On Additively Geometric Graphs

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Abstract

Let N and Z denote respectively the set of all nonnegative integers and the set of all integers. A (p, q)-graph $G = (V, E)$ is said to be additively (a, r)-geometric if there exists an injective function $f : V \rightarrow Z$ such that $f^+(E) = \{a, ar, \ldots, ar^{q-1}\}$ where $a, r \in N, r > 1$ and f^+ is defined by $f^+(uv) = f(u) + f(v)$ for all $uv \in E$. If further $f(v) \in N$ for all $v \in V$, then G is said to be additively $(a, r)^*$-geometric. In this paper we characterise graphs which are additively geometric and additively*-geometric.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither multiple edges nor loops. For graph theoretic terminology we refer to Harary [3]. The order and size of G are denoted by p and q respectively.

Several practical problems in real life situations have motivated the study of labelings of the vertices(edges) of graphs with real numbers or subsets of sets, which are required to obey variety of conditions. There is an enormous literature built up on several kinds of labelings of graphs over the past three decades or so. For a survey of various graph labeling problems one may refer to Gallian [2].

Acharya and Hegde [1] introduced the concept of additively arithmetic graphs and multiplicatively geometric graphs. Let $G = (V, E)$ be a graph. For a real valued function f defined on V, let $f^+(uv) = f(u) + f(v)$ and $f^\times(uv) = f(u)f(v)$, for all $uv \in E$. The function f is called an additively arithmetic function if $f^+(E) = \{k, k+d, \ldots, k+(q-1)d\}$ and f is called a multiplicatively geometric function if $f^\times(E) = \{a, ar, \ldots, ar^{q-1}\}$. In this paper we introduce the concept of additively geometric labeling and characterize graphs which admit such a labeling.

2 Main Results

Definition 2.1. A (p, q)-graph $G = (V, E)$ is said to be additively (a, r)-geometric if there exists an injective function $f : V(G) \to Z$ such that $f^+(E(G)) = \{a, ar, ar^2, \ldots, ar^{q-1}\}$ where Z denotes the set of all integers, f^+ is defined by $f^+(uv) = f(u) + f(v)$, for all $uv \in E(G)$ and a, r are positive integers with $r > 1$. If further $f(v) \in N$ for all $v \in V$, then G is said to be additively $(a, r)^*$-geometric. A graph G is said to be additively geometric (additively* geometric) if G is additively (a, r)-geometric (additively $(a, r)^*$-geometric) for some integers a and r.

We observe that if f is an additively (a, r)-geometric numbering of a graph G and k is any positive integer, then $g = kf$ is an additively (ka, r)-geometric numbering of G. Hence any additively (a, r)-geometric graph is additively (ka, r)-geometric for any positive integer k.

Theorem 2.2. *Any tree is additively $(a, r)^*$ - geometric for all a and for all r.*

Proof. Root the tree T at a vertex v with $\deg v \geq 2$. Let n denote the height of the tree. We prove the result by induction on n. Let $n_i, 1 \leq i \leq n$, denote the number of vertices at level i. We order the vertices at level i as $v_{i_1}, v_{i_2}, \ldots, v_{i_{n_i}}$, so that v_{ij} has a child if and only if v_{is} has a child for all $s < j$. Now, if $n = 1$, then $T = K_{1,n_1}$ and $f : V(K_{1,n_1}) \rightarrow N$ defined by $f(v) = 0, f(v_{1,i}) = ar^{i-1}, i = 1, 2, \ldots, n_1$, is an additively (a,r)-geometric numbering of T.

If $n = 2$, we define f for the vertices at level 2 by $f(v_{2,j}) = ar^{n_1+j-1} - f(p(v_{2,j}))$, where $1 \leq j \leq n_2$ and $p(x)$ denote the parent of the vertex x. We claim that $f(v_{2,j}) > f(v_{1,i})$ for all i and j. If $f(v_{2,s}) \leq f(v_{1,t})$ for some s and t,then $f(v_{2,s}) \leq ar^{n_1-1}$. Hence, $ar^{n_1} \leq f(v_{2,s}) + f(p(v_{2,s})) \leq 2ar^{n_1-1}$, so that $r \leq 2$. Since $r \neq 1$, it follows that $r = 2$ and and hence $f(v_{2,s}) + f(p(v_{2,s})) = a2^{n_1}$. Also $p(v_{2,s}) = v_{1,1}$ and $f(p(v_{2,s})) = a$, so that $a2^{n_1} = f(v_{2,s}) + f(v_{1,1}) \leq a2^{n_1-1} + a$, which is a contradiction. Further $f(v_{2,j})$ is a strictly increasing function of j and hence f is injective and is an additively (a,r)-geometric numbering of T.

We now assume that any tree T of height k has an additively (a,r)-geometric numbering f such that $f(v_{m+1,j}) > f(v_{m,i})$ for all i and $j, m = 1, 2, \ldots, k-1$. Let T be a tree with height $k+1$. Let T_1 be the tree of height k obtained by removing all vertices of T at level $k+1$. Let f_1 be an additively (a,r)-geometric numbering of T_1. Now f defined on $V(T)$ by $f(v) = f_1(v)$ if $v \in V(T_1)$ and $f(v_{k+1,j}) = ar^{n_1+n_2+\cdots+n_k+j-1} - f_1(p(v_{k+1,j}))$ is an additively (a,r)-geometric numbering of T. Since the vertex values are increasing, they are nonnegative.

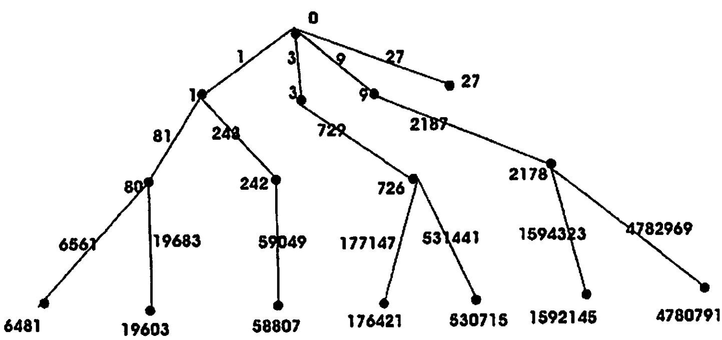

Figure 2.1.
Additively (1,3) - geometric numbering for a tree

Remark 2.3. Trees are arbitrarily additively*-geometric, in the sense that they are additively $(a,r)^*$-geometric for all values of a and r. Further if T is a tree rooted at a vertex v, then there exists an additively (a,r) - geometric numbering of T in which the root is assigned any integer less than $\frac{a}{2}$ as its label.

Theorem 2.4. *The odd cycle C_{2n+1} is additively (a,r)-geometric if and only if a is even. Also if f is an additively (a,r)-geometric numbering of C_{2n+1}, then $f(u)$ is negative for at least one $u \in V(C_{2n+1})$.*

Proof. Let $C_{2n+1} = (v_1, v_2, \dots, v_{2n+1}, v_1)$. Suppose a is even. Define the function $f : V(C_{2n+1}) \to Z$ by

$$f(v_1) = \tfrac{1}{2}(a - ar + ar^2 - ar^3 + ar^4 - \dots - ar^{2n-1} + ar^{2n}),$$
$$f(v_2) = \tfrac{1}{2}(a + ar - ar^2 + ar^3 - ar^4 + \dots + ar^{2n-1} - ar^{2n}),$$
$$f(v_3) = \tfrac{1}{2}(-a + ar + ar^2 - ar^3 + ar^4 - \dots - ar^{2n-1} + ar^{2n}),$$
$$\vdots$$
$$f(v_{2n}) = \tfrac{1}{2}(a - ar + ar^2 - ar^3 + ar^4 - \dots + ar^{2n-2} + ar^{2n-1} - ar^{2n}) \text{ and}$$
$$f(v_{2n+1}) = \tfrac{1}{2}(-a + ar - ar^2 + ar^3 - ar^4 + \dots - ar^{2n-2} + ar^{2n-1} + ar^{2n}).$$

Since $r > 1$ and a is even, it follows that all the $f(v_i)$'s are distinct nonzero integers and $f^+(E(G)) = \{a, ar, ar^2, \dots, ar^{2n}\}$. Hence f is an additively (a,r)-geometric numbering of C_{2n+1}.

To prove the converse, let f be an additively (a,r)-geometric numbering of C_{2n+1} and let $f(v_i) = k_i, 1 \le i \le 2n+1$. Then $\{k_1+k_2, k_2+k_3, \dots, k_{2n}+k_{2n+1}, k_{2n+1}+k_1\} = \{a, ar, ar^2, \dots, ar^{2n}\}$. Without loss of generality, we may assume that $k_1 + k_2 = a$.

Let $k_i + k_{i+1} = ar^{s_{i-1}}, 2 \le i \le 2n$ and $k_{2n+1} + k_1 = ar^{s_{2n}}$, where $\{s_1, s_2, \dots, s_{2n}\} = \{1, 2, \dots, 2n\}$. This system of equations has a unique solution given by

$$k_1 = \tfrac{1}{2}(a - ar^{s_1} + ar^{s_2} - ar^{s_3} + ar^{s_4} - \dots - ar^{s_{2n-1}} + ar^{s_{2n}}),$$
$$k_2 = \tfrac{1}{2}(a + ar^{s_1} - ar^{s_2} + ar^{s_3} - ar^{s_4} + \dots + ar^{s_{2n-1}} - ar^{s_{2n}}),$$
$$\vdots$$
$$k_{2n+1} = \tfrac{1}{2}(-a + ar^{s_1} - ar^{s_2} + ar^{s_3} - ar^{s_4} + \dots + ar^{s_{2n-1}} + ar^{s_{2n}}).$$

Since each k_i is an integer, it follows that a is even. We claim that either k_1 or k_2 is negative.

Now, $k_1 = \frac{a}{2}(1 - rx)$ and $k_2 = \frac{a}{2}(1 + rx)$,where $x = r^{s_1-1} - r^{s_2-1} + r^{s_3-1} - r^{s_4-1} + \dots + r^{s_{2n-1}-1} - r^{s_{2n}-1}$. Since $k_1 \ne k_2$, it follows that $x \ne 0$. Further r is an integer with $r > 1$ and hence $k_1 \ne 0$. Suppose $k_1 > 0$. Then $1 > rx$, so that $x < 0$. Since r and x are integers and $r > 1$, it follows that $r(-x) > 1$. Thus $1 - rx > 2$. Hence $k_1 > a$ and since $k_1 + k_2 = a$, we get $k_2 < 0$. Thus either $f(v_1) < 0$ or $f(v_2) < 0$.

Corollary 2.5. *If a is any even integer, C_{2n+1} is additively (a,r)-geometric but not additively $(a,r)^*$-geometic.*

Theorem 2.6. *Any even cycle is not additively geometric.*

Proof. Suppose the even cycle $C_{2n} = (v_1, v_2, \ldots, v_{2n}, v_1)$ admits an additively (a,r)-geometric numbering f.
Let $f(v_i) = k_i$. We may assume without loss of generality that

$$\begin{aligned} &k_1 + k_2 = a \\ &k_2 + k_3 = ar \\ &k_3 + k_4 = ar^2 \\ &\vdots \\ &k_{2n-1} + k_{2n} = ar^{2n-2} \\ &k_{2n} + k_1 = ar^{2n-1}. \end{aligned}$$

Hence $a - ar + ar^2 - ar^3 + \cdots + ar^{2n-2} - ar^{2n-1} = 0$, so that $1 - r + r^2 - r^3 + \cdots + r^{2n-2} - r^{2n-1} = 0$. Thus $r|1$ which is a contradiction.

Theorem 2.7. *An additively geometric graph G does not contain even cycles.*

Proof. Let f be an additively (a,r) - geometric labeling of G so that $f^+(E(G)) = \{a, ar, ar^2, \ldots, ar^{q-1}\}$. Suppose G contains an even cycle $C_{2n} = (v_1 v_2 \ldots v_{2n} v_1)$. Let $f(v_i) = k_i$. Then we have

$$\begin{aligned} &k_1 + k_2 = ar^{s_1} \\ &k_2 + k_3 = ar^{s_2} \\ &k_3 + k_4 = ar^{s_3} \\ &\vdots \\ &k_{2n-1} + k_{2n} = ar^{s_{2n-1}} \\ &k_{2n} + k_1 = ar^{s_{2n}}, \text{ where } \{s_1, s_2, \ldots, s_{2n}\} \subset \{0, 1, 2, \ldots, q-1\}. \end{aligned}$$

Then as in the proof of Theorem 2.6, we get the equation $ar^{s_1} + ar^{s_3} + \cdots + ar^{s_{2n-1}} = ar^{s_2} + ar^{s_4} + \cdots + ar^{s_{2n}}$ and dividing by $\min\{ar^{s_i}\}$ we get $r|1$, which is a contradiction.

Corollary 2.8. *K_n is additively geometric if and only if $n \leq 3$.*

Corollary 2.9. *$K_{m,n}$ is additively geometric if and only if $m = 1$ or $n = 1$.*

Corollary 2.10. *Any cycle with at least one chord is not additively geometric.*

Corollary 2.11. *Any additively geometric graph which is not a forest contains an odd cycle as an induced subgraph.*

Theorem 2.12. *A connected additively (a,r)-geometric graph G is either a tree or a unicyclic graph.*

Proof. Let f be an additively (a,r)-geometric numbering of G. Suppose G contains two cycles $C_m = (u_1u_2\dots u_mu_1)$ and $C_n = (v_1v_2\dots v_nv_1)$. It follows from Theorem 2.7 that m and n are odd and C_m and C_n cannot have a common edge.

We claim that $|V(C_m)\bigcap V(C_n)| \le 1$. Otherwise there exists two vertices $u,v \in V(C_n)\bigcap V(C_m)$ such that if P_1 and Q_1 are the $u-v$ sections of C_m and P_2 and Q_2 are the $u-v$ sections of C_n, then each of the pairs $(P_1,P_2),(P_1,Q_2)$ and (P_2,Q_1) are internally disjoint paths. Let n_1,m_1,n_2 and m_2 be the lengths of P_1,Q_1,P_2 and Q_2 respectively so that $n_1+m_1=n$ and $n_2+m_2=m$. Since n and m are odd, we may assume with out loss of generality that n_1 and n_2 are even and m_1 and m_2 are odd. Now P_1 followed by P_2^{-1} is an even cycle in G, which is a contradiction. Hence $|V(C_m)\bigcap V(C_n)| \le 1$.

We consider two cases.

Case 1. $|V(C_m)\cap V(C_n)| = 0$.

Since G is connected, we may assume that there exists a path $P_k = (u_1w_1w_2\dots w_{k-2}v_1)$ such that $V(P_k)\bigcap V(C_m) = \{u_1\}$ and $V(P_k)\bigcap V(C_n) = \{v_1\}$.

Let $f(u_i) = k_i, i = 1,2,\dots,m,\quad f(v_i) = l_i, i = 1,2,\dots,n$ and $f(w_i) = s_i, i = 1,2,\dots,k-2$.

If $k=2$, then we obtain the following system of equations.

$k_i + k_{i+1} = ar^{t_i}, i = 1,2,\dots,m-1,$
$k_m + k_1 = ar^{t_m},$
$k_1 + l_1 = ar^{t_{m+1}},$
$l_i + l_{i+1} = ar^{t_{m+i+1}}, i = 1,2,\dots,n-1,$
$l_n + l_1 = ar^{t_{m+n+1}}$, where $t_j \in \{0,1,2,\dots,m+n\}$.

From the above system we get

$$ar^{t_1} - ar^{t_2} + ar^{t_3} - ar^{t_4} + \dots, -ar^{t_{m-1}} + ar^{t_m} - 2ar^{t_{m+1}} + ar^{t_{m+2}} - ar^{t_{m+3}} + \dots - ar^{t_{m+n}} + ar^{t_{m+n+1}} = 0.$$

On dividing the above equation by $\min\{ar^{t_i}\}$, we get a polynomial equation in r with constant term 1 or 2. If the constant term is 1, then $r \mid 1$ which is a contradiction. If the constant term is 2, then $r \mid 2$, so that $r=2$ and by dividing the corresponding equation by 2 we get an equation with constant term 1 which is again a contradiction.

If $k \ge 3$, then we obtain the following system of equations

$k_i + k_{i+1} = ar^{t_i}, i = 1,2,\dots,m-1,$

$k_m + k_1 = ar^{t_m}$,
$k_1 + s_1 = ar^{t_{m+1}}$,
$s_i + s_{i+1} = ar^{t_{m+i+1}}, i = 1, 2, \ldots, k-3$,
$s_{k-2} + l_1 = ar^{t_{m+k-1}}$,
$l_i + l_{i+1} = ar^{t_{m+k+i-1}}, i = 1, 2, \ldots, n-1$,
$l_n + l_1 = ar^{t_{m+n+k-1}}$, where $t_j \in \{0, 1, 2, \ldots, m+n+k-2\}$.

From the above system we get the equation

$ar^{t_1} - ar^{t_2} + ar^{t_3} - ar^{t_4} + \cdots - ar^{t_{m-1}} + ar^{t_m} - 2ar^{t_{m+1}} + 2ar^{t_{m+2}} - 2ar^{t_{m+3}} + \cdots - 2ar^{t_{m+k-2}} + 2ar^{t_{m+k-1}} - ar^{t_{m+k}} + ar^{t_{m+k+1}} - ar^{t_{m+k+2}} + \cdots + ar^{t_{m+n+k-2}} - ar^{t_{m+n+k-1}} = 0$, if k is odd and
$ar^{t_1} - ar^{t_2} + ar^{t_3} - ar^{t_4} + \cdots - ar^{t_{m-1}} + ar^{t_m} - 2ar^{t_{m+1}} + 2ar^{t_{m+2}} - 2ar^{t_{m+3}} + \cdots + 2ar^{t_{m+k-2}} - 2ar^{t_{m+k-1}} + ar^{t_{m+k}} - ar^{t_{m+k+1}} + ar^{t_{m+k+2}} - \cdots - ar^{t_{m+n+k-2}} + ar^{t_{m+n+k-1}} = 0$, if k is even.

In both the cases we get a polynomial equation in r with constant term 1 or 2, which is a contradiction.

Case 2. $|V(C_m) \cap V(C_n)| = 1$.
Take the vertex labels as in case.1 and let $l_1 = k_1$. Then we have

$k_i + k_{i+1} = ar^{t_i}, i = 1, 2, \ldots, m-1$,
$k_m + k_1 = ar^{t_m}$,
$k_1 + l_2 = ar^{t_{m+1}}$,
$l_i + l_{i+1} = ar^{t_{m+i}}, i = 2, 3, \ldots, n-1$,
$l_n + k_1 = ar^{t_{m+n}}$, where $t_j \in \{0, 1, 2, \ldots, m+n-1\}$.

As in Case 1, we get the equation $ar^{t_1} - ar^{t_2} + ar^{t_3} - ar^{t_4} + \cdots - ar^{t_{m-1}} + ar^{t_m} - ar^{t_{m+1}} + \cdots + ar^{t_{m+n-1}} - ar^{t_{m+n}} = 0$ and proceeding as above we get contradiction.

Corollary 2.13. *If G is a disconnected additively (a,r) - geometric graph, then each component of G is either a tree or a unicyclic graph with unique odd cycle.*

Theorem 2.14. *Let G be a connected unicyclic graph with unique odd cycle C_{2m+1} and $G \neq C_{2m+1}$. Then G is additively (a,r)-geometric if and only if a is even or a is odd and r is even. If f is an additively (a,r)-geometric numbering of such a graph G, then $f(u)$ is negative for at least one $u \in V(G)$.*

Proof. Let $C_{2m+1} = (v_1 v_2 \ldots v_{2m+1} v_1)$. Let $v_{i,1}, v_{i,2}, \ldots, v_{i,t}$ be the vertices on C_{2m+1} with degree ≥ 3 and let T_j be the tree rooted at $v_{i,j}$ and let k_j be the height of $T_j, 1 \leq j \leq t$. Let $u^{(j)}_{i,1} u^{(j)}_{i,2} u^{(j)}_{i,3} \ldots u^{(j)}_{i,n^{(j)}_i}$ be the vertices of T_j at level $i, 1 \le i \le k_j$.Then the tree T_j has $n_1^{(j)} + n_2^{(j)} + \cdots + n_{k_j}^{(j)} = N_j$ vertices other than $v_{i,j}$. Hence the size of G is $2m+1+N_1+N_2+\cdots+N_t$. We now define $f : V(G) \to Z$ as follows.

$$f(v_1) = \tfrac{1}{2}(ar - ar^2 + ar^3 - ar^4 + ar^5 - \cdots + ar^{2m-1} - ar^{2m} + ar^{2m+1}),$$
$$f(v_2) = \tfrac{1}{2}(ar + ar^2 - ar^3 + ar^4 - ar^5 + \cdots - ar^{2m-1} + ar^{2m} - ar^{2m+1}),$$
$$f(v_3) = \tfrac{1}{2}(-ar + ar^2 + ar^3 - ar^4 + ar^5 - \cdots + ar^{2m-1} - ar^{2m} + ar^{2m+1}),$$
$$\vdots$$
$$f(v_{2m}) = \tfrac{1}{2}(ar - ar^2 + ar^3 - ar^4 + ar^5 - \cdots + ar^{2m-1} + ar^{2m} - ar^{2m+1}),$$
$$f(v_{2m+1}) = \tfrac{1}{2}(-ar + ar^2 - ar^3 + ar^4 - ar^5 + \cdots - ar^{2m-1} + ar^{2m} + ar^{2m+1}),$$
$$f(u_{1,1}^{(1)}) = a - f(v_{i,1})$$
$$f(u_{1,s}^{(1)}) = ar^{2m+s} - f(v_{i,1}), s = 2, 3, \ldots, n_1^{(1)}$$
$$f(u_{i,s}^{(1)}) = ar^{2m+n_1^1+n_2^1+\cdots+n_{i-1}^1} + s - f(p(u_{i,s}^{(1)})), s = 1, 2, \ldots, n_i^{(1)}$$
$$i = 2, 3, \ldots, k_1.$$

For $j = 2, 3, \ldots, t$, $f(u_{1,s}^{(j)}) = ar^{2m+N_1+N_2+\cdots+N_{j-1}+s} - f(v_{i,j}), s = 1, 2, \ldots, n_1^{(j)}$
$f(u_{i,s}^{(j)}) = ar^{2m+N_1+N_2+\cdots+N_{j-1}+n_1^{(j)}+n_2^{(j)}+\cdots+n_{i-1}^{(j)}+s} - f(p(u_{i,s}^{(j)})), s = 1, 2, \ldots, n_i^{(j)}$
$i = 2, 3, \ldots, k_j$.

Then $f(v_i)$ is a polynomial in r with coefficients $+\frac{1}{2}a$ or $-\frac{1}{2}a$. Also we may find that the other vertex values are also polynomial expressions in r and each contains terms in $r, r^2, r^3, \ldots, r^{2m+1}$ with coefficients $+\frac{1}{2}a$ or $-\frac{1}{2}a$ and other terms in other powers of r with coefficients $+a$ or $-a$. So if we equate $f(w_i)$ and $f(w_j)$ for any two vertices w_i and w_j of G, we get an equation in the form $g(r) = 0$ where $g(r)$ is a polynomial in r with constant term 1, which is a contradiction. Hence f is injective.

If a is even or if a is odd and r is even, then all the vertex values are distinct integers and $f^+(E(G)) = \{a, ar, ar^2, \ldots, ar^{2m+N_1+N_2+\cdots+N_t}\}$. Hence f is an additively (a, r) - geometric numbering for G.

To prove the converse, suppose that there exists an additively (a, r) - geometric numbering f for G. Let $ar^{s_1} = min\{f^+(e) : e \in E(C_{2m+1})\}$. With out loss of generality assume that $f(v_1) + f(v_2) = ar^{s_1}$. Let

$$f(v_2) + f(v_3) = ar^{s_2},$$
$$f(v_3) + f(v_4) = ar^{s_3},$$
$$\vdots$$
$$f(v_{2m}) + f(v_{2m+1}) = ar^{s_{2m}} \text{ and}$$
$$f(v_{2m+1}) + f(v_1) = ar^{s_{2m+1}}, \text{ where } \{s_1, s_2, \ldots, s_{2m+1}\} \subset \{0, 1, 2, \ldots, q-1\}.$$

Now proceeding as in Theorem 2.4, it can be proved that either $f(v_1) < 0$ or $f(v_2) < 0$.

The above theorems lead to the following characterisation of additively geometric and additively* geometric connected graphs.

Theorem 2.15. *A connected graph G is additively geometric if and only if G is either a tree or a unicyclic graph in which the unique cycle is odd.*

Theorem 2.16. *A connected graph G is additively* geometric if and only if G is a tree.*

We now proceed to characterise disconnected graphs which are additively geometric and additively*-geometric.

Theorem 2.17. *Let G be a disconnected graph in which each component is either a tree or a unicyclic graph with unique odd cycle. Then for all even a, $a \geq 4$, there exists an additively (a, r) - geometric numbering f of G such that $f(u)$ is positive if u is a vertex of a tree component and $f(u)$ is negative for at least one vertex u of each cycle.*

Proof. Let G_1, G_2, $\ldots$, G_k be the components of the graph G and let $q_1 \leq q_2 \leq \cdots \leq q_k$. Let a be even and $a \geq 4$. Let f_i be an additively (a, r) - geometric numbering of G_i, $1 \leq i \leq k$, obtained using Theorem 2.2, Theorem 2.4 and Theorem 2.14 in such a way that $f_i(v_i)$ is positive if v_i is the root of the tree G. (Such a labeling exists by Remark 2.3). Define $f : V(G) \to Z$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1) \\ r^{q_1+q_2+\cdots+q_{i-1}} f_i(v) & \text{if } v \in V(G_i),\ 2 \leq i \leq k. \end{cases}$$

Then f is injective and $f^+(E(G)) = \{a, ar, ar^2, \ldots, ar^{q_1+q_2+\cdots+q_k-1}\}$. Hence f forms an additively (a, r) - geometric numbering of G. It follows from Theorem 2.2 that if G is a forest, then all the vertex values are positive. Otherwise, it follows from Theorem 2.4 and Theorem 2.14 that at least one vertex value is negative.

Thus we have the following theorems giving a characterisation of disconnected additively geometric and additively* geometric graphs.

Theorem 2.18. *A disconnected graph G is additively geometric if and only if each component of G is either a tree or a unicyclic graph with odd cycle.*

Theorem 2.19. *A disconnected graph G is additively* geometric if and only if G a forest.*

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