On Balance Index Sets of L-Products with Cycles and Complete Graphs

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Abstract

Let G be a graph with vertex set V and edge set E. A labeling $f : V \to \{0,1\}$ induces a partial edge labeling $f^* : E \to \{0,1\}$ defined by $f^*(xy) = f(x)$ if and only if $f(x) = f(y)$ for each edge $xy \in E$. The balance index set of G, denoted BI(G), is defined as $\{|f^{*-1}(0) - f^{*-1}(1)| : |f^{-1}(0) - f^{-1}(1)| \leq 1\}$. In this paper, we study the balance index sets of graphs which are L-products with cycles and complete graphs.

1 Introduction

Liu, Tan and the second author [7] considered a new labeling problem in graph theory. A vertex labeling of a graph $G = (V, E)$ is a mapping f from V into the set $\{0,1\}$. For each vertex labeling f of G, we define a partial edge labeling f^* of G in the following way. For each edge uv in E, define

$$f^*(u,v) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

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Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled 0 and 1 respectively under the mapping f. Similarly, denote by $e_f(0)$ and $e_f(1)$, respectively, the number of edges of G that are labeled 0 and 1 respectively under the induced partial function f^*. In other words, for $i = 0, 1$,

$$\begin{aligned} v_f(i) &= |\{u \in V : f(u) = i\}|, \\ e_f(i) &= |\{uv \in E : f^*(uv) = i\}|. \end{aligned}$$

For brevity, when the context is clear, we will simply write $v(0)$, $v(1)$, $e(0)$ and $e(1)$ without any subscript.

Definition 1.1. A vertex labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \leq 1$, and ***balanced*** if both $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

It is clear that not all the friendly graphs are balanced. Lee, Lee and Ng [6] introduced the following notion in [3] as an extension of their study of balanced graphs.

Definition 1.2. The ***balance index set*** of the graph G is defined as

$$\mathrm{BI}(G) = \{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}.$$

Example 1. Figure 1 shows a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$. □

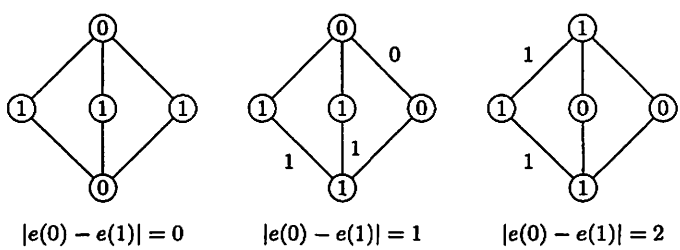

Figure 1: The friendly labelings of a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$.

Example 2. For a cycle C_n with vertex set $\{x_1, x_2, \ldots, x_n\}$, we denote by $C_n(t)$ the cycle with a chord x_1x_t. The balance index sets of $C_4(3)$, $C_6(4)$ and $C_6(5)$ are shown in Figure 2. All of them equal to $\{0, 1\}$. □

We note here that not every graph has a balance index set consisting of an arithmetic progression.

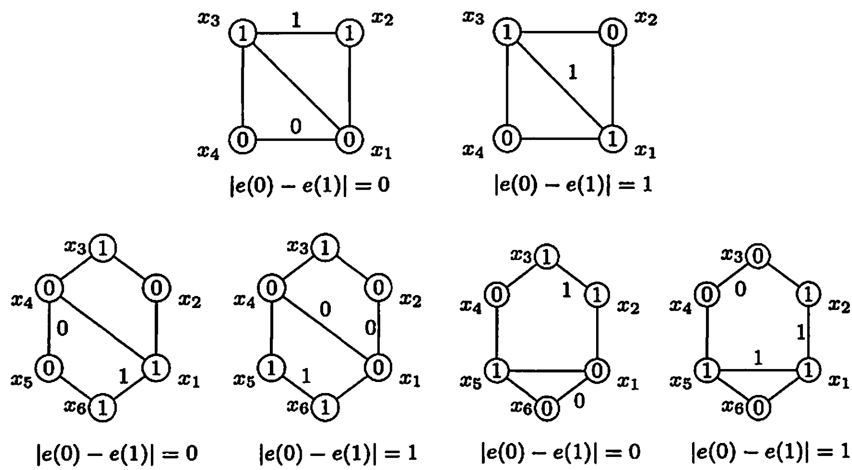

Figure 2: The balance index sets of $C_4(3)$, $C_6(4)$ and $C_6(5)$.

Example 3. The graph $\Phi(1,3,1,1)$ is composed of $C_4(3)$ with a pendant edge appended to each of x_1, x_3 and x_4, and three pendant edges appended to x_2. Figure 3 shows that $\mathrm{BI}(\Phi(1,3,1,1)) = \{0,1,2,3,4,6\}$. Note that 5 is missing from the balance index set. □

In general, it is difficult to determine the balance index set of a given graph. Most of existing research on this problem have focused on some special families of graphs with simple structures, see [1, 2, 6, 8]. Here are a couple of examples:

$$\mathrm{BI}(C_n(t)) = \begin{cases} \{0,1\} & \text{if } n \text{ is even,} \\ \{0,1,2\} & \text{if } n \text{ is odd.} \end{cases}$$

and

$$\mathrm{BI}(\mathrm{St}(n)) = \begin{cases} \{k\} & \text{if } n = 2k+1, \\ \{k-1,k\} & \text{if } n = 2k. \end{cases}$$

The balance index sets of the graph which are formed by the amalgamation of complete graphs, stars, and generalized theta graphs were studied in [4, 5]. In [10], the second author, with Zhang, Ho and Wen, investigated some trees of diameter at most four.

2 Generalized L-Product

Let H be a connected graph with a distinguished vertex s. Construct a new graph $G \times_L (H,s)$ as follows: take $|V(G)|$ copies of (H,s) and identify each

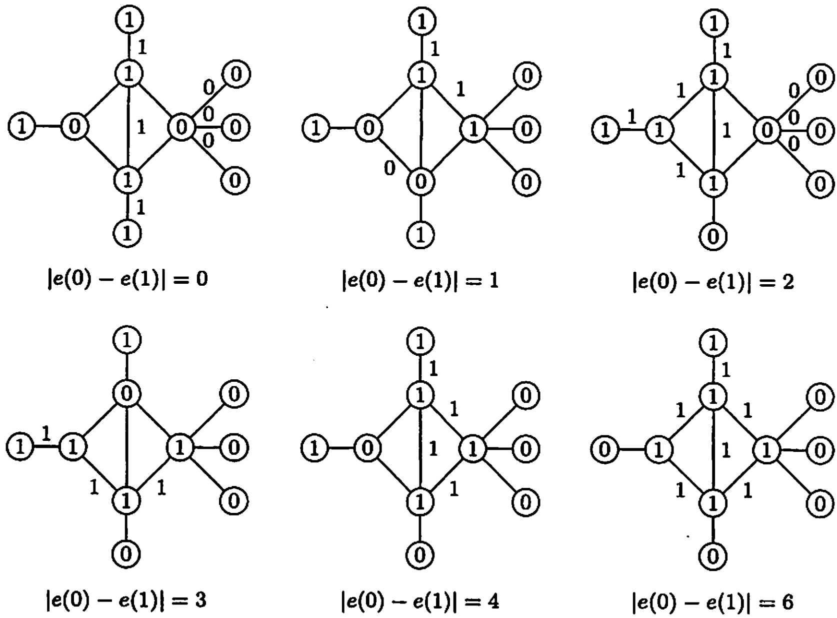

Figure 3: The six friendly labelings of $\Phi(1,3,1,1)$.

vertex of G with s of a single copy of H. We call the resulting graph the ***L-product*** of G and (H, s). More generally, the n copies of the graphs to be identified with the vertices of G need not be identical. Let Gph^* be the family of pairs (H, s), where H is a connected graph with a distinguished vertex s. For any graph G and any mapping $\Phi : V(G) \to Gph^*$, we construct the ***generalized L-product*** of G and Φ, denoted by $G \times_L \Phi$, by identifying each $v \in V(G)$ with s of the respective $\Phi(v)$.

Example 4. Figure 4 shows that $\mathrm{BI}(C_4 \times_L (K_5, s)) = \{0, 2, 4\}$. □

Example 5. Figure 5 shows that the generalized L-products of a cycle C_3 with a mapping $\Phi : V(G) \to Gph^*$, where $\Phi(c_1) = K_5$, $\Phi(c_2) = K_3$ and $\Phi(c_3) = K_4$. □

Example 6. The balance index set of a graph depends on its topological structure. For example, let the vertices on P_3 be u_1, u_2 and u_3, and denoted by $\mathrm{St}(m)$, the star with center c and m pendant vertices. We find that $\mathrm{BI}(P_3 \times_L \Phi) = \{1, 2, 4\}$ if $\Phi(u_1) = \Phi(u_2) = (\mathrm{St}(2), c)$, and $\Phi(u_3) = (\mathrm{St}(3), c)$; but $\mathrm{BI}(P_3 \times_L \Phi) = \{0, 2, 4\}$ if $\Phi(u_1) = \Phi(u_3) = (\mathrm{St}(2), c)$, and $\Phi(u_2) = (\mathrm{St}(3), c)$. See Figure 6. □

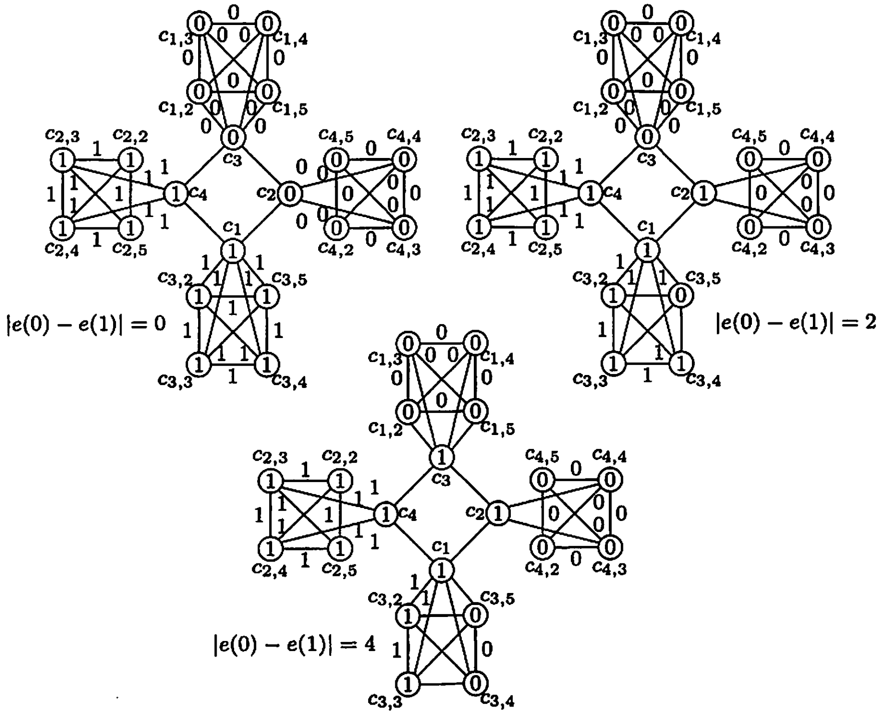

Figure 4: The balance index sets of $C_4 \times_L (K_5, s)$.

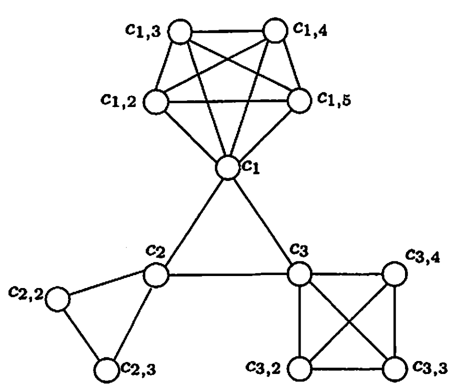

Figure 5: The balance index sets of $C_3 \times_L \Phi$.

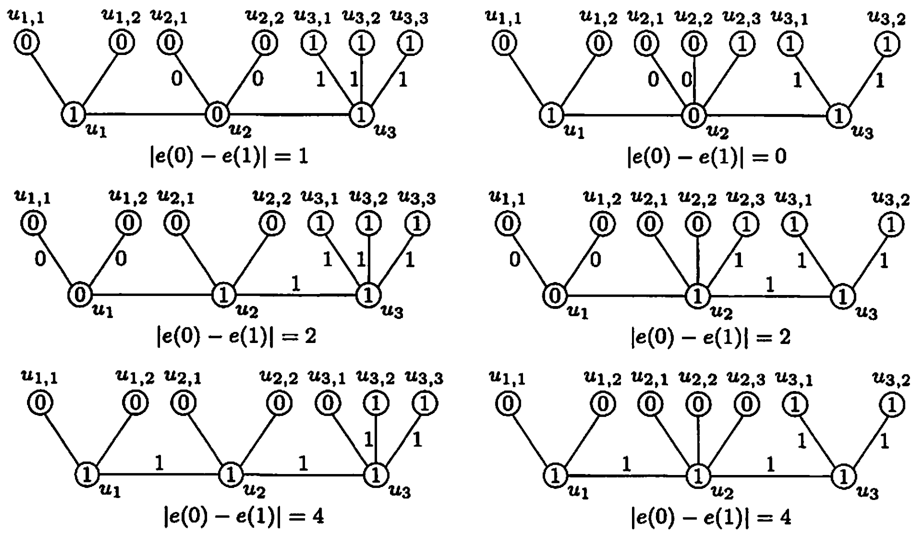

Figure 6: The balance index sets of $P_3 \times_L \Phi$.

3 Balance Index Sets of the Generalized L-Product of C_n with Cycles

The proofs of the next two results can be found in [5]. Nonetheless, we provide the alternate proofs below.

Lemma 3.1 *For any (not necessarily friendly) vertex labeling of* C_m*, we have* $e(0) - e(1) = v(0) - v(1)$.

Proof. It is straightforward to verify that switching the labels of two adjacent vertices does not alter the value of $e(0) - e(1)$ or $v(0) - (1)$. Hence, we may assume the 0-vertices (vertices that are labeled 0) are adjacent to each other, and so are the 1-vertices. The result follows immediately from the observation that $e(0) = v(0) - 1$, and $e(1) = v(1) - 1$. □

Lemma 3.2 *If a graph contains a cycle of length* m *as a subgraph, which has* z *vertices labeled* 0 *and* $m - z$ *vertices labeled* 1*, then, restricted to that cycle,* $e(0) - e(1) = 2z - m$.

Proof. It follows from the proof of Lemma 3.1 that $e(0) - e(1) = v(0) - v(1) = z - (m - z) = 2z - m$. □

Theorem 3.3 *For any* n *and* Φ *such that* $\Phi(v)$ *is a cycle for any vertex* v*, assume* $|V(C_n \times_L \Phi)| = 2q + r$*, where* $0 \leq r \leq 1$*. Then*

$$BI(C_n \times_L \Phi) = \begin{cases} \{n+r, n+r-2, n+r-4, \ldots, 1\} & \text{if } n+r \text{ is odd,} \\ \{n+r, n+r-2, n+r-4, \ldots, 0\} & \text{if } n+r \text{ is even.} \end{cases}$$

Proof. Let $v_i(0)$, $v_i(1)$, $e_i(0)$ and $e_i(1)$ denote the respective values restricted to the ith cycle $\Phi(c_i)$, and let z be the number of 0-vertices on C_n. Then

$$\begin{aligned} e(0)-e(1) &= 2z-n+\sum_{i=1}^{n}[e_i(0)-e_i(1)] \\ &= 2z-n+\sum_{i=1}^{n}[v_i(0)-v_i(1)] \\ &= 2z-n+v(0)-v(1), \end{aligned}$$

where $0 \le z \le n$. Notice that this formula does not depend on how we label the vertices of $\Phi(c_i)$. Hence, one can easily obtain a friendly labeling with any z-value between 0 and n. If $r=0$, we need $v(0)-v(1)=0$; hence

$$\{e(0)-e(1) \mid 0 \le z \le n\} = \{-n, -n+2, -n+4, \ldots, n-2, n\}.$$

In a similar manner, if $r=1$, then $v(0)-v(1)=\pm 1$; hence

$$\{e(0)-e(1) \mid 0 \le z \le n\} = \{-n-1, -n+1, -n+3, \ldots, n-1, n+1\}.$$

The result follows immediately. □

Corollary 3.4 *For any n and Φ such that $\Phi(v)$ is a cycle for any vertex v, the values in $BI(C_n \times_L \Phi)$ always form an arithmetic progression.*

Example 7.

$$\begin{aligned} \mathrm{BI}(C_3 \times_L \Phi) &= \begin{cases} \{1,3\} & \text{if } |V(C_3 \times_L \Phi)| \text{ is even,} \\ \{0,2,4\} & \text{if } |V(C_3 \times_L \Phi)| \text{ is odd;} \end{cases} \\ \mathrm{BI}(C_4 \times_L \Phi) &= \begin{cases} \{0,2,4\} & \text{if } |V(C_4 \times_L \Phi)| \text{ is even,} \\ \{1,3,5\} & \text{if } |V(C_4 \times_L \Phi)| \text{ is odd;} \end{cases} \\ \mathrm{BI}(C_5 \times_L \Phi) &= \begin{cases} \{1,3,5\} & \text{if } |V(C_5 \times_L \Phi)| \text{ is even,} \\ \{0,2,4,6\} & \text{if } |V(C_5 \times_L \Phi)| \text{ is odd.} \end{cases} \end{aligned}$$

4 Balance Index Sets of the Generalized L-Product of G with Cycles

Theorem 3.3 can be extended to the L-product of any graph G with cycles. Given any friendly labeling f of $G \times_L \Phi$, where $\Phi(v)$ is a cycle for any $v \in V(G)$, denoted by $e_f^*(0)$ and $e_f^*(1)$ the restriction of $e(0)$ and $e(1)$ on G; that is, $e_f^*(0)$ and $e_f^*(1)$ represent the number of edges in G that are labeled by 0 and 1 respectively.

Theorem 4.1 *For any Φ such that $\Phi(v)$ is a cycle for any vertex v, let $p = |V(G \times_L \Phi)|$, and let F denote the set of friendly labelings of $G \times_L \Phi$. Then*

$$BI(G \times_L \Phi) = \begin{cases} \{|e_f^*(0) - e_f^*(1)| : f \in F\} & \text{if } |V(G \times_L \Phi)| \text{ is even,} \\ \{|e_f^*(0) - e_f^*(1) \pm 1| : f \in F\} & \text{if } |V(G \times_L \Phi)| \text{ is odd.} \end{cases}$$

Proof. Let $v_i(0)$, $v_i(1)$, $e_i(0)$ and $e_i(1)$ denote the respective values restricted to the ith cycle $\Phi(c_i)$. Then

$$\begin{aligned} e(0) - e(1) &= e_f^*(0) - e_f^*(1) + \sum_{i=1}^{n}[e_i(0) - e_i(1)] \\ &= e_f^*(0) - e_f^*(1) + \sum_{i=1}^{n}[v_i(0) - v_i(1)] \\ &= e_f^*(0) - e_f^*(1) + v(0) - v(1). \end{aligned}$$

The result follows immediately. □

To determine $\mathrm{BI}(G \times_L \Phi)$, we need to go over all friendly labelings f of $G \times_L \Phi$, study their restrictions on G, and gather the values of $e_f^*(0) - e_f^*(1)$ to form the balance index set.

Corollary 4.2 *For any Φ such that $\Phi(v)$ is a cycle for any vertex v, let $p = |V(St(n) \times_L \Phi)|$. Then*

$$BI(St(n) \times_L \Phi) = \begin{cases} \{0, 1, 2, \ldots, n\} & \text{if } p \text{ is even,} \\ \{0, 1, 2, \ldots, n+1\} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Without loss of generality, we may assume the center c of the star $\mathrm{St}(n)$ is labeled 0. If z of the n pendant vertices of $\mathrm{St}(n)$ are labeled 0, then $e_f^*(0) = z$, and $e_f^*(1) = 0$. Thus, $e_f^*(0) - e_f^*(1) = z$. It is easy to verify that $0 \le z \le n$, because we can label the remaining vertices of $\mathrm{St}(n) \times_L \Phi$ such that the overall labeling is friendly. The result follows immediately from Theorem 4.1. □

Corollary 4.3 *For any Φ such that $\Phi(v)$ is a cycle for any vertex v, let $p = |V(P_n \times_L \Phi)|$. Then*

$$BI(P_n \times_L \Phi) = \begin{cases} \{0, 1, 2, \ldots, n+1\} & \text{if } p \text{ is even,} \\ \{0, 1, 2, \ldots, n+2\} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Let the two pendant vertices of P_n be u and v. Without loss of generality, we may assume that u is labeled 0. Using an argument similar to the ones used in proving Lemma 3.2, one can show that

$$e_f^*(0) - e_f^*(1) = \begin{cases} 2z - n & \text{if } f(v) = 1, \\ 2z - n + 1 & \text{if } f(v) = 0. \end{cases}$$

If $f(v) = 1$, we have $0 \le z \le n - 1$, hence

$$2z - n = -n, -n + 2, \dots, n - 4, n - 2.$$

If $f(v) = 0$, we have $0 \le z \le n$, hence

$$2z - n - 1 = -n - 1, -n + 1, \dots, n - 3, n - 1.$$

The result follows from Theorem 4.1. □

Corollary 4.4 *For any Φ such that $\Phi(v)$ is a cycle for any vertex v, let $p = |V(K_n \times_L \Phi)|$. Then*

$$BI(K_n \times_L \Phi) = \begin{cases} \left\{ \left| \binom{n}{2} - (n-1)k \right| : 0 \le k \le n \right\} & \text{if } p \text{ is even}, \\ \left\{ \left| \binom{n}{2} - (n-1)k \pm 1 \right| : 0 \le k \le n \right\} & \text{if } p \text{ is odd}. \end{cases}$$

Proof. Let k be the number vertices in K_n that are labeled 1, then the $\binom{k}{2}$ edges among them are labeled 1. The other $n - k$ vertices in K_n are labeled 0, hence the $\binom{n-k}{2}$ edges among them are labeled 0. All other edges are unlabeled. Consequently, $e_f^*(0) - e_f^*(1) = \binom{n-k}{2} - \binom{k}{2} = \binom{n}{2} - (n-1)k$, and the result follows from Theorem 4.1. □

5 Balance Index Sets of the L-Products with Complete Graphs

Lemma 5.1 *For $C_n \times_L (K_m, s)$, where $n, m \ge 3$,*

$$e(0) - e(1) = 2z - n + \frac{1}{2}(m-1)[v(0) - v(1)],$$

where $0 \le z \le n$.

Proof. Let the vertices of C_n be $u_1, u_2, \dots, u_n$. Let z_i be the number of 0-vertices in $V(\Phi(u_i)) - V(C_n)$. Thus, the number of 1-vertices in the same set is $m - 1 - z_i$. In a similar manner, let z and $n - z$ be the number of 0- and 1-vertices of C_n, respectively. Then $v(0) = z + \sum_{i=1}^{n} z_i$ and $v(1) = n - z + \sum_{i=1}^{n}(m-1-z_i)$. Consequently,

$$v(0) - v(1) = 2z - n - n(m-1) + 2\sum_{i=1}^{n} z_i.$$

On the base cycle C_n, it is easy to verify that switching any two adjacent vertices does not alter the value of $e(0) - e(1)$. Hence, we may assume the 0-vertices are adjacent to each other, and likewise the 1-vertices form a block of adjacent vertices. Then, we have

$$e(0) = z - 1 + \sum_{i=1}^{z} \binom{z_i+1}{2} + \sum_{i=z+1}^{n} \binom{z_i}{2},$$

and

$$e(1) = n - z - 1 + \sum_{i=1}^{z} \binom{m-1-z_i}{2} + \sum_{i=z+1}^{n} \binom{m-z_i}{2}.$$

It follows from

$$\begin{aligned} & 2\left[\binom{z_i+1}{2} - \binom{m-1-z_i}{2}\right] \\ = \quad & (z_i+1)z_i - [(m-1)-z_i]\,[(m-1)-(z_i+1)] \\ = \quad & -(m-1)^2 + (m-1)(2z_i+1) \end{aligned}$$

and

$$\begin{aligned} 2\left[\binom{z_i}{2} - \binom{m-z_i}{2}\right] &= z_i(z_i-1) - [(m-1)-(z_i-1)][(m-1)-z_i] \\ &= -(m-1)^2 + (m-1)(2z_i-1) \end{aligned}$$

that

$$\begin{aligned} & 2[e(0) - e(1)] \\ = \quad & 2(2z-n) - n(m-1)^2 + (m-1)(2z-n) + 2(m-1)\sum_{i=1}^{n} z_i \\ = \quad & 2(2z-n) + (m-1)[v(0) - v(1)]. \end{aligned}$$

Since the result does not depend on how the vertices of each copy of K_m are labeled, we have $0 \le z \le n$. The proof is now complete. □

Theorem 5.2 *For any integer $n, m \ge 3$,*

$$BI(C_n \times_L (K_m, s)) = \begin{cases} \{|2z-n| : 0 \le z \le n\} & \textit{if } mn \textit{ is even,} \\ \{\left|2z - n \pm \tfrac{1}{2}(m-1)\right| : 0 \le z \le n\} & \textit{if } mn \textit{ is odd.} \end{cases}$$

Proof. The result follows from Lemma 5.1 and the fact that $C_n \times_L (K_m, s)$ has mn vertices. □

Example 8. We find

$$\mathrm{BI}(C_4 \times_L (K_5, s)) = \{|2z-4| : 0 \le z \le 4\} = \{0,2,4\},$$

which is confirmed in Example 4. We also find

$$\mathrm{BI}(C_3 \times_L (K_4, s)) = \{|2z-3| : 0 \le z \le 3\} = \{1,3\},$$

and

$$\mathrm{BI}(C_5 \times_L (K_3, s)) = \{|2z-5 \pm 1| : 0 \le z \le 5\} = \{0,2,4,6\}. \qquad \square$$

What if $\Phi(u_i)$ is the complete graph on m_i vertices, where the m_is are not the same? The argument is almost identical, except that we no longer have a nice simple formula. In particular, we find that $v(0) = z + \sum_{i=1}^n z_i$ and $v(1) = n - z + \sum_{i=1}^n (m_i - 1 - z_i)$. Hence,

$$v(0) - v(1) = 2z - n + \sum_{i=1}^n (2z_i - m_i + 1) = 2z + \sum_{i=1}^n (2z_i - m_i).$$

We also find

$$e(0) = z - 1 + \sum_{i=1}^z \binom{z_i+1}{2} + \sum_{i=z+1}^n \binom{z_i}{2},$$

and

$$e(1) = n - z - 1 + \sum_{i=1}^z \binom{m_i - 1 - z_i}{2} + \sum_{i=z+1}^n \binom{m_i - z_i}{2}.$$

It follows from

$$2\left[\binom{z_i+1}{2} - \binom{m_i-1-z_i}{2}\right] = -(m_i-1)^2 + (m_i-1)(2z_i+1)$$

and

$$2\left[\binom{z_i}{2} - \binom{m_i-z_i}{2}\right] = -(m_i-1)^2 + (m_i-1)(2z_i-1)$$

that

$$e(0) - e(1) = -\frac{1}{2}\sum_{i=1}^n (m_i-1)^2 + \sum_{i=1}^n (m_i-1)z_i + \frac{1}{2}\left[\sum_{i=1}^z m_i - \sum_{i=z+1}^n m_i\right],$$

subject to the conditions that

$$0 \le z_i \le m_i \quad \text{and} \quad \left|2z + \sum_{i=1}^n (2z_i - m_i)\right| \le 1.$$

Since we cannot factor out $(m_i - 1)$, it is not an easy task to find a simple formula. Nevertheless, we do have a generalization of Theorem 5.2.

Theorem 5.3 *For any integer $m \geq 3$, and any graph G with n vertices, and let F denote the set of friendly labelings of $G \times_L \Phi$. Then*

$$BI(G \times_L (K_m, s)) = \begin{cases} \{|e^*(0) - e^*(1)| : f \in F\} & \text{if } mn \text{ is even,} \\ \{|e^*(0) - e^*(1) \pm \frac{1}{2}(m-1)| : f \in F\} & \text{if } mn \text{ is odd,} \end{cases}$$

where $e^*(0)$ and $e^*(1)$ are the restriction of $e(0)$ and $e(1)$ on the graph G.

Proof. The proof is almost identical to that of Theorem 5.2. The difference occurs at the base graph G, hence we only need to replace $2z - n$ with $e^*(0) - e^*(1)$. □

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