4$ serial C programs that place the fifth zero in all possible $2n-4$ places. These programs operate with binary variables and therefore are much more efficient that the original exhaustive search programs on triadic variables.

By observing carefully the exhaustive search results of all these programs, we see that the fifth zero can actually be located in only 2 places in the solution vector, out of the $2n-4$ possibilities. The location of the fifth zero is determined as either the middle of the vector $a_1 \ldots a_{n-2}$ or the middle of the vector $b_3 \ldots b_n$.

In summary, when the 4 zeros are located as described in (1) then the fifth zero can only appear in precisely two different places, as detailed in the two configurations below

$$\underbrace{a_1 \star \ldots \star}_{\frac{n-3}{2}\text{ terms}} \quad 0 \quad \underbrace{\star \ldots \star a_{n-2}}_{\frac{n-3}{2}\text{ terms}} \quad 0 \quad 0 \quad 0 \quad 0 \quad \underbrace{b_3 \star \ldots \star b_n}_{n-2\text{ terms}} \tag{2}$$

$$\underbrace{a_1 \star \ldots \star a_{n-2}}_{n-2\text{ terms}} \quad 0 \quad 0 \quad 0 \quad 0 \quad \underbrace{b_3 \star \ldots \star}_{\frac{n-3}{2}\text{ terms}} \quad 0 \quad \underbrace{\star \ldots \star b_n}_{\frac{n-3}{2}\text{ terms}} \tag{3}$$

where $a_i, b_j, \star$ denote binary $\{-1, +1\}$ variables.

In the sequel, we work with configurations of type (2) exclusively.

The following conjecture implies conjecture 1.

Conjecture 2: For all permissible odd values of n, there is a weighing matrix $W(2n, 2n-5)$ constructed from two circulants, such that the 5 zeros are located according to the pattern of configuration (2) given above.

Using conjecture 2, we transform the problem of looking for a weighing matrix $W(2n, 2n-5)$ constructed from two circulants, from a problem in $2n$ triadic variables into a problem in $2n-5$ binary variables. The computational gain is expressed by the ratio $\frac{3^{2n}}{2^{2n-5}}$ which is equal to $32(2.25)^n$. This allows us to tackle previously computationally intractable values of n, see [2, 6].

4 Algorithms and results

To verify conjecture 2 for the permissible odd values of $n \in \{5, 7, 9, 11, 15, 17, 21, 23, 25, 27\}$ we used Maple to automatically generate serial C programs.

Maple is an ideal tool to generate reliable C code for this type of problems, via its `CodeGeneration` package. Moreover, using Maple one can easily and accurately describe the combinatorial and diophantine characteristics of these problems.

To verify conjecture 2 for the permissible odd values of $n \in \{29, 33, 35, 39, 43, 45\}$ we parallelized the corresponding automatically generated serial C programs. We constructed ten new weighing matrices from two circulants

$$W(2 \cdot 23, 41),\ W(2 \cdot 25, 45),\ W(2 \cdot 27, 49),\ W(2 \cdot 29, 53),\ W(2 \cdot 33, 61),$$

$$W(2 \cdot 35, 65),\ W(2 \cdot 39, 73),\ W(2 \cdot 43, 81),\ W(2 \cdot 45, 85),\ W(2 \cdot 47, 89).$$

See [1, 2, 3, 6] for lists of known and unknown weighing matrices. Here are the first rows of the new weighing matrices $W(2n, 2n-5)$ constructed from two circulants that we found, in the format $a_1, \ldots, a_n, b_1, \ldots, b_n$.

```
W(2*23,41) solution

-1 -1 -1 -1 -1 1  1 -1 1 -1 0  1  1  1 -1 -1  1 -1 -1 1 -1 0  0

 0  0 -1 -1  1 1 -1 -1 1  1 1 -1 -1 -1 -1 -1 -1  1 -1 1 -1 1 -1

W(2*25,45) solution

1 1  1  1 1 -1 -1 1 1 -1 1  0 1 -1  1 -1 -1 -1 -1 1 1  1  1 0 0

0 0 -1 -1 1 -1 -1 1 1  1 1 -1 1  1 -1  1 -1  1 -1 1 1 -1 -1 1 1

W(2*27,49) solution

1 1  1  1  1 1 -1 -1 -1 1 -1 -1  0 -1 -1 1  1 -1 -1  1 -1  1 -1
1 -1 0 0

0 0 -1 -1 -1 1  1 -1  1 1 -1 -1 -1 -1  1 1 -1 -1 -1 -1  1 -1 -1
1 -1 1 -1

W(2*29,53) solution

-1 -1 -1  1 1  1 -1 -1 1 -1 -1 1 -1 0 -1 -1 1  1 1 -1 -1  1 1 1
 1 1 1 0  0

 0  0 -1 -1 1 -1  1  1 1 -1  1 1  1 1 -1  1 1 -1 1 -1  1 -1 1 1
 1 -1 1 1 -1
```

W(2*33,61) solution

-1 -1 -1 -1 -1 1 -1 -1 -1 -1 1 -1 1 -1 1 0 -1 1 1 -1 1 -1 -1 -1
-1 1 1 1 1 1 -1 0 0

0 0 -1 -1 1 -1 -1 -1 1 -1 1 1 1 -1 -1 -1 1 1 1 -1 -1 1 1 -1
-1 1 -1 1 -1 -1 1 -1 -1

W(2*35,65) solution

-1 -1 -1 -1 1 1 -1 1 1 1 -1 1 1 1 1 1 0 1 1 -1 -1 1 -1 1 1 1 -1
-1 1 1 1 -1 1 0 0

0 0 1 -1 -1 -1 -1 1 1 -1 1 -1 -1 -1 1 -1 1 -1 -1 1 1 1 1 -1 -1
1 -1 1 -1 1 1 -1 1 1 1

W(2*39,73) solution

-1 -1 1 1 1 -1 -1 1 -1 1 -1 1 1 1 1 -1 -1 1 0 1 1 1 1 1 1 1 1 -1 1
-1 -1 -1 1 -1 1 1 -1 0 0

0 0 -1 1 -1 -1 1 1 1 -1 1 1 1 1 -1 -1 1 -1 -1 1 1 -1 -1 1 1 1 1 -1
1 -1 1 1 -1 -1 -1 -1 1 -1 1

W(2*43,81) solution

-1 -1 -1 1 1 -1 1 -1 -1 1 1 -1 1 -1 -1 -1 1 -1 1 1 0 1 -1 1 1 1 -1
1 -1 1 1 -1 -1 -1 1 -1 -1 1 1 1 -1 0 0

0 0 -1 -1 1 -1 -1 -1 -1 1 -1 1 1 -1 1 1 1 -1 -1 -1 1 1 1 -1 1 1 1
-1 1 -1 -1 1 1 1 1 1 1 -1 1 1 1 1 1

W(2*45,85) solution

-1 -1 -1 -1 1 1 1 1 1 -1 -1 -1 -1 1 -1 1 -1 -1 -1
1 -1 0 1 -1 1 1 1 -1 1 1 1 1 -1 1 1 -1 -1 1 1 1 -1
1 -1 0 0

0 0 1 1 -1 1 1 -1 1 1 1 -1 1 1 -1 1 -1 1 1 -1
-1 -1 -1 1 -1 -1 1 1 1 1 -1 1 -1 1 1 1 -1 -1 -1 1 1
1 -1 1 1

W(2*47,89) solution

-1 1 1 -1 -1 1 -1 1 -1 -1 1 -1 -1 1 1 -1 -1 1 1 1 1 1 0 1 1 1 -1 1
-1 -1 -1 -1 1 1 1 1 1 1 -1 -1 1 -1 1 1 1 0 0

0 0 -1 -1 1 -1 1 -1 1 -1 -1 1 1 -1 -1 1 -1 1 -1 1 1 -1 1 -1 1 1 1 -1
-1 1 1 1 1 -1 -1 1 1 1 1 1 1 1 -1 -1 -1 1 -1

The ten weighing matrices $W(2 \cdot 23, 41)$, $W(2 \cdot 25, 45)$, $W(2 \cdot 27, 49)$, $W(2 \cdot 29, 53)$, $W(2 \cdot 33, 61)$, $W(2 \cdot 35, 65)$, $W(2 \cdot 39, 73)$, $W(2 \cdot 43, 81)$, $W(2 \cdot 45, 85)$, $W(2 \cdot 47, 89)$ constructed from two circulants are given here for the first time. These ten weighing matrices are listed as open problems in [6]. These ten weighing matrices have been communicated to the editors of the second edition of the Handbook of combinatorial designs [3]. A number of other solutions for $W(2n, 2n-5)$ constructed from two circulants for $n = 23, 25, 27, 29, 33, 35, 39, 43, 45, 47$ appear in the web page `http://www.cargo.wlu.ca/weighing/` .

We give a summary of the results for weighing matrices $W(2n, 2n-5)$ constructed from two circulants in the following table.

n	$2n$	2n-5	Reference for W(2n,2n-5)
3	6	1	Trivial or [5]
5	10	5	[5]
7	14	9	[5]
9	18	13	[5]*or*[6]
11	22	17	[5]
13	26	21	Does not exist, 21 is not a sum of two squares
15	30	25	[6]
17	34	29	[6]
19	38	33	Does not exist, 33 is not a sum of two squares
21	42	37	[6]
23	46	41	New, Conjecture 2
25	50	45	New, Conjecture 2
27	54	49	New, Conjecture 2
29	58	53	New, Conjecture 2
31	62	57	Does not exist, 57 is not a sum of two squares
33	66	61	New, Conjecture 2
35	70	65	New, Conjecture 2
37	74	69	Does not exist, 69 is not a sum of two squares
39	78	73	New, Conjecture 2
41	82	77	Does not exist, 77 is not a sum of two squares
43	86	81	New, Conjecture 2
45	90	85	New, Conjecture 2
47	94	89	New, Conjecture 2
49	98	93	Does not exist, 93 is not a sum of two squares
51	102	97	Undecided

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6 Conclusion

In this paper we find new weighing matrices $W(2n, 2n-5)$ constructed from two circulants, by establishing simple patterns for the locations of the 5 zeros in a potential solution. Even if the conjectures made in this paper may turn out not to be true in general, their importance lies in the fact that they were the driving force behind the discovery of new results and that they exploit successfully the idea of establishing patterns for the locations of the zeros in weighing matrices constructed from two circulants. It should be worthwhile to try to extend these ideas to discover other patterns in the case of weight $2n-5$ and other weights as well.

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