Neighborhoods of Unbordered Words in the n-Cube

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Abstract

The n-cube is the graph whose vertices are all binary words of length $n > 1$ and whose edges join vertices that differ in exactly one entry; i.e., are at Hamming distance 1 from each other. If a word has a non-empty prefix, not the entire word, which is also a suffix then it is said to be bordered. A word that is not bordered is unbordered. Unbordered words have been studied extensively and have applications in synchronizable coding and pattern matching. The neighborhood of an unbordered word $\mathbf{w}$ is the word itself together with the set of words at Hamming distance 1 from $\mathbf{w}$. Over the binary alphabet the neighborhood of an unbordered word $\mathbf{w}$ always contains two bordered words obtained by complementing the first and last entries of $\mathbf{w}$. We determine those unbordered words $\mathbf{w}$ whose neighborhoods otherwise contain only only unbordered words.

1 Introduction

Words have been given various names in the literature: strings, sequences, lists, texts, or vectors. Formally, A *word* is a finite ordered sequence of elements chosen from a finite alphabet. For emphasis words will be written in boldface. The alphabet used in this paper is the binary set $\{0, 1\}$. The *n-cube*, Q_n, $n > 1$, is the graph whose vertices are words of length n over the alphabet $\{0, 1\}$ with two vertices joined by an edge if they differ in exactly one entry, i.e., two vertices are joined by an edge if the Hamming distance between them is 1. The *Hamming distance* between two words in Q_n is the number of entries in which they differ. The *complement* of $x \in \{0,1\}$ is denoted by $\overline{x}$; i.e., $\overline{0} = 1, \overline{1} = 0$. If $\mathbf{w} = w_1 \dots w_n$ is a word with entries w_i, then a *border* of w is a non-empty word that is both a proper prefix and a suffix of w. That is, w has a border if there exists an integer p, $1 \leq p < n$, such that $w_1 \dots w_p = w_{n-p+1} \dots w_n$. A word without a border is said to be *unbordered.* In the engineering literature, a border of a word in the n-cube is called a bifix and an unbordered word is called bifix-free [8]. Lothaire[6, Chapter 8] refers to unbordered words as primary words.

Every bordered word has a shortest border which is necessarily unbordered. The shortest border of a word is the only border of the word that is itself unbordered [4, Lemma 2]. By Nielsen's lemma given below as Lemma 1, the shortest border of a word is not longer than half the word length.

Let B_n denote the set of bordered binary words of fixed length n over $\{0,1\}$ and U_n the set of unbordered binary words of length n. Every binary word of length $n > 1$ which both begins and ends in 0 or begins and ends in 1 is bordered. Accordingly, unbordered words must either begin in 0 and end in 1 or begin in 1 and end in 0. We study the subgraphs of Q_n induced by unbordered words.

Definition 1 *Define the following induced subgraphs of Q_n:*

$$\begin{aligned} Q_n^{01} &= \{\mathbf{w} \in Q_n | \mathbf{w} = 0 \cdots 1\} \\ B_n^{01} &= \{\mathbf{w} \in Q_n^{01} | \mathbf{w} \in B_n\} \\ U_n^{01} &= \{\mathbf{w} \in Q_n^{01} | \mathbf{w} \in U_n\}. \end{aligned}$$

and similarly define Q_n^{10}, B_n^{10}, and U_n^{10}. If $n = 2$ then $B_2 = \{00, 11\}$ and $U_2 = \{01, 10\}$ while B_2^{01} is empty and $U_2^{01} = \{01\}$.

The *neighborhood* of an unbordered word w in Q_n is the word itself together with the set of words at Hamming distance 1 from w and will be denoted simply $N(\mathbf{w})$. The set of words at Hamming distance 1 from a vertex w in the subgraph Q_n^{01} together with w itself will be denoted $N^{01}(\mathbf{w})$; i.e.

$$N^{01}(\mathbf{w}) = N(\mathbf{w}) \cap Q_n^{01}.$$

Similarly define $N^{10}(\mathbf{w})$. For example, if $\mathbf{w} = 01$ then $N(\mathbf{w}) = \{01, 00, 11\}$ while $N^{01}(\mathbf{w}) = \{01\}$.

The vertices which are unbordered words in the n-cube are partitioned into two disjoint subgraphs:

$$U_n = U_n^{01} \cup U_n^{10}.$$

It was shown in [1] that the induced subgraphs U_n^{01}, U_n^{10} in Q_n are connected. It is immediate that the Hamming distance between a word in Q_n^{01} and a word in Q_n^{10} is at least 2.

Since U_n^{01} and U_n^{10} are isomorphic as graphs under both reversal and complementation, results are stated only for U_n^{01} in the following .

2 Generating Unbordered Words

The following lemma of Nielsen [8] ensures that any algorithm to determine whether a word of length n is unbordered need not check for borders of length greater than $\lfloor \frac{n}{2} \rfloor$.

Lemma 1 (Nielsen, 1973) *A word* $\mathbf{w}$ *of length* n *over an arbitrary alphabet is unbordered if and only if it has no borders of lengths* $1, 2, \ldots \lfloor \frac{n}{2} \rfloor$.

Nielsen [8] also gave a recursive method for generating unbordered words over any finite alphabet. We restrict attention to the binary case since our interest is the geometrical relations between bordered and unbordered words in Q_n.

Definition 2 *If* $n = 2k$, $k > 0$, *and* $\mathbf{w} = w_1 \cdots w_{2k} \in Q_n$ *define for each* $x \in \{0, 1\}$ *vertex mappings* $\sigma_x : Q_{2k} \longrightarrow Q_{2k+1}$ *by*

$$\sigma_x(\mathbf{w}) = w_1 \cdots w_k x w_{k+1} \cdots w_{2k}.$$

Clearly, each σ_x is injective but not surjective.

Definition 3 *If* $n = 2k$, $k > 0$, *and* $\mathbf{w} = w_1 \cdots w_{2k} \in Q_n$ *define for* $a, b \in \{0, 1\}$ *vertex mappings* $\tau_{ab} : Q_{2k} \longrightarrow Q_{2k+2}$ *by*

$$\tau_{ab}(\mathbf{w}) = w_1 \cdots w_k ab w_{k+1} \cdots w_{2k}.$$

As with σ_x, τ_{ab} is injective but not surjective.

Theorem 1 (Nielsen, 1973) *If* $\mathbf{w} = w_1 \cdots w_n \in Q_n$ *and* $n = 2k$ *then*

$$\sigma_x(\mathbf{w}) \in U_{n+1} \Longleftrightarrow \mathbf{w} \in U_n \quad (1)$$

and

$$\tau_{ab}(\mathbf{w}) \in U_{n+2} \Longrightarrow \mathbf{w} \in U_n. \quad (2)$$

The converse of the latter implication holds if and only if

$$w_2 \cdots w_k = w_{k+1} \cdots w_{n-1} \Longrightarrow w_1 \neq b \text{ or } w_n \neq a.$$

Comment: Any border of $\tau_{ab}(\mathbf{w}) = w_1 \cdots w_k ab w_{k+1} \cdots w_n$ with length less than or equal to $k+1$ is either a border of $\mathbf{w}$ or is $w_1 \cdots w_k a = b w_{k+1} \cdots w_n$.

3 Pure Neighborhoods of Unbordered Words

If $n = 3$ then the neighbors of 001 in Q_3 are 001, 101, 011, 000 and the neighbors of 011 in Q_3 are 011, 111, 001, 010. However, when restricted to the induced subgraph, Q_3^{01}, $N^{01}(001) = \{001, 011\} = N^{01}(011)$.

Proposition 1 *If* $\mathbf{w}$ *is any unbordered word in* U_n^{01} *and* $n > 1$ *then* $\mathbf{w}$ *has at least two bordered neighbors in* B_n.

Proof: If $\mathbf{w} \in U_n^{01}$ then $\mathbf{w} = 0w_2 \cdots w_{n-1}1$ for $w_i \in \{0,1\}$. Both $0w_2 \cdots w_{n-1}0$ and $\mathbf{w} = 1w_2 \cdots w_{n-1}1$ are neighboring bordered words.

Proposition 2 *Let* $\mathbf{w} \in U_n^{01}, n = 2k,\ k > 1,$ *and* $\mathbf{w}' \in N^{01}(\mathbf{w})$. *If* $\mathbf{w} \neq \mathbf{w}'$, *then*

$$H(\sigma_x(\mathbf{w}), \sigma_x(\mathbf{w}')) = 1 \quad (3)$$

and

$$H(\mathbf{w}, \mathbf{w}') = H(\tau_{ab}(\mathbf{w}), \tau_{ab}(\mathbf{w}')) = 1 \quad (4)$$

where H *denotes Hamming distance.*

Proof: Since $\mathbf{w}$ and $\mathbf{w}'$ agree in all entries except one, the same is true for $\sigma_x(\mathbf{w})$ and $\sigma_x(\mathbf{w}')$ by Definition 2. The argument is similiar for τ_{ab}.

Proposition 3 *If* $\mathbf{w} \in Q_n^{01}$, $n = 2k$, $k > 1$ *then*

$$\sigma_x(N^{01}(\mathbf{w})) \subset N^{01}(\sigma_x(\mathbf{w})) \tag{5}$$

and

$$\tau_{ab}(N^{01}(\mathbf{w})) \subset N^{01}(\tau_{ab}(\mathbf{w})). \tag{6}$$

Proof: Let $\mathbf{w}' \neq \mathbf{w} \in N^{01}(\mathbf{w})$. By (3), $\sigma_x(\mathbf{w}') \in N^{01}(\sigma_x(\mathbf{w}))$. Therefore, (5) holds. By (4), $\tau_{ab}(\mathbf{w}') \in N^{01}(\tau_{ab}(\mathbf{w}))$. Therefore, (6) holds.

Definition 4 *An unbordered word* $\mathbf{w} \in U_n^{01}$ *has a* pure neighborhood *if*

$$N^{01}(\mathbf{w}) \cap B_n^{01} = \phi.$$

There are no pure neighborhoods for unbordered words in Q_n itself as was noted in Proposition 1.

Lemma 2 *Let* $\mathbf{w} \in U_n^{01}$ *where* $n = 2k, k > 1$. *Then,* $N^{01}(\mathbf{w})$ *is a pure neighborhood if and only if* $N^{01}(\sigma_x(\mathbf{w}))$ *is pure.*

Proof: Assume $N^{01}(\mathbf{w})$ is pure and let $\mathbf{w} = w_1 \cdots w_n$. If $\mathbf{v}$ is any word in $N^{01}(\sigma_x(\mathbf{w}))$ not $\sigma_x(\mathbf{w})$ then $\mathbf{v}$ differs from $\sigma_x(\mathbf{w})$ in exactly one entry.

Case 1. If $\mathbf{v}$ differs from $\sigma_x(\mathbf{w})$ in entry $k+1$ then

$$\mathbf{v} = w_1 \cdots w_k \overline{x} w_{k+1} \cdots w_n = \sigma_{\overline{x}}(\mathbf{w}) \tag{7}$$

since the alphabet is $\{0, 1\}$. If $\mathbf{v}$ has a border then it has a border of length less than or equal to $k+1$ by Lemma 1. Thus, by Definition 2, any such border of $\mathbf{v}$ is necessarily a border of $\mathbf{w}$, contradicting the assumption that $\mathbf{w} \in U_n^{01}$. Therefore, $N^{01}(\sigma_x(\mathbf{w}))$ is pure in this case.

Case 2. Now suppose

$$\mathbf{v} = w_1 \cdots \overline{w_i} \cdots w_k x w_{k+1} \cdots w_n \tag{8}$$

or

$$\mathbf{v} = w_1 \cdots w_k x w_{k+1} \cdots \overline{w_i} \cdots w_n \tag{9}$$

for some i, $1 \leq i \leq n$. Since $\mathbf{v} \in N^{01}(\sigma_x(\mathbf{w}))$, $1 < i < n$.

If (8) then $\mathbf{v} = \sigma_x(\mathbf{w}^*)$ where $\mathbf{w}^* = w_1 \cdots \overline{w_i} \cdots w_n$. Clearly, $\mathbf{w}^* \in N^{01}(\mathbf{w})$, which is pure by hypothesis. Thus, by Theorem 1, $\mathbf{v} \in U_n^{01}$. Since $\mathbf{v}$ was an arbitrary element of $N^{01}(\sigma_x(\mathbf{w}))$ we conclude the latter is pure. The argument is similiar if (9). In conclusion,

$$N^{01}(\sigma_x(\mathbf{w})) = \sigma_x(N^{01}(\mathbf{w}) \cup \{\sigma_{\overline{x}}(\mathbf{w})\}. \tag{10}$$

Conversely, suppose $N^{01}(\sigma_x(\mathbf{w}))$ is pure and let $\mathbf{w}' \in N^{01}(\mathbf{w}), \mathbf{w}' \neq \mathbf{w}$. If $\mathbf{w}'$ is bordered then it has a border of length shorter than $\lfloor \frac{n}{2} \rfloor$ by Lemma 1. By the definition of $\sigma_x, \sigma_x(\mathbf{w}')$ would have the same border. But $\mathbf{w}' \in N^{01}(\mathbf{w})$ implies $\sigma_x(\mathbf{w}') \in N^{01}(\sigma_x(\mathbf{w}))$ by Proposition 2 and $N^{01}(\sigma_x(\mathbf{w}))$ is pure by assumption. Thus, by Theorem 1, $\mathbf{w}'$ is unbordered. Hence $N^{01}(\mathbf{w})$ is pure.

Lemma 3 *Let $\mathbf{w} = w_1 \cdots w_n \in U_n^{01}$ where $n = 2k, k > 1$. If $N^{01}(\tau_{ab}(\mathbf{w}))$ is pure then $N^{01}(\mathbf{w})$ is pure. Conversely, if $N^{01}(\mathbf{w})$ is pure and $w_1 \neq b$, $w_n \neq a$ then $N^{01}(\tau_{ab}(\mathbf{w}))$ is pure.*

Proof: First assume $N^{01}(\tau_{ab}(\mathbf{w}))$ is pure and let $\mathbf{w}' \in N^{01}(\mathbf{w})$, $\mathbf{w}' \neq \mathbf{w}$. By (6), $\tau_{ab}(\mathbf{w}') \in N^{01}(\tau_{ab}(\mathbf{w}))$ which is pure by assumption. If $\mathbf{w}'$ were bordered then $\tau_{ab}(\mathbf{w}')$ would be by Lemma 1, a contradiction. Therefore, $N^{01}(\mathbf{w})$ is pure.

Conversely assume $N^{01}(\mathbf{w})$ is pure and first suppose $\mathbf{v} \in N^{01}(\tau_{ab}(\mathbf{w}))$ differs from $\tau_{ab}(\mathbf{w})$ in some entry other than the two central ones, say, $\mathbf{v} = w_1 \cdots \overline{w_j} \cdots w_k ab w_{k+1} \cdots w_n$, since the alphabet is binary. Then, $\mathbf{v} = \tau_{ab}(\mathbf{u})$ where $\mathbf{u} = w_1 \cdots \overline{w_j} \cdots w_k w_{k+1} \cdots w_n \in N^{01}(\mathbf{w})$. Since $N^{01}(\mathbf{w})$ is pure, $\mathbf{u}$ is unbordered and therefore $\mathbf{v} = \tau_{ab}(\mathbf{u})$ is unbordered since it would inherit all borders of $\mathbf{u}$ with length strictly less than k. But the possibility remains of a border with length exactly k; i.e., $w_1 \cdots \overline{w_j} \cdots w_k a = b w_{k+1} \cdots w_n$. However, in that case, , $w_1 = b$ and $w_n = a$, contrary to the hypothesis.

Now assume that $\mathbf{v}$ differs from $\tau_{ab}(\mathbf{w})$ in entry $k+1$ or $k+2$ so that $\mathbf{v}$ is one of:

$$\mathbf{v} = w_1 \cdots w_k \overline{a} b w_{k+1} \cdots w_n = \tau_{\overline{a}b}(\mathbf{w}) \tag{11}$$

$$\mathbf{v} = w_1 \cdots w_k a \overline{b} w_{k+1} \cdots w_n = \tau_{a\overline{b}}(\mathbf{w}). \tag{12}$$

If, say, (11) then any border of $\mathbf{v}$ of length less than or equal to $k+1$ is either a border of $\mathbf{w}$ or is

$$w_1 \cdots w_k \overline{a} = b w_{k+1} \cdots w_n. \tag{13}$$

But if (13) holds then $w_1 = b$, contrary to hypothesis. Similiarly, if (12) holds then $w_n = a$, again contradicting the hypothesis. Accordingly, $N^{01}(\tau_{ab}(\mathbf{w}))$ is pure.

It will be convenient to use the notation of formal language theory and write words from U_n^{01} in the form $0^{a_1}1^{b_1}\cdots 0^{a_t}1^{b_t}, 0 < a_i, b_i; i = 1, \ldots t$. where x^c denotes the concatenation of x with itself c times.

If $t = 1$ it is easy to see that any word $\mathbf{w}$ of the form 0^a1^b, $a, b > 0$ is unbordered. In particular, $0^{n-1}1$ is unbordered. If $n = 2$ then $N^{01}(\mathbf{w})$ is the singleton $\{01\}$ and 01 is unbordered.

Lemma 4 *For every $n > 2$, none of the words $0^{n-1}1 \in U_n^{01}$ have pure neighborhoods.*

Proof: Let $\mathbf{w} = 0^{n-1}1$. For $n > 2$ the neighbors 0^n and $10^{n-2}1$ of $\mathbf{w}$ in Q_n are bordered but are not in U_n^{01}. The word $0^{n-1}1$ has $n-2$ other neighbors in Q_n^{01} all which have the form 0^a10^b1 where $a + b = n - 2$ and $a > b \geq 0$. These are the $n - 2$ words

$$0^{n-2}11, 0^{n-3}101, \ldots, 010^{n-3}1.$$

If $a \leq b$ in 0^a10^b1 then 0^a1 is a border of 0^a10^b1 and any border of 0^a10^b1 must have this form. Thus, the other unbordered neighbors of $0^{n-1}1$ in Q_n^{01} are of the form $0^a10^b1, a > b \geq 0$. If n is even the others are the $\frac{n-2}{2}$ words

$$0^{n-2}11, 0^{n-3}101, \ldots, 0^{\frac{n}{2}}10^{\frac{n-4}{2}}1.$$

If n is odd then the other unbordered neighbors of $0^{n-1}1$ in $U_n^{01}$are the $\frac{n-1}{2}$ words

$$0^{n-2}11, 0^{n-3}101, \ldots, 0^{\frac{n+1}{2}}10^{\frac{n-1}{2}}1.$$

The argument is similar for words of the form $1^{n-1}0 \in U_n^{10}$.

Definition 5 *A word* $\mathbf{v} \in Q_n$ *is a* descendant of a word $\mathbf{u} \in Q_m$, $m < n$, if there exists a sequence $f_1, f_2, \cdots f_k, f_i \in \{\sigma_x, \tau_{ab} | x, a, b \in \{0, 1\}\}$ such that $f_k \circ f_{k-1} \circ \cdots \circ f_1(\mathbf{u}) = \mathbf{v}$.

Note that the empty sequence ensures that every word is considered a descendant of itself. Since the images of σ_x have odd length the compositions $\sigma_x \circ \sigma_x$ and $\tau_{ab} \circ \sigma_x$ never occur in a sequence of descendants. Thus, the possible descendant sequences are the words $\sigma_x \circ \tau_{ab}^k$, and $\tau_{ab}^k, k > 0$, for $x, a, b \in \{0, 1\}$.

If $n = 4$ then it is easy to see that $U_4^{01} = \{0001, 0011, 0111\}$. Of these three unbordered words only 0011 has a pure neighborhood, since by Lemma 4, 0001 and 0111 cannot have pure neighborhoods. In fact, both 0001 and 0111 are adjacent to the bordered word 0101. $N^{01}(0011) = \{0011, 0111, 0001\}$ is the only pure neighborhood when $n = 4$.

Example 1 *Assume* $\mathbf{w} \in Q_5$ *is a descendant of* 0011. *Then,* $\mathbf{w} = 00a11$ *where either* $a = 0$ *or* $a = 1$. *These two words and their* Q_5^{01} *pure neighborhoods are*

0	0	0	1	1		0	0	1	1	1
0	1	0	1	1		0	1	1	1	1
0	0	1	1	1		0	0	0	1	1
0	0	0	0	1		0	0	1	0	1

The following theorem shows that the only unbordered words in Q_n^{01} with pure neighborhoods are descendants of 0011 and only descendants of these length 4 words satisfying a mild restriction have pure neighborhoods. Because of Theorem 1, the inductive proof naturally falls into two parts depending on whether f_1 in Definition 5 is a σ_x or a τ_{ab}.

Theorem 2 *If* $n \geq 5$ *is odd any unbordered word* $\mathbf{w} \in U_n^{01}$ *has a pure neighborhood if and only if it is a descendant of* $0011 \in U_4^{01}$.

Proof: Since n is odd, let $n = 2k + 1$, $k > 1$ and $\mathbf{w} \in U_n^{01}$. Theorem 1 implies there is $\mathbf{v} \in U_{n-1}^{01}$ such that $\mathbf{w} = \sigma_{w_{k+1}}(\mathbf{v})$, where $\mathbf{w} = w_1 \cdots w_n$. Assume $N^{01}(\mathbf{w}) \subset U_n^{01}$ is pure. By Lemma 2, $N^{01}(\mathbf{v}) \subset U_{n-1}^{01}$. By induction, $\mathbf{v}$ is a descendant of 0011. Hence $\mathbf{w}$ is a descendant of 0011 since the first two and last two entries of $\mathbf{w}$ are the same as those of $\mathbf{v}$ by the definition of σ_x.

Conversely, let $\mathbf{w} \in U_n^{01}$ be a descendant of 0011. By Theorem 1, $\mathbf{w} = \sigma_x(\mathbf{u})$ for some $x \in \{0, 1\}$ and $\mathbf{u} \in U_n^{01}$. Since $\mathbf{w}$ is a descendant of 0011 by hypothesis, the unbordered word $\mathbf{u}$ is a descendant of 0011 by definition. By induction, $N^{01}(\mathbf{u})$ is pure. By Lemma 2 $N^{01}(\sigma_x(\mathbf{w}))$ is pure for each $x \in \{0, 1\}$.

Theorem 3 *Let* $n \geq 6$ *be even and* $\mathbf{w} \in U_n^{01}$. *Necessarily,* $\mathbf{w} = \tau_{ab}(\mathbf{u})$ *for some* $\mathbf{u} \in U_{n-2}^{01}$ *and* $a, b \in \{0, 1\}$. *If* $\mathbf{w}$ *has a pure neighborhood then it is a descendant of* 0011. *Conversely, if* $\mathbf{w}$ *is a descendant of* 0011 *then it has a pure neighborhood provided only that* $w_1 \neq b$ *and* $w_n \neq a$.

Proof: Let $\mathbf{w} = w_1 \cdots w_n$. Since n is even, Theorem 1 implies $\mathbf{w} = \tau_{ab}(\mathbf{u})$ for $a, b \in \{0,1\}$ and $\mathbf{u} \in U^{01}_{n-2}$. By the definition of τ_{ab}, $\mathbf{u}$ is unbordered since $\mathbf{w}$ is. Assume $N^{01}(\mathbf{w}) \subset U^{01}_n$ is a pure neighborhood. By Lemma 3, $N^{01}(\mathbf{u})$ is pure. By induction $\mathbf{u}$ is a descendant of 0011 and hence $\mathbf{w}$ is.

Conversely, suppose $\mathbf{w} \in U^{01}_n$ is a descendant of 0011. If $u = u_1 \cdots u_n$ then $u_1 = w_1$ and $u_n = w_n$ by the definition of τ_{ab}. Accordingly, The condition $w_1 \neq b$ and $w_n \neq a$ necessarily applies to $\mathbf{u}$. By induction, $N^{01}(\mathbf{u})$ is a pure neighborhood. Therefore, $N^{01}(\mathbf{w})$ is pure by Lemma 3.

Example 2 *The six arrays below are all the pure neighborhoods in* U^{01}_7. *The first line of each array is the central word.*

$$\begin{array}{ccccccc} 0&0&0&0&0&1&1\\ 0&1&0&0&0&1&1\\ 0&0&1&0&0&1&1\\ 0&0&0&1&0&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1 \end{array} \qquad \begin{array}{ccccccc} 0&0&0&0&1&1&1\\ 0&1&0&0&1&1&1\\ 0&0&1&0&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&1&0&1 \end{array}$$

$$\begin{array}{ccccccc} 0&0&0&1&1&1&1\\ 0&1&0&1&1&1&1\\ 0&0&1&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&1&0&1&1\\ 0&0&0&1&1&0&1 \end{array} \qquad \begin{array}{ccccccc} 0&0&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&1&0&1&1&1\\ 0&0&1&1&0&1&1\\ 0&0&1&1&1&0&1 \end{array}$$

$$\begin{array}{ccccccc} 0&0&0&1&0&1&1\\ 0&1&0&1&0&1&1\\ 0&0&1&1&0&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&1&0&0&1 \end{array} \qquad \begin{array}{ccccccc} 0&0&1&0&1&1&1\\ 0&1&1&0&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&1&1&1&1&1\\ 0&0&1&0&0&1&1\\ 0&0&1&0&1&0&1 \end{array}$$

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