Strength Six Orthogonal Arrays and Their Non-Existence

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Abstract

In this paper, we consider the non-existence of some bi-level orthogonal arrays (O-arrays) of strength six, with m constraints ($6 \leq m \leq 32$), and with index set μ ($1 \leq \mu \leq 512$). The results presented here tend to improve upon the results available in the literature.

1 Introduction and Preliminaries

For ease of reference, we first list some basic concepts and definitions.

Definition. An *array* T with m *constraints* (rows), N *runs* (columns, treatment-combinations), and two *symbols* is merely a matrix T of size $m \times N$ with two elements (say, 0 and 1).

An array T assumes great importance when we impose some combinatorial structure on it. One such combinatorial structure leads us to the concept of a

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balanced array (B-array).

Definition. T is called a *B-array* of *strength* t $(1 \le t \le m)$ if it satisfies the following conditions: In every $(t \times N)$ submatrix T^* of T (clearly there are $\binom{m}{t}$ such submatrices), every $(t \times 1)$ vector $\underline{\alpha}$ of weight i $(0 \le i \le t$; the weight of a vector $\underline{\alpha}$ is the number of 1s in it) appears with the same frequency μ_i (say). The vector $\underline{\mu}' = (\mu_i;\ i = 0, 1, 2, \ldots, t)$ and m are called the *parameters* of the array T.

The preceding definition can easily be extended to B-arrays with s symbols. Also, note that $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$. Thus, N is known if we are given $\underline{\mu}'$.

Definition. A B-array T is called an *orthogonal* array (O-array) if $\mu_i = \mu$, for each i. In this special case, $N = 2^t \mu$.

Thus, an O-array is a special case of a B-array. Also, the incidence matrix of a balanced incomplete block design (BIBD) is a special case of a B-array with $t = 2$. B-arrays have been shown to be related to various other combinatorial structures.

B-arrays and O-arrays have been extensively used to construct fractional factorial designs in statistical design of experiments, and O-arrays have found great use in coding theory, information theory, and statistical quality control. Under different values of t, these combinatorial arrays assist us in the resolution of different kinds of problems in factorial designs.

In this paper, we restrict ourselves to arrays with $t = 6$. Such arrays, under certain conditions, would allow us to estimate all the effects up to and including three-factor interactions (when higher order interactions are assumed to be negligible). The problem of constructing such arrays, for a given $\underline{\mu}'$ with the maximum possible value of m, is very important both in combinatorics and in the statistical design of experiments. Such problems for O-arrays have been studied, among others, by Bose and Bush [1], Chopra, Low, Dios [5], Hedayat, Sloane and Stufken [6], Rao [8, 9], Seiden and Zemach [12], and Yamamoto, et. al [14]; while the corresponding problem for B-arrays has been studied, among others, by Chopra [4].

A related and important problem in the study of O-arrays is to obtain the minimal number of runs N for any O-array, for given values of m and t. In this paper, we consider the first type of problem for the existence of O-arrays (ie. to obtain the maximum value of m, for a given μ and t). The results obtained here go on to improve upon not only the results given in Table 12.1 in Hedayat, Sloane and Stufken [6], but also those given in Table 3 in Chopra, Low and Dios [5].

Definition. A B-array T with m rows and index set $\underline{\mu}' = (\mu-1, \mu, \mu, \mu, \mu, \mu, \mu-1)$ is called a *near* O-array. Here, $N = 64\mu - 2$.

Note that if we juxtapose to T two vectors (one of weight 0 and another of weight m), we would obtain an O-array of index set μ with m rows. To gain further insight into the importance of O-arrays and B-arrays, the interested reader may consult the list of references (by no means, exhaustive) at the end of this paper, and the further references listed therein.

2 Main Results with Discussion

Definition. Two columns of a B-array T with m rows are said to have i *coincidences* $(0 \leq i \leq m)$ if the symbols appearing in these two columns are the same in i of the rows.

The following two lemmas are obvious.

Lemma 1. *A near O-array T with $t = 6$ and $m = 6$ always exists.*

Lemma 2. *A near O-array T of strength t $(= 6)$ is also of strength t' $(0 < t' \leq t = 6)$. Considered as an array of strength t', its index set is given by $(\mu_j^{t'}, j = 0, 1, 2, \ldots, t')$, where $\mu_j^{t'} = 2^{t-t'}\mu - 1$, for $j = 0, t'$ and $\mu_j^{t'} = 2^{t-t'}\mu$, for $j = 1, 2, \ldots, t' - 1$.*

The next two results are from Chopra and Dios [3].

Lemma 3. *Consider a B-array T having a column (say, the first one) of weight l. Let x_j be the number of columns (other than the first) having exactly j $(0 \leq j \leq m)$ coincidences with the the first one. Then, the following results hold:*

$$\sum_{j=0}^{m} x_j = N - 1, \tag{2.1}$$

$$\sum j^k x_j = \sum_{t'=1}^{k} \left[g(k;t') \sum_{i=0}^{t'} \binom{l}{i} \binom{m-l}{t'-i} (\mu_i^{t'} - 1) \right],$$

where $t' \leq k$, and $1 \leq k \leq 6$.

Remark. The above result can be easily obtained by counting the number of coincidences in two different ways. The constants $g(k;t')$ are known for each $(k;t')$ when we derive (2.1) above. These constants for $k = 1, 2, 3, 4, 5, 6$ are respectively: 1, (1, 2), (1, 6, 6), (1, 14, 36, 24), (1, 30, 150, 240, 120), and (1, 62, 540, 1560, 1800, 720).

Theorem 1. *Consider a near O-array of size $(m \times N)$ with $t = 6$ and $\underline{\mu}' = (\mu - 1, \mu, \mu, \mu, \mu, \mu, \mu - 1)$. Then, the following inequality is true:*

$$\begin{aligned}
&L_2 L_6 \geq L_4^2 + L_2 L_3^2, \quad \textit{where} \qquad (2.2)\\
&\quad L_2 = (N-1)B_2 + B_1^2,\\
&\quad L_3 = (N-1)^2 B_3 - 3(N-1)B_2 B_1,\\
&\quad L_4 = (N-1)^3 B_4 - 4(N-1)^2 B_3 B_1 + 6(N-1)B_2 B_1^2 - 3B_1^4,\\
&\quad L_6 = (N-1)^5 B_6 - 6(N-1)^4 B_5 B_1 + 15(N-1)^3 B_4 B_1^2\\
&\qquad\qquad - 20(N-1)^2 B_3 B_1^3 + 15(N-1) B_2 B_1^4 - 5B_1^6, \quad \textit{and}\\
&\quad B_k = \sum j^k x_j.
\end{aligned}$$

Discussion and explanation: Here we describe how Theorem 1 (true for near O-arrays) is used to obtain corresponding results for the corresponding O-arrays. It is quite obvious that the max(m) for near O-array is k if (2.2) is contradicted for $m = k+1$ (say). Now we attach to the near O-array T $(\mu-1, \mu, \mu, \mu, \mu, \mu, \mu-1)$ two k-vectors, one having all zeros and the other having all ones. This would give us the corresponding O-array T^* with index set μ and number of constraints k. If we want to add another row to T^*, it is quite obvious we would not be able to accomplish it under the constraint that the resulting T^* has two $(k+1)$-vectors, one of all zeros and another with all ones. Thus we are looking at a special class of O-arrays having two runs, one of all zeros and another one of all ones.

Remark. It is obvious that (2.2) is merely a polynomial inequality involving l, μ and m. For given values for μ and l, it becomes an inequality in m. It is straight-forward to check now if (2.2) is or is not satisfied for any value of m (≥ 6). If (2.2) is contradicted for $m = m^* + 1$ (say), then the maximum value of m for the array is m^*. A computer program was prepared to check (2.2) for all μ satisfying $1 \leq \mu \leq 512$, and m satisfying $6 \leq m \leq 41$.

3 Tables 1–3 with Explanations, Comments, and Illustrations

The entries of Table 2 list the maximum possible value of m $(6 \leq m \leq 32)$ for each μ $(1 \leq \mu \leq 512)$ and $l = 0$ for which (2.2) is satisfied. The entries of Table 1 [extracted from Table 12.1 with $t = 6$, given in Hedayat, Sloane and Stufkin [6]] give the smallest index μ possible for an O-array with $t = 6$, and m rows. There are two kinds of entries: (i) exact value of μ if it is known, and (ii) an interval μ_0–μ_1 indicating that such an O-array must have index at least μ_0, and that an O-array with index μ_1 is known to exist. For example, the entry for $m = 12$, $t = 6$ is 12–16 which means that an O-array with $m = 12$ and $N = (16)(2^6) = 1024$ runs is known to exist, and that any such O-array must contain at least $(12)(2^6) = 768$ runs, but the existence of each O-array with $\mu = 12, 13, 14$, and 15 is unknown. In design language, any entry of the type μ_0–μ_1 in Table 1 means that there is a fractional factorial design of resolution 7 with m factors and $N = \mu_1 \cdot 2^6$ runs.

In Table 1, all unlabeled lower bounds are obtained from the trivial observation that the non-existence of an O-array of index μ and m rows implies the non-existence of an O-array of index μ and $(m+1)$ rows. All unlabeled upper bounds are consequences of the following observations: (i) an O-array of index $\mu = 1$ and $m = t+1$ always exists, and (ii) an O-array of index μ, strength t, and m rows implies the existence of an O-array of index μ, strength t, and $(m-1)$ rows, an O-array of index μ, strength $t-1$, and $m-1$ rows, and an O-array of index 2μ, strength t, and $(m+1)$ rows.

Remark. In practice, it means that if an upper bound μ_1 is unlabeled, then the justification for it can be obtained by following the table downwards and possibly diagonally downwards to the right, until an entry μ_1 is reached which carries a label. Thus, some upper bounds (for $t = 6$) in Table 1 are obtained by appealing to the existence of O-arrays with $t > 6$ and or with larger values of m. The entries given in our Table 2 are obtained by using results dealing with O-arrays having strength $t = 6$.

Table 3 entries are obtained by using Table 2 to revise Table 1 entries. Table 3 entries clearly demonstrate that numerous intervals given in Table 1, for certain kinds of O-arrays, have been considerably shortened. Below, we provide some illustrations outlining the arguments used to achieve this reduction.
Illustrations.

1. Let us take $m = 31$ in Table 1, for which the interval for μ is 96–256. Now for $\mu = 96$, from Table 2 we have $m \leq 23$ which implies the O-array with $m = 31$ and $\mu = 96$ is not possible. Thus, we remove $\mu = 96$ from Table 1. This argument applies to each μ satisfying $96 \leq \mu \leq 212$ which means the interval for $m = 31$ is 213–256 (a significant reduction). In Chopra, Low and Dios [5], the reduced interval for $m = 31$ is 142–256. Thus, the present interval is also a significant improvement over the one given in [5].

2. For $m = 16$, the interval for μ from Table 1 is 21–32. For $\mu = 21, 22$ (in Table 2), we have $m \leq 14$ and $m \leq 15$ for all μ in 23–27. Thus for $m = 16$, the new interval for μ is 28–32. We have been able to eliminate 7 values of μ from the interval. In Chopra, Low and Dios [5], this case did not show any reductions.

3. For $m = 15$ in Table 1, we have only one value of μ, namely $\mu = 16$, and it is labeled. If we check Hedayat, Sloane and Stufkin [6], we find that there exists an O-array with index $\mu = 16$, $m = 16$, and $t = 7$. This implies the existence of an O-array with $\mu = 16$, $m = 15$, and $t = 6$.

Thus all entries in Table 1 are not obtained by appealing to strength six arrays but also, among others, by appealing to higher strength arrays. We have one advantage over Table 1. Table 2 is obtained by appealing only to $t = 6$ arrays. In constructing Table 3, we have mostly picked up those entries from Table 1 in which there is only one value of μ, since those problems have been resolved. Our main concern has been those entries where μ's appear as intervals, where research problems occur.

In our discussion, we are not considering O-arrays of strength $t > 6$. Furthermore, we have to keep in mind that an O-array of strength 6 could come from an O-array of strength 7, but every O-array of strength 6 may not be of strength 7. Using arguments similar to the ones above, we are able to revise Table 1 by eliminating those O-arrays from various intervals which do not exist. This results in the reduction of intervals.

m	μ	m	μ
6	1	20	29–32
7	1	21	32
8	2	22	32
9	4	23	32*
10	6*–8	24	41–64
11	8*	25	51–128
12	12–16	26	58–128
13	16	27	66–128*
14	16	28	73–256
15	16*	29	74–256
16	21–32	30	87–256
17	26–32	31	96–256*
18	29–32	32	108–512
19	29–32		

Table 1: Minimal possible index μ of orthogonal arrays having 2 symbols, m factors, and strength 6.

μ	m	μ	m
1	6	97–109	24
1	7	110–123	25
2–3	8	124–138	26
4–6	9	139–154	27
7–8	10	155–172	28
9–10	11	173–191	29
11–14	12	192–212	30
15–17	13	213–234	31
18–22	14	235–257	32
23–27	15	258–282	33
28–32	16	283–308	34
33–39	17	309–336	35
40–46	18	337–365	36
47–54	19	366–397	37
55–63	20	398–430	38
64–73	21	431–465	39
74–84	22	466–501	40
85–96	23	502–539	41

Table 2: For a given μ ($1 \leq \mu \leq 512$), the maximum possible value of m for orthogonal arrays of strength 6 with 2 symbols.

m	μ	m	μ
6	1	20	32
7	1	21	32
8	2	22	32
9	4	23	32
10	7–8	24	64
11	8	25	110–128
12	12–16	26	124–128
13	16	27	128
14	16	28	155–256
15	16	29	173–256
16	28–32	30	192–256
17	32	31	213–256
18	32	32	235–512
19	32		

Table 3: (Revised) Minimal possible index μ for a given m, for orthogonal arrays of strength 6 with 2 symbols.

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