On designs constructed by group actions

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Abstract

Using the action of the linear fractional groups $L_2(q)$, $q = 8$, 25, 27, 29, 31 and 32 some 1-designs are constructed. It is shown that subgroups of the automorphism group of $L_2(q)$ appear as the full automorphism group of the contructed designs. In the cases $q = 8$ and 32 it is shown that the symmetric groups $\mathbb{S}_9$ and $\mathbb{S}_{33}$ respectively appear as the automorphism group of one of the constructed designs.

1 Introduction

Key and Moori [5] constructed designs, codes and graphs that have the Janko group J_1 and J_2 as automorphism groups. The method of construction is to consider the primitive permutation representations of the simple group $G \cong J_1$ or J_2. If G acts on a set Ω primitively and $\Delta \neq \{\omega\}$ is an orbit of G_ω on Ω, then Δ^G is the block set for a 1-design. The designs constructed from the groups J_1 and J_2 in this way all had $Aut(G)$ as their

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automorphism group. Observing the results of [5] the authors conjectured that in certain cases the automorphism groups of the designs obtained by a primitive representation of a general simple group G will have the automorphism group $Aut(G)$ as its full automorphism group. But later in [6] they found counter examples to the above conjecture. Motivated by the works in [5] and [6] we also found another counter example for the conjecture in [1]. However it is interesting to apply the above method to certain groups and their primitive representations to construct designs and find their automorphism groups. Our intention in this paper is to continue the research work in [1] and consider the linear fractional groups $L_2(q)$ for certain q. Primitive permutation representations of these groups are given in [4] and our computations use the GAP system described in [7].

2 Definitions

Let F_q denote the Galois field with $q = p^n$ elements, where p is a prime number and $n \in \mathbb{N}$. For a detailed definition of the linear fractional group $L_2(q)$ we may refer the reader to [3]. The group of all the invertible 2 by 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries in F_q is denoted by $GL_2(q)$. The projective linear transformation associated with $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined as follows. We join the symbol ∞ to F_q and define the following permutation f_α of $F_q \cup \infty$ by:

$$f_\alpha(x) = \begin{cases} \frac{ax+b}{cx+d} & \text{if } x \in F_q \text{ and } cx+d \neq 0, \\ \infty & \text{if } x \in F_q \text{ and } cx+d = 0, \\ \frac{a}{c} & \text{if } x = \infty \text{ and } c \neq 0, \\ \infty & \text{if } x = \infty \text{ and } c = 0. \end{cases}$$

Then the set of all such mappings is denoted by $PGL_2(q)$, i.e.

$$PGL_2(q) = \{x \longmapsto \frac{ax+b}{cx+d} \mid ad - bc \neq 0\}.$$

The projective special linear group in dimension 2 is also called the linear fractional group and it is denoted by either $PSL_2(q)$ or $L_2(q)$ and consist of the following elements

$$L_2(q) = \{x \longmapsto \frac{ax+b}{cx+d} \mid ad-bc \in F_q^{\Box}\}$$

where $F_q^{\Box}$ denotes the set of non-zero squares in F_q.

For the structures of the maximal subgroups of the groups $L_2(q)$ we refer the reader to [4]. The notations for designs are standard as in [2], but we remind a few definitions. Let $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure with point set $\mathcal{P}$ and block set $\mathcal{B}$ were $\mathcal{I}$ is a subset of $\mathcal{P} \times \mathcal{B}$ called the incidence relation. For $p \in \mathcal{P}$ and $B \in \mathcal{B}$ we will write $p\mathcal{I}B$ if and only if $(p, B) \in \mathcal{I}$. Then D is called a $t-(v,k,\lambda)$ design if $|\mathcal{P}| = v$, $|\mathcal{B}| = k$ for each $B \in \mathcal{B}$, and every t points of $\mathcal{P}$ is incident with precisely λ blocks of $\mathcal{B}$. The design is called symmetric if the number of points v is equal to the number of blocks b. The number of blocks through a set of s points is denoted by λ_s and is independent of the set if $s \leq t$. It is easy to deduce that

$$\lambda_s = \lambda_t \binom{v-s}{t-s} / \binom{k-s}{t-s}$$

where $\lambda = \lambda_t$. Here we remark that D is also a $s-(v,k,\lambda_s)$ design as well.

A $t-(v,k,\lambda)$ design is called trivial if every subset of $\mathcal{P}$ with cardinality k is a block of $\mathcal{B}$, i.e. $b = \binom{v}{k}$. The dual of the incidence structure $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is $D^t = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, hence if D is a $t-(v,k,\lambda)$ design, then D^t is a design with b points such that the size of every block is λ_1. The incidence matrix of a structure $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $|\mathcal{P}| \times |\mathcal{B}|$ matrix A whose rows are labeled by points in $\mathcal{P}$ and whose columns are labeled by blocks in $\mathcal{B}$ and the entry $(p, B) \in \mathcal{P} \times \mathcal{B}$ is equal to 1 if and only if p is incident with $\mathcal{B}$, otherwise it is zero. Therefore A is a matrix with entries 0 or 1 and the incidence matrix of D^t is A^t which is the transpose of A. Two structures $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $D' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ are called isomorphic if there is a one to one correspondence $\theta : \mathcal{P} \longrightarrow \mathcal{P}'$ with the following property:

$$p\mathcal{I}B \Longleftrightarrow \theta(p)\mathcal{I}'\theta(B), p \in \mathcal{P}, B \in \mathcal{B}.$$

In this case we will write $D \cong D'$. The structure D is called self-dual if $D \cong D^t$. An isomorphism of D onto itself is called an automorphism of D.

The set of all the automorphisms of D is a group and it is denoted by *Aut* D. If the incidence matrix of the structure D is A, then *Aut* D consists of precisely the pairs (P, Q), where P is a permutation of the rows of A and Q is a permutation of the columns of A such that $PAQ = A$.

3 Method

Construction of the 1-designs uses Proposition 1 of [5]. Here we will describe the general case as mentioned in page 155 of [3]. Let G act on a set Ω of size n. Let $B \subseteq \Omega$ with $|B| \geq 2$. Then $D = (\Omega, B^G, \in)$ is an incidence structure, where $B^G = \{B^g \mid g \in G\}$. If the action of G on Ω is t-homogeneous, i.e. for any t-subsets B_1 and B_2 of Ω there is $g \in G$ such that $B_1^g = B_2$, then for a subset B of Ω with $|B| \geqslant t$, $D = (\Omega, B^G, \in)$ is a $t-(v, k, \lambda)$ design with parameters $v = n$,

$$\lambda = \lambda_t = b\binom{k}{t}/\binom{v}{t} = (|G|\binom{k}{t})/(|G_B|\binom{v}{t}).$$

Where G_B denoted the global stabilizer of B under G.

Now in the special case that the action of G on Ω is transitive we have $t = 1$ and we will obtain a 1-design D with parameters $1-(n, k, r)$, where $r = [G : G_B]\frac{k}{n}$. Furthermore if for a point $\omega \in \Omega$ we have $|G_B| = |G_\omega|$, which happens if the action of G on Ω is primitive and $B \neq \{\omega\}$ is an orbit of G_ω on Ω, then the design D has parameters $1-(n, k, k)$.

Now the construction of the 1-designs continues as follows. Consider a group which acts primitively on a set Ω of size n. Take $\omega \in \Omega$ and consider an orbit Δ, $|\Delta| = k > 1$ of the stabilizer G_ω on Ω. Then Δ^G is the block set of a symmetric design with parameters $1-(n, k, k)$. If the action of G on Ω is 2-transitive, then G_ω, $\omega \in \Omega$, has only two orbits $\{\omega\}$ and $\Omega - \{\omega\}$ on Ω and the design obtained in this way is trivial. In our investigations we will not consider trivial designs.

Note that according to [5] the above construction gives a self-dual block design and G acts as an automorphism group of this design i.e. $G \leq$ *Aut* D and as a permutation group on the set of blocks, G is primitive as well.

4 Primitive actions of the groups $L_2(q)$, $q = 8, 25, 27, 29, 31$ and 32

First we will establish some notations concerning the tables at the end of this section. The maximal subgroups of $L_2(q)$ up to conjugacy are in [4] whose isomorphic types are listed in the second column of the table. Degree denotes the index of each maximal subgroup in the group, # indicates the number of orbits of the stabilizer of a point in the action of the group on the set of right cosets of a maximal subgroup. The rest of the entries after # are devoted to the lengths of the orbits of a point stabilizer and an entry like $m(n)$ indicates n orbits of length m. The number beneath each orbit length is the order of the automorphism group of the design. Note that if $\# = 2$, then this means that the action of the group on the set of the cosets of the subgroup is 2-transitive and hence the design obtained is trivial and will not be considered. All the calculations have been carried out using GAP.

Now Tables 1-6 give information we need about the groups $L_2(q)$, $q = 8, 25, 27, 29, 31$ and 32. According to these tables we have the following results.

1. $L_2(8)$ is the full automorphism group of the design with parameters $1-(36,7,7)$ and $1-(28,9,9)$. The full automorphism group of the design with parameters $1-(36,14,14)$ is $\mathbb{S}_9$ which will be proved at the end of this section.

2. $L_2(25)$ is the full automorphism group of the designs with parameters $1-(300,13,13)$ and $1-(325,6,6)$, but $L_2(25):2$, a subgroup of index 2 in $Aut(L_2(25))$, is the full automorphism group of the designs with parameters $1-(65,10,10)$, $1-(65,24,24)$, $1-(65,30,30)$, $1-(300,26,26)$ and $1-(325,12,12)$. Also, $Aut(L_2(25))$ is the full automorphism group of the design with parameters $1-(300,26,26)$.

3. $L_2(27)$ is the full automorphism group of the designs with parameters $1-(351,7,7)$, $1-(378,13,13)$ and $1-(819,4,4)$. The group $L_2(27):2$ is the full automorphism group of the designs with parameters $1-(351,14,14)$, $1-(378,26,26)$ and $1-(819,6,6)$. The group $Aut(L_2(27) = L_2(27):6$ is the full automorphism group of the design with parameters $1-(378,26,26)$.

The group $L_2(27):3$ is the full automorphism group of the designs with parameters $1-(819,12,12)$, $1-(378,26,26)$ and $1-(351,28,28)$.

4. $L_2(29)$ is the full automorphism group of the designs with parameters $1-(203,12,12)$, $1-(203,20,20)$, $1-(203,30,30)$, $1-(203,60,60)$, $1-(406,15,15)$ and $1-(435,7,7)$. The group $Aut(L_2(29))=L_2(29):2$ is the full automorphism group of the designs with parameters $1-(406,30,30)$ and $1-(435,14,14)$.

5. $L_2(31)$ is the full automorphism group of the designs with parameters $1-(248,5,5)$, $1-(248,12,12)$, $1-(248,20,20)$, $1-(248,30,30)$, $1-(248,60,60)$, $1-(465,8,8)$ and $1-(465,15,15)$. The group $Aut(L_2(31))=L_2(31):2$ is the full automorphism group of the designs with parameters $1-(465,16,16)$ and $1-(496,30,30)$.

6. $L_2(32)$ is the full automorphism group of the designs with parameters $1-(528,31,31)$ and $1-(496,33,33)$. For full automorphism group of the designs with parameter $1-(528,62,62)$ we have the following proposition. But first we need a Lemma.

Lemma 1 *If G is a simple group and $|G|=|A_{33}|$, then $G\cong A_{33}$.*

Proof. According to the classification of finite simple groups, G is isomorphic to one of the groups: An alternating group A_n, $n\geq 5$, a sporadic group or a simple group of Lie type. If $G\cong A_n$, then it is clear that $n=33$ and we are done. From the list of orders of the sporadic groups in [4] we see that G cannot be a sporadic group. Therefore we examine the possibility of G being isomorphic to a simple group of Lie type defined over a finite field of characteristic p. Since we have assumed $|G|=|A_{33}|$, hence $\pi(G)=\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$, and therefore p may be one of the numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 or 31. From the list of orders of simple groups of Lie type given in [4] we observe that the order of G is divisible by numbers of the form $p^k\pm 1$, $k\in\mathbb{N}$. We consider the following cases.

Case 1 $p=3, 7, 11, 13, 17$ or 19. In the cases $p=3, 11, 13, 17$ the smallest k for which $31\mid p^k+1$ is 15 and the smallest l for which $31\mid p^l-1$ in the cases $p=7, 19$ is 15. Examinations of the above p's and considering the numbers $p^r\pm 1$, $r\leq 15$ we obtain primes other than primes in $\pi(G)$ which is a contradiction.

Case 2 $p = 23$ or 29. In this case the smallest k for which $31 \mid p^k + 1$ is 5. But then $p^2 + 1$ contains a prime not in $\pi(G)$, again a contradiction.

Case 3 $p = 2$ or 5. In this case 9 is the least positive integer k for which $19 \mid 2^k + 1$ and $19 \mid 5^k - 1$. But then considering the numbers of the form $p^l \pm 1$ for $l \leqslant 9$ we obtain a contradiction.

Case 4 $p = 31$. From [4] we see that the order of a simple group of Lie type is divisible by p for some $k \geqslant 1$. Hence in this case we must have $k = 1$ and $G \cong PSL_2(17)$ a contradiction.

Therefore $G \cong A_{33}$ and the Lemma is proved. ∎

Proposition 1 *Let Ω denote the set of right cosets of a subgroup of $G = L_2(32)$ isomorphic to D_{62}. Then for $\omega \in \Omega$, the point stablizer G_ω has an orbit Δ of length* 62 *and orbiting Δ under G we obtain a design with parameters* $1-(528, 62, 62)$ *whose full automorphism group is isomorphic to S_{33}.*

Proof. It is clear that $|\Omega| = [L_2(32) : D_{62}]$. By Table 6 there is an orbit of length 62 under the action of G_ω, $\omega \in \Omega$, on Ω. Using a program in GAP we have obtained $|Aut\ D| = 868331761881188649551819440128000000 = 2^{31}3^{15}5^{7}7^{4}11^{3}.13^{2}.17.19.23.29.31$ Using GAP we found that $Aut\ D$ has only one non-trivial normal proper subgroup N whose order is $\frac{1}{2}|Aut\ D|$. But again using GAP we found out N is a simple group. Since $|N| = |A_{33}|$ by previous Lemma we deduce that $N \cong A_{33}$. But $\frac{Aut\ D}{N} \cong \mathbb{Z}_2$ and $Aut\ D$ does not contain elements of order 34, hence $Aut\ D = S_{33}$. ∎

Remark 1 *The design D constructed in the above proposition is another counter example to the conjecture made in [2].*

Now we return to Table 1 and prove the automorphism group of the 1-design obtained by orbiting the orbit of length 14 is isomorphic to $\mathbb{S}_9$.

Proposition 2 *Let Ω be the set of all the right cosets of a subgroup of $G = L_2(9)$ isomorphic to D_{14}. Then for $\omega \in \Omega$, the point stabilizer G_ω has an orbit Δ of length* 14 *and orbiting Δ under G we obtain a design with parameters* $1-(36, 14, 14)$ *whose full automorphism group is isomorphic to $\mathbb{S}_9$.*

Proof. By Table 1 there is an orbit Δ of length 14 under the action of G_w, $w \in \Omega$. If we orbit Δ under the action of G we will obtain a 1-design D with parameters $1-(36, 14, 14)$. The order of the full automorphism group of D is obtained by a program in GAP and it turns out to be $|AutD| = |362880| = |9!|$. Using the GAP again it is verified that $AutD$ has only one normal subgroup N of order 2 and $\frac{AutD}{N}$ is a simple group. But $|\frac{AutD}{N}| = \frac{1}{2}9!$ and it is well-known that any simple group of order $\frac{1}{2}9!$ is isomorphic to $\mathbb{A}_9$. Therefore $\frac{AutD}{N} \cong \mathbb{A}_9$ and since $AutD$ does not contain elements of order 18 we obtain $AutD \cong \mathbb{S}_9$ and the Proposition is proved. ■

After examination of the automorphism groups of the design obtained from the primitive action of the group $L_2(q)$ for $q = 8$, 25, 27, 29, 31 and 32 we put forward the following conjectures.

Conjecture 1 *(see [1]) For q in an odd prime number, the full automorphism group of the designs obtained in the manner described so far, is either* $L_2(q)$ *or* $Aut(L_2(q)) = L_2(q) : 2$.

Conjecture 2 *For q an odd prime power all subgroups of* $Aut(L_n(q))$ *can appear as automorphism groups of the 1-designs obtained from the primitive action of* $L_2(q)$.

Conjecture 3 *For* $q = 2^n$, *if n is odd, then the automorphism group of the 1-designs under consideration is either* $L_2(q)$ *or* $\mathbb{S}_{q+1}$. *If n is even, all subgroups of* $Aut(L_2(q)) = L_2(q) : n$ *or* $\mathbb{S}_{q+1}$ *can appear as the automorphism group of the constructed 1-designs, where : denotes semi-direct product of groups.*

The full details of the computer programming can be found at the web site:

http://www.fos.ut.ac.ir/~darafsheh

Table 1: Orbits of the point-stabilizer of $L_2(8)$

no.	*maximal subgroup*	*degree*	#	*lenght and* $\lvert Aut \rvert$
1	D_{18}	28	4	9(3) 504
2	D_{14}	36	5	14(1) 7(3) 362880 504

Table 2: Orbits of the point-stabilizer of $L_2(25)$

no.	*maximal subgroup*	*degree*	#	*lenght and* $\lvert Aut \rvert$
1	S_5	65	4	10(1) 24(1) 30(1) 15600
2	S_5	65	4	10(1) 24(1) 30(1) 15600
3	D_{24}	325	21	6(2) 12(10) 24(8) 7800 15600
4	D_{26}	300	18	13(11) 26(6) 7800 15600 31200

Table 3: Orbits of the point-stabilizer of $L_2(27)$

no.	*maximal subgroup*	*degree*	#	*length and* $\lvert Aut \rvert$
1	A_4	819	76	4(8) 6(3) 12(64) 9828 19656 29484
2	D_{26}	378	22	13(13) 26(8) 9828 19659 29484 58968
3	D_{28}	351	21	7(2) 14(12) 28(6) 9828 19656 29484

Table 4: Orbits of the point-stabilizer of $L_2(29)$

no.	maximal subgroup	degree	#	length and |Aut|
1	D_{28}	435	25	7(2) 14(12) 28(9) 12180 24360
2	D_{30}	406	21	15(13) 30(7) 12180 24360
3	A_5	203	8	12(1) 20(2) 30(3) 60(1) 12180
4	A_5	203	8	12(1) 20(2) 30(3) 60(1) 12180

Table 5: Orbits of the point-stabilizer of $L_2(31)$

no.	maximal subgroup	degree	#	length and |Aut|
1	D_{30}	496	25	15(15) 30(9) 14880 29760
2	D_{32}	465	24	8(2) 16(14) 32(7) 14880 29760
3	A_5	248	9	5(1) 12(1) 20(1) 30(3) 60(2) 14880
4	A_5	248	9	5(1) 12(1) 20(1) 30(3) 60(2) 14880

Table 6: Orbits of the point-stabilizer of $L_2(32)$

no.	maximal subgroup	degree	#	length and |Aut|
1	D_{62}	528	17	31(15) 62(1) 32768 33!
2	D_{66}	496	16	33(15) 32768

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