Group Divisible Designs With Two Groups and Block Size Five With Fixed Block Configuration

Spencer P. Hurd*
The Citadel, Dept. Math/CS, Charleston, SC, 29409
hurds@citadel.edu

Nutan Mishra
Dept. Math. Stastist., Univ. South Alabama, Mobile, AL
nmishra@usouthal.edu

Dinesh G. Sarvate
College of Charleston, Dept. Math, Charleston, SC, 29424
sarvated@cofc.edu

Abstract

We present constructions and results about GDDs with two groups and block size five in which each block has Configuration (s, t), that is, in which each block has exactly s points from one of the two groups and t points from the other. After some results for a general k, s and t, we consider the $(2, 3)$ case for block size 5. We give new necessary conditions for this family of GDDs and give minimal or near-minimal index examples for all group sizes $n \geq 4$ except for $n = 24s + 17$.

1 Introduction

A group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k-element subsets of a v-set $\mathbf{X}$ called blocks which satisfies the following properties: each point of $\mathbf{X}$ appears in r of the b blocks; the $v = nm$ elements of $\mathbf{X}$ are partitioned into m subsets (called groups) of size n each; points within the same group are called first associates of each other and appear together in

*The first author wishes to thank The Citadel Foundation for its support during this research.

λ_1 blocks; any two points not in the same group are second associates and appear together in λ_2 blocks. [15], [17].

Designs of the type discussed here are known as GDD's as well as group divisible PBIBD's (partially balanced incomplete block designs). In [15] GDD refers exclusively to the case when $\lambda_1 = 0$, and if $\lambda_1 \neq 0$, then PBIBD is used [17]. PBIBD's were introduced as generalizations of BIBD's (balanced incomplete block designs). BIBD's are known to be universally optimal, and the optimality of PBIBD's with two groups is established in [3], and, in the extensive tables of PBIBDs given in [4], very few non-trivial examples with two groups are listed there. PBIBD's are applied in plant breeding work [14] and in group testing [5]. For further readings on two class PBIBDs see Chapter 11 of [16], and for extensive cross-connections and an introduction to other types of PBIBD's see [17] .

In [8] the authors there settled the existence problem for group divisible designs with first and second associates with block size $k = 3$ and with m groups each of size n with $m, n \geq 3$. The problem of necessary and sufficient conditions for $m = 2$ or $n = 2$, and block size three, was established in [7]. Similar partial results were established for GDDs with block size four in [9]. [10], and [11].

The purpose of this note is to establish similar results for GDDS with block size five and two groups. To this end, we consider designs which we denote by Configuration (s, t). These are GDDs for which each block intersects one of the groups in exactly s point (and hence intersects the other group in $t = k - s$ points). We consider in detail the $(2, 3)$ case with block size 5 in this paper. This case achieves the greatest separation (Theorem 5) between indices when $\lambda_2 > \lambda_1$, namely,

$$\frac{\lambda_2}{\lambda_1} = \frac{3(n-1)}{2n}.$$

The equation implies that, for any given n, there is a least value for the pair (λ_1, λ_2) and any other GDD with that n and configuration will have indices $(w\lambda_1, w\lambda_2)$ for some positive integer w. Consequently, we focus on constructing such a "minimal" GDD since we may then say that the necessary conditions are sufficient for the existence for any indices with that n and configuration. We describe as near-minimal a design which has indices exactly twice the minimal size.

In what follows, we construct minimal or near-minimal examples for all GDD $(n, 2, 5; \lambda_1, \lambda_2)$ except for $n = 24s + 17$, for which we have a design with indices 4 times the minimal values. Our complete results from Section 2 are summarized in the table below.

Configuration $(2,3)$	Summary Of minimality Results
$n \equiv 1 \pmod 6$	all minimal
$n \equiv 2 \pmod 6$	$n = 8$, minimal $n \equiv 12s+2$, minimal $n \equiv 12s+8$, near-minimal
$n \equiv 3 \pmod 6$	all minimal
$n \equiv 4 \pmod 6$	$n = 4$, $12s+10$, minimal $n \equiv 12s+4$, near-minimal
$n \equiv 5 \pmod 6$	$n = 5$, $12s+11$, minimal $n \equiv 24s+5$, near-minimal $n \equiv 24s+17$, w $= 4$
$n \equiv 0 \pmod 6$	$n = 12s+6$, minimal $n = 12s$, near-minimal

It is well-known [17] for these designs that the replication number r and number of blocks b satisfy:

$$r = (\lambda_1(n-1) + n\lambda_2)/4$$
$$b = n(\lambda_1(n-1) + n\lambda_2)/10 = 2nr/5.$$

These two necessary conditions on b and r determine possibilities for the parameter n and the indices λ_1 and λ_2. For example, if $\lambda_1 \equiv 1 (\mathrm{mod}\, 20)$ and $\lambda_2 \equiv 0 (\mathrm{mod}\, 20)$ then $n \equiv 1, 5 (\mathrm{mod}\, 20)$.

Example 1 *We show a GDD$(4,2,5;8,9)$, the smallest (fewest blocks) Configuration $(2,3)$ GDD. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c,d\}$. Then the 24 blocks are listed in the array below.*

$$\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
d & d & d & c & c & c & b & b & b & a & a & a \\
a & b & c & a & b & d & a & c & d & b & c & d \\
 & & & & & & & & & & & \\
a & a & a & a & a & a & a & a & a & b & b & b \\
b & b & b & b & b & b & c & c & c & c & c & c \\
c & c & c & d & d & d & d & d & d & d & d & d \\
1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 3 & 2 & 2 & 3 \\
2 & 3 & 3 & 2 & 4 & 4 & 3 & 4 & 4 & 3 & 4 & 4
\end{array}.$$

There are (at least) two other necessary conditions:

Theorem 1 *For any GDD$(n,2,5;\lambda_1,\lambda_2)$,*
(a) $b \geq \max\{2r - \lambda_1, 2r - \lambda_2)$;
(b) $2n\lambda_2 \leq 3(n-1)\lambda_1$.

Proof For part (a), consider the set of blocks containing points x and y. There are r blocks containing x and $r-\lambda_i$ blocks which contain y and do not contain x. So there are at least $2r-\lambda_i$ blocks. For part b, let b_5 be the number of blocks with 5 points from the same group, b_4 the number of blocks from with 4 points from one group and the $5th$ from the other group, and let b_3 denote the number of blocks with 3 points from one of the groups. Counting the contribution of these blocks towards the number of pairs of points from the same group in blocks together gives: $10b_5+6b_4+4b_3=2\lambda_1\binom{n}{2}=n(n-1)\lambda_1$. Counting the pairs of points from different groups gives $4b_4+6b_3=n^2\lambda_2$. By subtraction we have

$2b_3-10b_5=n^2\lambda_2-n^2\lambda_1+n\lambda_1\leq 2b_3\leq 2b=n[(n-1)\lambda_1+n\lambda_2]/5.$

$5n^2\lambda_2-5n^2\lambda_1+5n\lambda_1\leq n^2\lambda_1-n\lambda_1+n^2\lambda_2.$

$2n\lambda_2\leq 3(n-1)\lambda_1.$ ∎

Condition (b) shows that while $\lambda_2\geq\lambda_1$ is possible, it turns out that we always have $\lambda_2<1.5\lambda_1$. The inequality in (b) is sharp since the extreme bounds for λ_2 are achieved by the Configuration $(2,3)$ designs in Section 2. The following theorem is a direct application of Theorem 1.

Theorem 2 *The family GDD$(n,2,5;2u,3uv)$ does not exist for any integers $u,v>0$.*

In our notation, $s+t=k$, and if each block has Configuration (s,t), then the number of blocks with s points from the first group is exactly the number of blocks with t points from the second group (it is the same set of blocks). It is convenient to state this, and a bit more, as a theorem.

Theorem 3 *Suppose a GDD$(n,2,k;\lambda_1,\lambda_2)$ has Configuration (s,t). Then the number of blocks with s points (respectively t) from the first group is equal to the number of blocks with s points (respectively t) from the second group. Consequently, for any s and t, the number of blocks b is necessarily even.*

Proof. Let A and B denote the two groups and A_s and A_t (B_s and B_t, respectively) denote the number of blocks from group A (resp., group B) with s points and t points. Note $A_s+A_t=b=B_s+B_t$, the number of blocks. The number of pairs of points contributing to λ_1 from group A is $\binom{s}{2}A_s+\binom{t}{2}A_t=\binom{n}{2}\lambda_1$. The corresponding number from group B is $\binom{s}{2}B_s+\binom{t}{2}B_t=\binom{n}{2}\lambda_1$. We note that $B_s=A_t$ and $B_t=A_s$ since they count the same blocks. Thus by substitution and subtraction, $A_s\left[\binom{s}{2}-\binom{t}{2}\right]=A_t\left[\binom{s}{2}-\binom{t}{2}\right]$ and the result follows. ∎

Theorem 4 *For any GDD$(n,2,k;\lambda_1,\lambda_2)$ with Configuration (s,t), the second index is given by $\lambda_2=\frac{\lambda_1(n-1)}{n}\left[\frac{k(k-1)-2\beta}{2\beta}\right]$ where $\beta=\binom{s}{2}+\binom{t}{2}$.*

Proof. For these designs the equation $bk = vr$ becomes $bk = 2nr = 2n\{\lambda_1(n-1)+n\lambda_2\}/(k-1)$. Counting the contribution to the first index by same-group pairs, we note that there are β pairs per block. Hence $\beta b = 2\lambda_1\binom{n}{2}$. Eliminating b from the last two equations gives the result. ∎

We close this section with some combinatorial comments. With N as the incidence matrix of a block design, the following is true for a GD design: $|NN'| = rk(rk - v\lambda_2)^{(m-1)}(r - \lambda_1)^{m(n-1)}$. A GD design is said to be singular (SGD) if $r = \lambda_1$, semiregular (SRGD) if $r > \lambda_1$ and $rk = v\lambda_2$, or regular (RGD) if $r > \lambda_1$ and $rk > v\lambda_2$.The next two theorems are well known.

Theorem 5 *For any prime block size, in particular for $k = 5$, singular GDD do not exist.*

Proof. For the singular class, a block intersects an entire group or misses it entirely. It is well-known [15] that that a singular GD design is always derivable from corresponding BIBD, by replacing each treatment by a group of n-treatments. Conversely, if we collapse each group to a point, the resulting blocks are those of a BIBD. Thus there is a one-to-one correspondence between existence of a BIB design and a singular GD design. Obviously the block size of the singular GD design is nk if the blocksize of the BIB design is k Thus block size of a singular GD is always a composite number. Hence singular GDDs with $k = 5$ do not exist. ∎

Theorem 6 *An SRGD design with $k = 5$ exists if and only if $m = 5$ and $\lambda_1 = 0$*

Proof. Let d be a SRGD design with $k = 5$, then the fact that, for a SRGD design, k is divisible by m, implies that m has to be equal to five. And hence each block contains a unique treatment from every group resulting in $\lambda_1 = 0$. Let d be a GD design with $k = 5$, $m = 5$ and $\lambda_1 = 0$, then by definition, it is a SRGD design. ∎

In view of Theorem 5 and Theorem 6, all GDDs with $m = 2$ constructed here are RGDs. There do not seem to be any designs in Clatworthy [4] of the type we consider in Section 2. The only three we noticed in [4] are listed in the array below and have block size 5 and two groups.

GDDs from Clatworthy [4]:

Design	v	r	b	m	n	λ_1	λ_2	Block-intersection type
R133	8	5	8	2	4	4	2	(1,4) type
R135	8	10	16	2	4	8	4	(1,4) type
R141	10	10	20	2	5	5	4	mixed-type; half each configuration

From the list of blocks for RGD 133 given in [4], all blocks are of the type $(1, 4)$. However, it is clear that for any GDD$(2, n, n+1; n, 2)$, all the blocks are of type $(1, n)$. To see this, apply the formula in Theorem 4 with $k = n + 1$ and $(\lambda_1, \lambda_2) = (n, 2)$ and solve for β.

For convenience, we list several well known constructions.

Theorem 7 *If a BIBD(mn, k, λ) and a BIBD(n, k, μ) exists, then a GDD $(n, m, k;\ \lambda + \mu, \lambda)$ exists. If a BIBD $(v = kt, k, \lambda)$ exists then the GDD $(t, k, k; \lambda + i, \lambda)$ exists (the example R141 just above is of this type). If a BIBD $(5t, 5, \lambda)$ exists then a GDD$(t, k, k; \lambda + i, \lambda)$ exists. If a BIBD $(2n, 5, \lambda_2)$ and a BIBD $(n, 5, \lambda_2 - \lambda_1)$ exist, then GDD $(2, n, 5; \lambda_1, \lambda_2)$ exists.*

2 Configuration (2, 3) GDD's

In this section we consider GDDs such that each block intersects each of the two groups in two points or three points. We refer to these designs as Configuration $(2, 3)$ GDD's. First theorem of this section shows that the inequality in Theorem 1 is an equality for Configuration $(2, 3)$ GDD's.

Theorem 8 *For any Configuration $(2, 3)$ GDD$(n, 2, 5;\ \lambda_1, \lambda_2)$, we have $\lambda_2 = 3(n-1)\lambda_1/2n$. Further, if n is even, and n is not a multiple of 3, then $\lambda_1 \geq 2n$ and $\lambda_2 \geq 3(n-1)$.*

Proof. Each block has one pair from one group and three pairs from the other group. There are six pairs per block from different groups. Thus, $6b = n^2\lambda_2$. This can be solved for b, equated with the other expression for b, and simplified to the desired result. ∎

Since $\gcd(n, n-1) = 1$, $\lambda_1 \geq n/3$. Hence, it follows from this theorem that $\lambda_2 \geq (n-1)/2$. Now consider any Configuration $(2, 3)$ GDD with point x from group A and point y from group B. If λ_2 is even then it may be possible that the set of λ_2 blocks which contain both x and y may be divided into two categories with the same number of blocks. In the first category, say, each block has three points from group A and in the second category has three points from group B. In this situation it may be possible to use certain symmetries in the construction of the blocks of the GDD. When λ_2 is odd, this division of blocks into two categories is not possible, and the constructions are much harder (see $n = 8$ in Section 2.2 in this regard).

In this section some of the constructions use BIBDs on n points. For this and other reasons we put the residue classes of n mod 6 in different subsections. The next construction illustrates other possibilities. A Configuration (2, 3) GDD(5, 2, 5; 5, 6) satisfies most necessary conditions but does not exist - it would have only 25 blocks and the number of blocks must be even (Theorem 3). A constuction with 50 blocks turns out to be minimal, i.e., with the fewest possible number of blocks; however, if only 25 blocks have group A triples and 25 have group B triples, it is not convenient (perhaps not possible) to use a BIBD(5, 3, 3) with *ten* blocks as a constituent.

Example 2 *A Configuration* (2, 3) *GDD*(5, 2, 5; 10, 12). *The groups are* $A = \{1, 2, 3, 4, 5\}$ *and* $B = \{a, b, c, d, e\}$. *Half of the fifty blocks are shown as columns in the array.*

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1
3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2
a	c	e	b	d	c	e	b	d	a	e	b	d	a	c	b	d	a	c	e	d	a	c	e	b
c	e	b	d	a	e	b	d	a	c	b	d	a	c	e	d	a	c	e	b	a	c	e	b	d

It is easier to check the indices geometrically. The group A triples in the blocks shown determine pairs equivalent to 10 copies of a 5-cycle and 5 copies of a 5-pointed star. The 25 pairs from group B in these 25 blocks form five 5-stars (two-factors) on the points of B. Likewise, the 25 pairs in the 25 blocks not shown will be 5 more copies of the 5-star on group A. The collection of cycles and stars from group A can be recomposed to form 10 complete copies of the complete graph K_A (on the five points of group A), and likewise for group B. Thus, $\lambda_1 = 10$. Each of the triples from group A is paired with all 5 blocks from a 5-star (2-factor) from the complete graph K_B and vice-versa in the blocks not shown. The point 1, for example, occurs in 3 distinct triples (in a set of 5 blocks) and is matched with each point of group B twice corresponding to each triple, six matchings of 1 with each point from B. It follows that $\lambda_2 = 12$ as six more matchings occur with roles of the groups reversed. This design has the smallest indices possible for $n = 5$.

Recall that we occasionally use the term "near-minimal GDD" to mean one with exactly twice the minimal number of blocks.

2.1 Configuration (2, 3) for $n = 6t + 1$

In general, from the first part of Theorem 8, for $n = 6t+1$, a Configuration (2, 3) GDD must satisfy, for some positive integer w, the following:

$$\lambda_1 = nw,\ \lambda_2 = 3(n-1)w/2,$$
$$b = n^2(n-1)w/4.$$

In order that the number of blocks b is an integer, we must have for some integers s and t:

$n = 6t+1$	$24s+1$	$24s+7$	$24s+13$	$24s+19$
Minimum w	1	4	2	4

First, for $n = 7$, the necessary conditions give $7\lambda_2 = 9\lambda_1$. Assume therefore that $\lambda_1 = 7w$ and $\lambda_2 = 9w$ for some positive integer w. The number of blocks in a Configuration $(2,3)$ GDD$(7,2,5;7w,9w)$ is $b = 147w/2$. Since the number of blocks must be even (Theorem 3), take $w = 4$ (as in the table just above), and then $b = 2(147)$. There is a construction for $n = 7$ and $w = 4$ and obviously the indices are minimal.

Let $A = \{a_1, a_2, ..., a_7\}$ and $B = \{b_1, b_2, ..., b_7\}$ denote the groups. Take 21 copies of A, a BIBD$(7,3,1)$ based on the points of set A. There are 21 pairs of points for set B, and we augment each block of A (which appears 21 times) with each pair from B. This creates 147 of the needed blocks and the other 147 are created similarly reversing the roles of A and B. Each pair of points from set A appears in blocks together 21 times (since we used 21 copies of A) and 7 more times (since we used 7 complete two-factorizations of K_A and K_B, one per block of the BIBD). Thus $\lambda_1 = 28$. Since the replication number for a BIBD$(7,3,1)$ is $r = 3$, point a_1, say, appears with point b_1 three times for each of the six pairs from B containing b_1, for a total of 18. But a_1 and b_1 appear together another 18 times by reversing the roles of A and B. Thus, $\lambda_2 = 36$. This creates a Configuration $(2,3)$ GDD$(7,2,5;28,36)$.

Note that BIBD$(6t+1,3,1)$ exists for $t = 1,2,3,\ldots$. Thus we generalize the construction above in the following theorem.

Theorem 9 *Suppose* $n = 6t+1$. *Then there exists a Configuration* $(2,3)$ *GDD*$(n,2,5;4n,6(n-1))$.

Proof. Use the construction for $n = 7$. Take $3n$ copies of BIBD$(n,3,1)$ on each group and n two-factorizations of K_A and K_B. The parameter $w = 4$ in this case, and the design is not minimal for two of the four cases. When $n = 6t+1$, there are $n(n-1)/6$ blocks in a BIBD$(n,3,1)$. We will use $3n$ copies of A, a BIBD$(n,3,1)$ based on set $A = \{a_1, a_2, ..., a_n\}$. We note that K_n, the complete graph on n points, has a two-factorization, a decomposition in two-factors or subgraphs in which every vertex appears twice. Each two-factor has n pairs. We use the notation K_A (and K_B) for the complete graph on vertices in set A (and B). We will use $3t$ complete decompositions of K_A and K_B. In a two-factor, every vertex appears twice so there are $6t+1 = v$ edges in a two-factor. Each of the $3n$ copies of a block of A is to be augmented with one of the $3n$ pairs from three two-factors from K_B. Now, a_1 from A appears in $(v-1)/2 = 3t$ blocks and b_1, say, in B, appears twice in each of the three two-factors. Thus, a_1 and b_1 appear together in $18t$ blocks. They appear together in another $18t$ blocks with the roles of A and B reversed. Therefore, $\lambda_2 = 36t$. Since $3n$ copies of A and n decompositions of K_A were used (and similarly for B), $\lambda_1 = 4n$. ∎

The design constructed by this theorem has minimal indices exactly when $n = 24s+7$ and $n = 24s+19$.

Theorem 10 *For $n = 24s+7$ or $n = 24s+19$ the necessary conditions are sufficient for the existence of Configuration $(2,3)$ GDD$(n,2,5;\ \lambda_1,\lambda_2)$ with $\lambda_1 = 4vw$ and $\lambda_2 = 36tw = 6(v-1)w$ for some positive integer w.*

The necessary conditions referred to here are the two equations on page 2 and Theorem 8. A design is α-resolvable if its blocks can be partitioned into classes so that within a class, each point occurs α times. The necessary conditions, that $\alpha v \equiv 0 \pmod{k}$ and $\lambda(v-1) \equiv 0 \pmod{(k-1)\alpha}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$, are known to be sufficient for $k = 3$ [12]. In every binary equi-replicate design of constant block size k such that $bk = vr$ and $b = mv$, the points can be arranged into a k-by-b array so that (with colums as blocks) every point occurs in each row m times [2]. We are now in a position to prove the following:

Theorem 11 *For $n = 24s+1$, $s \geq 1$, there is a Configuration $(2,3)$ GDD$(n,2,5;\ n, 3(n-1)/2)$, and thus the necessary conditions are sufficient (with $w = 1$) for existence.*

Proof. Take n copies of A, a 3-resolvable BIBD$(24s+1,3,1)$ and n copies of an isomorphic BIBD, say B. There are $4s$ resolution classes in each copy with n blocks per class. We consider all n copies of any four of the classes, say C_1, C_2, C_3 and C_4. We decompose each triple of C_4 into pairs. First, applying [2], we may order the points in the blocks of C_4 so that each point of A (and of B, in the corresponding class) appears once in the first position, once in the second position and once in the third position. Say $\{x_1,x_2,x_3\}$ is a block of C_4 and $\{y_1,y_2,y_3\}$ is the corresponding block from group B. We will refer to $\{x_1,x_2\}$ as the first pair, $\{x_1,x_3\}$ as the second pair, and $\{x_2,x_3\}$ as the third pair (and correspondingly for B). For each of the n copies of block b of C_1, augment with each of the n "first pairs" determined by C_4 (but using the corresponding points in B). Do this for all n blocks of C_1. Every block of C_1 now occurs with each point of B twice. Since every point of A occurs 3 times in C_1, every point of A

meets every point of B six times. Repeat the process for C_2 using second pairs of C_4, and again for C_3 using third pairs of C_4. Thus every point of A now meets every point of B eighteen times. Since there are s groups of 4 classes, the points meet $18s$ times. Finally, repeat the process reversing the roles of A and B. It follows that $\lambda_2 = 36s = 3(n-1)/2$. Since we used n copies of the BIBD$(n,3,1)$ for A and B, $\lambda_1 = n$. ■

Theorem 12 *The necessary conditions are sufficient for Configuration* (2,3) *GDD*$(24s+13, 2, 5; \lambda_1, \lambda_2)$.

Proof. For $n = 24s+13$, $w = 2$ is necessary (as in the table above). Use BIBD$(n,3,2)$ as ingredients. With index 2, the BIBDs will have $r = n-1$. Thus, there will be $r/3 = 4t = 4(2s+1)$ appropriate 3-resolution classes and the previous construction for $n = 24s+1$ can be applied here. This creates a design with minimal indices and any other will have some multiple of these indices, and the result follows. ■

2.2 Configuration (2,3) for $n = 6t+2$

In this subsection we give an minimal design for $n = 8$ with odd λ_2, a Configuration (2,3) GDD$(8,2,5;16,21)$, and we show that there exists an minimal or near-minimal solution for all $6t+2$.

First, for $n = 6t+2$, it is necessary that $\lambda_1 = 2nw$ and $\lambda_2 = 3(n-1)w$ for some positive integer w. With these values for the indices, the number of blocks b is given by

$$b = n^2(n-1)w/2.$$

We now give a solution for $n = 8$, $w = 1$, $\lambda_1 = 16$ and $\lambda_2 = 21$. We take the groups to be $A = \{1,2,...,8\}$ and $B = \{b1, b2, ...b8\}$. Using two copies of a BIBD$(8,3,6)$ based on group A (112 blocks), we will augment appropriately with one-factors from K_B. The other 112 blocks are formed in the same way, reversing the roles of the groups. Let X denote a BIBD$(8,3,6)$ which we construct in the following way. We identify eight clusters of seven blocks each, C_1 to C_8. Cluster C_i will be missing point i and will be developed cyclically so that each cluster is a BIBD$(7,3,1)$. Cluster C_1 (missing point 1) is the set of blocks $\{2,3,5\}, \{3,4,6\} ... \{8,2,4\}$. The starter blocks used are indicated below:

C_1	{2,3,5}
C_2	{1,3,5}
C_3	{1,2,5}
C_4	{1,2,5}

C_5	{1,2,4}
C_6	{1,2,4}
C_7	{1,2,4}
C_8	{1,2,4}

It may be checked that 7 blocks appear four times, and 10 blocks appear twice. The remaining 8 blocks are shown below (each is listed twice since we are using two copies of the design X). We also show the companion blocks formed from (corresponding) triples using group B. It is necessary that, in these 32 blocks, each point from group A be matched with each point from group B exactly 3 times, and moreover, the pairs used must form four complete one-factors from each group.

1	1	1	3	2	2	2	4	1	1	1	3	2	2	2	4
3	5	3	5	4	4	6	6	3	5	3	5	4	4	6	6
7	7	5	7	6	8	8	8	7	7	5	7	6	8	8	8
$b2$	$b2$	$b4$	$b6$	$b1$	$b1$	$b3$	$b5$	$b6$	$b4$	$b2$	$b2$	$b5$	$b3$	$b1$	$b1$
$b4$	$b6$	$b8$	$b8$	$b3$	$b5$	$b7$	$b7$	$b8$	$b8$	$b6$	$b4$	$b7$	$b7$	$b5$	$b3$

$b1$	$b1$	$b1$	$b3$	$b2$	$b2$	$b2$	$b4$	$b1$	$b1$	$b1$	$b3$	$b2$	$b2$	$b2$	$b4$
$b3$	$b5$	$b3$	$b5$	$b4$	$b4$	$b6$	$b6$	$b3$	$b5$	$b3$	$b5$	$b4$	$b4$	$b6$	$b6$
$b7$	$b7$	$b5$	$b7$	$b6$	$b8$	$b8$	$b8$	$b7$	$b7$	$b5$	$b7$	$b6$	$b8$	$b8$	$b8$
1	1	3	3	2	2	4	4	3	3	1	1	4	4	2	2
5	7	7	5	6	8	8	6	7	5	5	7	8	6	6	8

The remaining blocks occur in sets of 4 or 8 (using two copies of X) and each four-some of blocks may be augmented by a single one-factor (4 pairs) from the other group. The 112 blocks with triples from group A need, in all, four complete one-factorizations (7 one-factors of 4 pairs each), and similarly for the other 112 blocks. Note, each pair from group A meets 12 times in triples from the two copies of X, and 4 more times from the four one-factorizations. Thus, $\lambda_1 = 12 + 4 = 16$. Each point from group A occurs in 36 blocks not pictured in sets of four copies and each set of four is augmented with a one-factor from group B. This counts 9 towards a total for λ_2. Another 9 comes from reversing the roles of the groups, and 3 more come from the blocks listed above. Thus, $\lambda_2 = 9 + 9 + 3 = 21$. Since these indices for the GDD are minimal and any such design with $n = 8$ must have indices a multiple of these, we may say:

Theorem 13 *The necessary conditions are sufficient for the existence of a Configuration* $(2,3)$ *GDD*$(8,2,5;\ 16w, 21w)$ *for any positive integer* w.

Theorem 14 *There exists a Configuration* $(2,3)$ *GDD*$(6t{+}2, 2, 5;\ 4n, 6(n-1))$ *with* $w = 2$, *and, consequently, the necessary conditions are sufficient for* $n = 12s + 2$.

Proof. We use $w = 2$ in the general case for $n = 6t+2$. Take $n/2$ copies of a BIBD$(n,3,6)$ based on group A. This gives $n^2(n-1)/2$ blocks. The number of pairs in n complete one-factorizations is $n\binom{n}{2} = n^2(n-1)/2$, exactly the number of blocks. The $n/2$ copies of each block need to be augmented with the $n/2$ pairs from 1 one-factor of group B. The replication number for the BIBD is $r = 3(n-1)$. Every point from group A meets every point from group B in this way $2r = \lambda_2 = 6(n-1)$ times, counting the other blocks with triples from group B. As $n/2$ copies of a design with index 6 are used, $3n$ is counted towards λ_1. But we are using n one-factorizations of each group, and each pair occurs n more times. Thus, $\lambda_1 = 3n+n = 4n$. This creates a Configuration $(2,3)$ GDD$(n, 2, 5; 4n, 6(n-1))$ for $n = 6t+2$, $t \geq 1$. Now, for any $n = 6t+2$, obesrve that the replication number $r = [\lambda_1(n-1) + \lambda_2 n]/4 = [2nw(n-1) + 3(n-1)wn]/4$. Since $2n$ is always a multiple of 4, r is an integer only if n is a multiple of 4 (or w is even). It follows that $w = 2$ is necessary for existence when $n = 12s+2$. ■

For $n = 12s+2$ there is thus a minimal construction, and for $n = 12s+8$ there is a near-minimal construction.

2.3 Configuration (2, 3) For n = 6t+3

We begin with a construction. It is well known that, when $n = 3s \geq 9$, there exist resolvable BIBD$(n, 3, \mu)$. The index μ can be 1 if $n = 6t+3$, or 2 if $n = 6t$. The number of resolution classes is just the replication number $r = \mu(n-1)/2$.

Theorem 15 *Suppose that $n \equiv 0$ (mod 3) and that the BIBD$(n, 3, \mu)$ is resolvable with r resolution classes. If $r \equiv 0$ (mod 4), then there exists a Configuration $(2,3)$ GDD$(n, 2, 5;\ \lambda_1, \lambda_2)$ where $\lambda_1 = n\mu/3$ and $\lambda_2 = r$.*

Proof. Let $n = 3s$. We use $n/3$ copies of X = BIBD$(n, 3, \mu)$ based on set $X = \{x_1, x_2, ..., x_n\}$ and another $n/3$ copies of an isomorphic copy of X, say, Y based on set $\{y_1, y_2, ..., y_n\}$. X and Y will be the two groups. Number the resolution classes of X arbitrarily by $R_1, R_2, ..., R_r$. We begin with the first 4 classes, and use the blocks of classes R_1, R_2 and R_3. Augment each by a suitable pair taken from Y. There are $n/3$ copies of these three classes, a total of n resolution classes. Consider the $n/3$ blocks in R_4. Each block determines 3 pairs, a total of n distinct pairs. There are $n/3$ "copies" of each of the n pairs since there are $n/3$ copies of R_4. Supposing $\{x_1, x_2\}$ is one of these pairs, augment each block in one resolution class with $n/3$ copies of $\{y_1, y_2\}$. For each such X-pair, we put the corresponding Y-pair in each of the $n/3$ blocks in one resolution class of X. Continue with the next 4 remaining resolution classes, if any, until there are no more classes for X. Then reverse the roles of X and Y. The blocks in each 4th resolution class are decomposed into pairs which are used to augment one of the resolution classes for the other group. The first index is $n\mu/3$ since we used (all the pairs of) $n/3$ copies of a design X with index μ. In any 4th resolution class, each point appears in two pairs determined by its one block. Thus, y_1, say, appears in blocks with every x twice in this way (2 resolution classes of X), and twice more when the roles are reversed. But the process occurs $r/4$ times. Therefore, $\lambda_2 = 4(r/4) = r$. ■

Theorem 16 *The necessary conditions are sufficient for the existence of a Configuration* $(2,3)$ *GDD*$(6t+3,2,5;\lambda_1,\lambda_2)$.

Proof. *In the general case for GDDs with two groups of size* $n = 6t+3$, *we have* $\lambda_1 = wn/3$ *and* $\lambda_2 = w(n-1)/2$ *for some positive integer* w. *The number of blocks is* $b = n^2(n-1)w/12$. *It is necessary to look at four cases mod 24. When* $n \equiv 9 \pmod{24}$, b *is even and the number of resolution classes in a BIBD*$(n,3,1)$ *is* $r = (n-1)/2$, *a multiple of* 4. *Thus, in the previous theorem, we may take* $\mu = 1 (= w)$. *The theorem gives us a Configuration* $(2,3)$ *GDD*$(24t+9,2,5;n/3,(n-1)/2)$, *and these indices are minimal. If* $n = 24t+3$, *then the number of blocks* b *is given by* $b = (24t+3)(24t+3)(24t+2)w/12$. *Thus,* b *is an even integer (and minimal) only if* $w = 4 (= \mu)$, *and the construction gives a Configuration* $(2,3)$ *GDD*$(24t+3,2,5;4n/3,2(n-1))$, *with minimal indices. When* $n = 24t+15$, $w = 4$ *is again necessary. For* $n = 24t+21$, $w = 2$ *is necessary. In each of these other* 3 *cases, the larger value of* w *insures as well that* r *is a multiple of* 4. *Thus, the previous theorem can be applied. Since any other such GDD (in either of these four cases) will have indices some multiple of these, the result follows.* ■

2.4 Configuration (2, 3) for n = 6t+4

For $n = 6t+4$ we have $\lambda_1 = 2nw$, $\lambda_2 = 3(n-1)w$, and $b = n^2(n-1)w/2$. The design in Example 1 is minimal ($n = 4$). For all other $n = 6t+4$ we give an minimal or near-minimal construction. Note that the replication number is $r = [2nw(n-1) + 3(n-1)wn]/4$. Therefore, for $n = 12s+10$, $w = 2$ is necessary for existence.

Theorem 17 *There exists a Configuration* $(2,3)$ *GDD*$(6t+4,2,5;\lambda_1,\lambda_2)$ *which is near-minimal* ($w = 2$) *for* $n = 12s{+}4$ *and minimal for* $n = 12s{+}10$.

Proof. Use $3n/2$ copies of a BIBD$(n,3,2)$ as ingredients for each group and n complete one-factorizations of K_A and K_B. Augment the $3n/2$ copies of each block with 3 one-factors with $n/2$ pairs each. The rest is clear. ■

2.5 Configuration (2, 3) for n = 6t+5

Applying Theorem 5 to this case, we find $\lambda_1 = nw, \lambda_2 = 3(n-1)w/2$, and

$b = n^2(n-1)w/4$ for some positive integer w. In order for the number of blocks to be even, we need several cases, which we put in the array below

$n = 6t+5$	$24s+5$	$24s+11$	$24s+17$	$24s+23$
Minimum w	2	4	1	4

Theorem 18 *For all $n = 6t+5$, there exists a Configuration* (2, 3) *GDD*$(n, 2, 5; \lambda_1, \lambda_2)$ *with* $\lambda_1 = 4n$ *and* $\lambda_2 = 6(n-1)$.

Proof. For each of groups A and B, take n copies of a BIBD$(n, 3, 3)$. For the n copies of a block with points from group A, augment with the pairs of one two-factor from group B. Do this for each block of points from A and then do likewise for group B. Using n copies of a BIBD$(n, 3, 3)$ contributes $3n$ towards λ_1, and using n complete two-factorizations of K_A (and K_B) contributes n more. Thus, $\lambda_1 = 3n+n = 4n$. For each time point x from group A appears in a block of the original BIBD, it eventually meets every point y from group B exactly twice. This contributes $2r$ towards λ_2. Reversing the roles of the groups, $\lambda_2 = 4r = 4[3(n-1)/2] = 6(n-1)$. There are $n(n-1)/2$ blocks in the BIBD, and we are using n copies of two BIBDs, so $b = n^2(n-1)$ blocks. These values for λ_1, λ_2 and b are the ones given by the formulas above (when $w = 4$). ∎

Corollary 1 *The necessary conditions are sufficient for Configuration* (2, 3) *GDD*$(12t+11, 2, 5; \lambda_1, \lambda_2)$.

Proof. The preceeding theorem gives the minimal possible indices for $n = 6t+11$, and any other Configuration (2, 3) GDD$(12t+11, 2, 5; \lambda_1, \lambda_2)$ will have parameters some multiple of these. ∎

COMMENT: The previous theorem gives a near-minimal construction for $n = 24s+5$. Example 2 gives an minimal construction for $n = 5$, but we can improve only slightly for the general case $24s+17$. There is a BIBD$(24s+17, 3, 3)$ which is 3-resolvable, and the number of resolution classes is a multiple of 4. Thus, the construction for $n = 24s+1$ can be applied. This reduces w from 4 to 3 since then $\lambda_1 = 3n$.

2.6 Configuration (2, 3) for n = 6t

In this case, the minimal indices are $(\lambda_1, \lambda_2) = (2n/3, n-1)$ and the minimal number of blocks is $b = n^2(n-1)/6$.

Example 3 *A GDD*$(6, 2, 5; 4, 5)$ *with* 30 *blocks should exist; however, in this case the replication number would fail to be an integer. We give an minimal example of a Configuration* (2, 3) *GDD*$(6, 2, 5; 8, 10)$ *with* 60 *blocks.*

We use groups $A = \{1,2,3,4,5,6\}$ *and* $B = \{a,b,c,d,e,f\}$, *and we begin with three isomorphic copies of a BIBD*(6, 3, 2) *on each group. We augment the three copies of each block as indicated in the array.*

1	1	1	*a*	*a*	*a*
2	2	2	*b*	*b*	*b*
3	3	3	*c*	*c*	*c*
a	*b*	*c*	4	5	6
b	*c*	*a*	5	6	4

1	1	1	*a*	*a*	*a*
2	2	2	*b*	*b*	*b*
4	4	4	*d*	*d*	*d*
a	*b*	*d*	3	5	6
b	*d*	*a*	5	6	3

1	1	1	*a*	*a*	*a*
3	3	3	*c*	*c*	*c*
5	5	5	*f*	*f*	*f*
a	*c*	*e*	2	4	6
c	*e*	*a*	4	6	2

1	1	1	*a*	*a*	*a*
4	4	4	*d*	*d*	*d*
6	6	6	*f*	*f*	*f*
a	*d*	*f*	2	3	5
d	*f*	*a*	3	5	2

1	1	1	*a*	*a*	*a*
5	5	5	*e*	*e*	*e*
6	6	6	*f*	*f*	*f*
a	*e*	*f*	2	3	4
e	*f*	*a*	3	4	2

2	2	2	*b*	*b*	*b*
3	3	3	*c*	*c*	*c*
6	6	6	*f*	*f*	*f*
b	*c*	*f*	1	4	5
c	*f*	*b*	4	5	1

2	2	2	*b*	*b*	*b*
4	4	4	*d*	*d*	*d*
5	5	5	*e*	*e*	*e*
b	*d*	*e*	1	3	6
d	*e*	*b*	3	6	1

2	2	2	*b*	*b*	*b*
5	5	5	*e*	*e*	*e*
6	6	6	*f*	*f*	*f*
b	*e*	*f*	1	3	4
e	*f*	*b*	3	4	1

3	3	3	*d*	*d*	*d*
4	4	4	*e*	*e*	*e*
5	5	5	*f*	*f*	*f*
d	*e*	*f*	1	2	6
e	*f*	*d*	2	6	1

3	3	3	*c*	*c*	*c*
4	4	4	*d*	*d*	*d*
6	6	6	*f*	*f*	*f*
c	*d*	*f*	1	2	5
d	*f*	*c*	2	5	1

Each 5-by-6 subarray shows the set of blocks "generated" for the GDD by a block of the BIBD. A direct count shows $\lambda_1 = 8$ and $\lambda_2 = 10$.

Theorem 19 *There exists a Configuration* $(2,3)$ *GDD*$(6t, 2, 5; \lambda_1, \lambda_2)$ *for every* $t \geq 2$ *which is near-minimal* ($w = 2$) *for* $n = 12s$ *and minimal for* $n = 12s + 6$.

Proof. Use $n/2$ copies of a BIBD$(6t, 3, 2)$ on each group and $n/3$ complete one-factorizations. Note $\lambda_1 = 2(n/2) + n/3 = 4n/3$. The rest is similar to previous proofs. minimality for $n = 12s + 6$ follows on observing that the replication number r is an integer only if w is even. ∎

References

[1] Abel, R.J.R., and S.C. Furino, Resolvable and Near Resolvable Designs, p.87-94 in *The CRC Handbook of Combinatorial Designs*, edited by C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, Fl, 1996.

[2] H. Agrawal, Some generalizations of distinct representatives with applications to statistical designs, *Ann. Math. Statis.* **37** (1966), 25-528.

[3] Cheng, C.-T and K. Hinklemann, minimality of some two-associate class partially balanced incomplete block designs, *Ann. Statist.* **19** (1991), 1667-1671.

[4] Clatworthy, W.H., *Tables of two-Associate-Class Partially Balanced Designs*, National Bureau of Standards (U.S.) Applied Mathematics Series No. 63, 1973.

[5] Colbourn, C.J., Group testing, p.564-565, in *The CRC Handbook of Combinatorial Designs*, edited by C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, Fl, 1996.

[6] C.J. Colbourn and A. Rosa, *Triple Systems*, Clarendon Press, Oxford, 1999.

[7] Fu, H.L. and Rodger, C.A., Group divisible designs with two associate classes: n = 2 or m = 2, *J. Combin. Theory Ser A* **83** (1998), No. 1, 94-117.

[8] Fu, H.L., C.A. Rodger, and D.G. Sarvate, The existence of group divisible designs with first and second associates, having block size 3, *Ars Combinatoria* **54** (2000), 33-50.

[9] D. Henson, D.G. Sarvate and S.P. Hurd, On group divisible designs with three groups and block size four, to appear in discrete math.

[10] S.P. Hurd and D.G. Sarvate, Odd and even group divisible designs with two groups and block size four, Discrete math. **284**(2004), 189-196.

[11] S.P. Hurd and D.G. Sarvate, On Group Divisible Designs With Two Groups and Small n, to appear in Discrete Math.

[12] D. Jungnickel, R.C. Mullin and S.A Vanstone, The spectrum of α-resolvable block designs with block size 3, *Discrete Math.* **97** (1991), 269-277.

[13] Lindner, C.C., and C.A Rodger, Design Theory, CRC Press, Boca Raton, 1997.

[14] Mayo, O., *The Theory of Plant Breeding,* Clarendon Press, Oxford, 1987.

[15] Mullin, R.C. and H.-D. O.F. Gronau, PBD's and GDD: The Basics, p. 185-213 in *The CRC Handbook of Combinatorial Designs*, edited by C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, Fl, 1996.

[16] Street, A.P. and D.J. Street, *Combinatorics of Experimental Design*, Clarendon Press, Oxford, 1987.

[17] Street, A.P. and D.J. Street, Partially balanced incomplete block designs, p.419-423, in *The CRC Handbook of Combinatorial Designs*, edited by C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, Fl, 1996.

[18] Zhu, L., Self-Orthogonal Latin Squares (SOLS), p.442-447 in *The CRC Handbook of Combinatorial Designs*, edited by C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, Fl, 1996.