On the nonexistence of graphs of diameter 2 and defect 2

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Abstract

In 1960, Hoffman and Singleton investigated the existence of Moore graphs of diameter 2 (graphs of maximum degree d and d^2+1 vertices), and found that such graphs exist only for $d = 2, 3, 7$ and possibly 57. In 1980, Erdős *et al.*, using eigenvalue analysis, showed that, with the exception of C_4, there are no graphs of diameter 2, maximum degree d and d^2 vertices. In this paper, we show that

graphs of diameter 2, maximum degree d and $d^2 - 1$ vertices do not exist for most values of d with $d \geq 6$, and conjecture that they do not exist for any $d \geq 6$.

1 Introduction

There are many famous and difficult graph-theoretical problems that arose over the past four decades from the design of interconnection networks (such as local area networks, parallel computers, switching system architecture in VLSI technology, and many others). Perhaps one of the most prominent problems is the *degree/diameter problem*, which is to determine, for each d and k, the largest order $n_{d,k}$ of a graph of maximum degree d and diameter at most k. It is easy to show that $n_{d,k} \leq M_{d,k}$ where $M_{d,k}$ is the *Moore bound* given by

$$n_{d,k} \leq M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$$

For a survey of the degree/diameter problem, see [9].

In this paper, we concentrate on the case when the diameter is equal to 2. Since a graph of diameter 2 and maximum degree d can have at most $d^2 + 1$ vertices, it was asked in [4]: Given non-negative numbers d and Δ (*defect*), is there a graph of diameter 2 and maximum degree d with $d^2 + 1 - \Delta$ vertices? It was proved in [6] that if $\Delta = 0$ then there are graphs corresponding to $d = 2, 3, 7$, and possibly $d = 57$. For each of the values $d = 2, 3$ and 7, there exists a unique such graph [6]. The case $\Delta = 1$ was solved by Erdös *et al.* [4]. In this paper, we consider the case $\Delta = 2$, and prove that graphs of defect 2 do not exist for infinitely many values of degree $d \geq 6$.

We refer to a graph of maximum degree d, diameter $k \geq 2$ and order $M_{d,k} - \Delta$ ($\Delta \geq 1$) as a (d, k, Δ)-*graph*. Let G be a (d, k, Δ)-graph.

Definition 1 Let u be a vertex in G. A vertex v in G is called a *repeat* of u with *multiplicity* $m_v(u)$ ($1 \leq m_v(u) \leq \Delta$) if there are exactly $m_v(u) + 1$ different paths of lengths at most k from u to v.

It is immediate that

Observation 1 *Vertex u is a repeat of v with multiplicity $m_u(v)$ if, and only if, v is a repeat of u with the same multiplicity.*

A repeat with multiplicity 1 will be called a *single* repeat, a repeat with multiplicity 2 will be called a *double* repeat, and a repeat with multiplicity Δ will be called a *maximal* repeat.

We denote by $R(u)$ the set of all repeats of a vertex u in G. Taking into account the multiplicities of repeats, we denote by $R_m(u)$ the multiset of all the repeats of a vertex u in G containing each repeat v of u exactly $m_v(u)$ times.

Let u be a vertex in G, we denote by $N(u)$ the set of all neighbours of u. If A is a multiset of vertices of G then $N(A)$ denotes the multiset of all the neighbours of the vertices of A. We use $R_m(A)$ to denote the multiset of all the repeats of all vertices in A.

Proposition 1 *If G is regular then, for all $u \in V(G)$,*

$$|R_m(u)| = \sum_{v \in R(u)} m_v(u) = \Delta.$$

□

In [8], Miller *et al.* proved the following:

Theorem 1 (Neighborhood Theorem [8]) *For every vertex u in a regular (d, k, Δ)-graph, $N(R_m(u)) = R_m(N(u))$.*

Definition 2 A subset S of $V(G)$ is called a *closed repeat set* if $R_m(S) = S$. A closed repeat set is *minimal* if none of its proper subsets is a closed repeat set.

Definition 3 A *repeat subgraph* H_S of a closed repeat set S of G is a multigraph whose vertex set $V(H_S) = S$ and the number of parallel edges between a vertex u and any of its repeats, say $v \in R_m(u)$, equals the multiplicity $m_v(u)$.

We observe that

Observation 2 *If $\Delta < 1 + (d-1) + \cdots + (d-1)^{k-1}$ then G is regular.*

It is also true that

Observation 3 *If G is regular then the repeat graph H_G of G is Δ-regular.*

For the purpose of this paper, we shall consider each pair of parallel edges in H_G as a cycle of length 2.

Observation 4 *H_G is the union of cycles of lengths ≥ 2, each cycle being a minimal closed repeat set of G.*

Note that instead of writing "a vertex x is adjacent to a vertex y", we write $x \sim y$. Unless explicitly shown where necessary, by u_i and u_j $(i \neq j)$ we shall mean two distinct vertices.

2 Structural properties of $(d, 2, 2)$-graphs

In this section, we consider graphs of diameter 2 with defect 2. For $d \leq 2$, the path on 3 vertices is the only such graph. Let G be a $(d, 2, 2)$-graph for $d \geq 3$. From Observation 2, we have that

Observation 5 *For $d \geq 3$, every $(d, 2, 2)$-graph is regular.*

Let us consider the possible repeat configurations in $(d, 2, 2)$-graphs. Let u be a vertex of a $(d, 2, 2)$-graph. Then,

(*i*) u has two single repeats, $r_i(u)$, $i = 1, 2$.

(*ii*) u has one double (maximal) repeat, $r(u) = r_1(u) = r_2(u)$ with multiplicity 2.

With respect to repeats in G, there are five possible repeat configurations, as depicted in Figure 1.

We will denote the set of vertices of each type as Type 0, Type 1, Type 2a, Type 2b and Type 2c, as shown in Figure 1. We denote by n_0, n_1, n_{2a}, n_{2b} and n_{2c} the number of vertices of the corresponding repeat types.

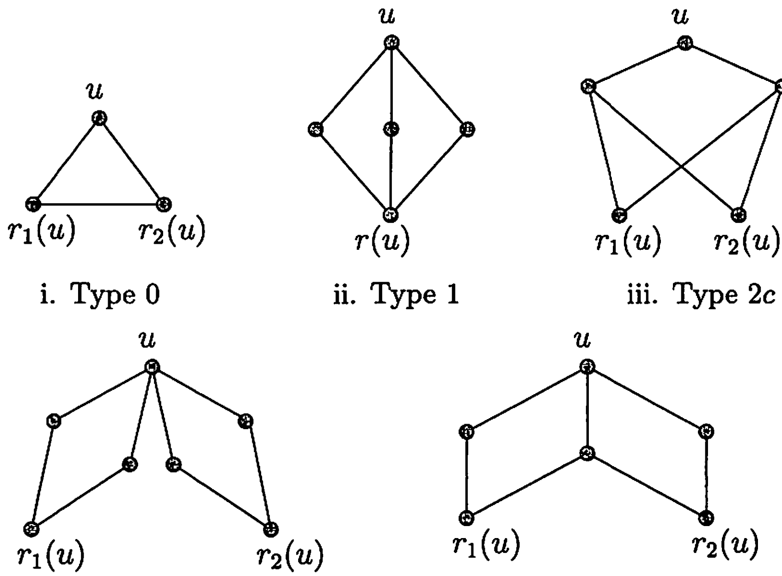
Figure 2 shows the only known non-isomorphic $(d, 2, 2)$-graphs for $d \geq 3$.

We observe the following

Observation 6 $n_0 + n_1 + n_{2a} + n_{2b} + n_{2c} = d^2 - 1$.

From now on, each cycle in H_G will be called a *repeat cycle*.

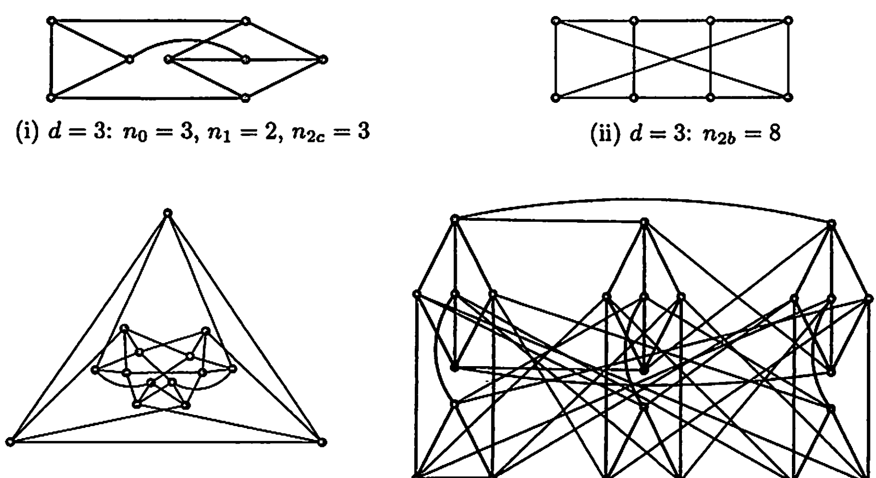
The following structural properties of G were proved in [10].



i. Type 0 ii. Type 1 iii. Type $2c$

iv. Type $2a$ v. Type $2b$

Figure 1: Possible repeat configurations for vertex u in a $(d, 2, 2)$-graph.



(i) $d = 3$: $n_0 = 3$, $n_1 = 2$, $n_{2c} = 3$ (ii) $d = 3$: $n_{2b} = 8$

(iii) $d = 4$: $n_0 = 3$, $n_{2b} = 12$ (iv) $d = 5$: $n_0 = 9$, $n_1 = 6$, $n_{2c} = 9$

Figure 2: All the known $(d, 2, 2)$-graphs.

Theorem 2 *[10] Let G be a $(d,2,2)$-graph G. Then,*

(i) If d is even then $n_0 = 3$ and $n_{2b} = d^2 - 4$. Furthermore, if $d = 4$ then G is the graph in Figure 2(iii), whose uniqueness was proved in [2].

(ii) If d is odd then either

- *$d = 3$ and G is the graph in Figure 2(i) or 2(ii); or*
- *$d = 5$ and G is the graph in Figure 2(iv); or*
- *$d \geq 7$ and $n_{2b} = d^2 - 1$.*

Corollary 1 *[10]* $n_{2b} \equiv 0 \pmod 2$.

$(d,2,2)$-graphs for even d

Let the vertices u_0, u_1, u_2 form a triangle in G, denoted by T, and let Υ_{2b} be the subset of all vertices of Type $2b$ in $N(u_0) \bigcup N(u_1) \bigcup N(u_2)$. Then Υ_{2b} is a minimal closed repeat set. We shall call Υ_{2b} the *outer repeat cycle* of T in H_G. Note that Υ_{2b} is the set of vertices at distance 1 from T, and $\Upsilon_{2b} \bigcap T = \emptyset$. The number of vertices of Υ_{2b} is $3(d-2)$.

Figure 3 illustrates a labeled partial structure of G in the case of even even d, and shows the cycle $u_0u_1u_2$ and its outer repeat cycle. Since all the vertices in Υ_{2b} are of Type $2b$, and Υ_{2b} is a minimal closed repeat set, by Corollary 1, there exists in H_G another cycle Υ'_{2b}, also of the same size as Υ_{2b}, that is, $3(d-2)$. Note that, in Figure 3, $u_3 \sim u_{3d-4}$ and $u_{3d-3} \sim u_{9d-16}$. This is because u_3 and u_{9d-16} belong to Υ_{2b}, whereas u_{3d-3} and u_{3d-4} belong to Υ'_{2b}.

Lemma 1 *[10] Let T be a triangle in G and let Υ_{2b} be the outer repeat cycle of T in H_G. Let C_t be any repeat cycle in H_G of length $t \geq 4$ such that there exists in G an edge between a vertex on Υ_{2b} and a vertex on C_t. Then either $t = \frac{1}{3}|\Upsilon_{2b}|$ or $t \equiv 0 \pmod{|\Upsilon_{2b}|}$.*

The methods used to deal with $(d,2,2)$-graphs for even d differ from the ones used for odd d. Therefore, we will analyze these classes of graphs in different sections.

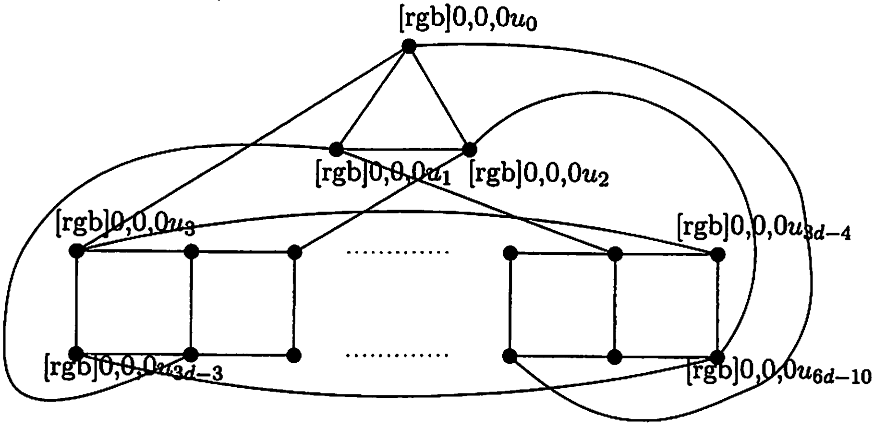


Figure 3: An illustration of the neighborhood of T in G for even d.

3 Nonexistence of infinitely many $(d, 2, 2)$-graphs for even d

In this section, we shall prove that for most values of even d, $(d, 2, 2)$-graphs do not exist.

From Theorem 2, it immediately follows that

Corollary 2 *G is not vertex-transitive for even degree d.*

Lemma 2 *For even $d \geq 6$, every cycle, other than the triangle in H_G, has length $3k(d-2)$ for some $k \geq 1$.*

Proof. As demonstrated in Section 2, if d is even then H_G contains one cycle of length 3 and at least two cycles of length $3(d-2)$. Let $u_1u_2u_3$ be the triangle T of G, and let $v_1 \ldots v_{3(d-2)}$ be the outer repeat cycle Υ_{2b} of T in H_G such that the repeats of v_j $(1 \leq j \leq 3(d-2))$ are $v_{(j-1)(\bmod (3(d-2)))}$ and $v_{(j+1)(\bmod (3(d-2)))}$. Without loss of generality, let us suppose that $u_1 \sim v_1$ and $u_2 \sim v_2$ in G.

Let the a_i, $i = 1, \ldots, b$, be the lengths of the cycles in H_G, and let $a_1 = 3$, $a_2 = a_3 = 3(d-2)$ correspond to T, Υ_{2b} and Υ'_{2b}, respectively. Thus, $f = \sum_{i=4}^{b} a_i = (d-2)(d-4)$.

Let C_{a_j} be an arbitrary cycle in H_G $(j \neq 1, 2, 3)$. Then, by Lemma 1, either $a_j = d-2$, or $a_j \equiv 0 \pmod{3(d-2)}$. Suppose that $a_j = d-2$.

Denote by $w_1, \ldots, w_{d-2}$ the vertices of C_{a_j} such that the repeats of w_k $(1 \leq k \leq d-2)$ are $w_{(k-1) \pmod{(d-2)}}$ and $w_{(k+1) \pmod{(d-2)}}$.

We know that the vertices of C_{a_j} must reach the vertices of T through the vertices of Υ_{2b}. Without loss of generality, suppose that $w_1 \sim v_1$ and $w_2 \sim v_2$. However, since $(d-2)$ is not divisible by 3 when d is even, by the Neighborhood Theorem, u_1 and w_1 would then have at least three common neighbors, namely v_1, v_{d-1} and v_{2d-3}. This is clearly impossible.

Therefore, each a_i $(4 \leq i \leq b)$ must be a multiple of $3(d-2)$. □

Theorem 3 *For even $d \geq 4$, there is no $(d, 2, 2)$-graph whenever $d \not\equiv 1$ (mod 3).*

Proof. Let b be the number of cycles in H_G. Let a_i, for $i = 1 \ldots b$, be the lengths of these cycles, denoting by a_1 the triangle. Then as $\sum_{i=1}^{b} a_i = d^2 - 1$, by Lemma 2, we have that $\sum_{i=2}^{b} a_i = 3(d-2)k = d^2 - 4 = (d-2)(d+2)$. Therefore, $d + 2 \equiv 0 \pmod 3$. □

By counting the total number N_5 of 5-cycles in G, we derive some further necessary conditions for the existence of G.

Theorem 4 *For even $d \geq 4$, if $N_5 = \frac{(d-2)(d^4+2d^2-2d-25)}{10}$ is not an integer then there is no $(d, 2, 2)$-graph.*

Proof. The number of 5-cycles is closely related to the number of edges involving vertices at level 2 from a particular vertex in a $(d, 2, 2)$-graph G. For computing such a number, it is necessary to classify each vertex u_0 in G into three types. By Theorem 2, we see that there is only one triangle T in G.

Type A. u_0 is one of the vertices of the triangle T.

Type B. u_0 is a vertex such that $u_0 \notin T$ and $N(u_0) \cap T \neq \emptyset$.

Type C. u_0 is a vertex such that $u_0 \notin T$ and $N(u_0) \cap T = \emptyset$.

Note that the number of vertices of Type A, B and C is 3, $3(d-2)$ and $(d-2)(d-1)$, respectively.

Let us now denote by N_α, N_β and N_γ the number of 5-cycles passing through a vertex of Type A, B and C, respectively. The number of vertices at distance 2 from u_0 is $(d-2)(d-1) + 2(d-2) = (d-2)(d+1)$.

Computing the number N_α

In this case, the number E of edges involving only vertices at distance 2 from u_0 is $\frac{(d-2)(d+1)(d-1)}{2}$. Each such edge determines one 5-cycle containing u_0. Therefore, $N_\alpha = E$; see Figure 4(a).

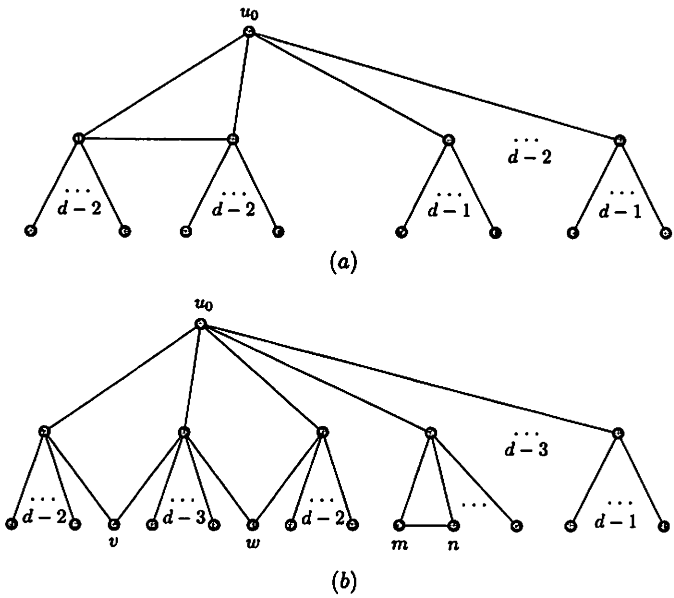


Figure 4: Auxiliary figure for Theorem 4.

For the remaining two cases, let u_0 be a vertex of Type 2b, and let v, w be the two repeats of u_0. The number E of edges involving only vertices at distance 2 from u_0 is $\frac{2(d-2)+[d(d-1)-4](d-1)}{2}$.

Computing the number N_β

For this case, refer to Figure 4(b). Let us denote by E' the number $E-1$; we do not count the edge mn.

We partition the number E' into E_1, E_2 and E_3, where E_1 and E_2 are the number of edges incident with the vertices v and w, respectively, and E_3 is the number of the remaining edges. Then $E' = E_1 + E_2 + E_3$.

Each edge counted in either E_1 or E_2 determines two 5-cycles, while each edge counted in E_3 determines only one 5-cycle. Therefore, given that

$E_1 = E_2 = d-2$, we have that the number of 5-cycles N_β passing through the vertex u_0 is

$$2E_1+2E_2+E_3 = 4(d-2)+\left(\frac{2(d-2)+[d(d-1)-4](d-1)}{2} - 1 - 2(d-2)\right)$$

Computing the number N_γ

In this case, the vertices of T are at distance 2 from u_0. We partition the number E into E_1, E_2 and E_3, where E_1 and E_2 are the number of edges incident with the vertices v and w, respectively, and E_3 is the number of the remaining edges. Then $E = E_1 + E_2 + E_3$.

Analogously, each edge counted in either E_1 or E_2 determines two 5-cycles, while each edge counted in E_3 determines only one 5-cycle. Therefore, given that $E_1 = E_2 = d-2$, we have that the number of 5-cycles N_γ passing through the vertex u_0 is

$$2E_1+2E_2+E_3 = 4(d-2)+\left(\frac{2(d-2)+[d(d-1)-4](d-1)}{2} - 2(d-2)\right)$$

Thus, the total number of 5-cycles is given by the following expression.

$$N_5 = \frac{3N_\alpha + 3(d-2)N_\beta + (d-2)(d-1)N_\gamma}{5} = \frac{(d-2)(d^4+2d^2-2d-25)}{10}$$

□

The results of Theorems 3 and 4 improve the upper bound for the order of $(d,2,2)$-graphs so that $n_{d,2} \leq d^2-3$ for infinitely many even degrees d. Indeed, if a value of d is ruled out by Theorems 3 and/or 4 then $n_{d,2} \leq d^2-3$. For $d \geq 10$, the first 50 values of d for which G might still exist are shown in Table 2.

4 Nonexistence of infinitely many $(d,2,2)$-graphs for odd d

Let us now get back to the graph H_G. For $d \geq 7$, from Theorem 2, we see that each vertex $u \in G$ has exactly two different repeats, that is, each component of H_G is a cycle of length at least 4.

10	22	34	40	52	64	70	82	94	100
112	124	130	142	154	160	172	184	190	202
214	220	232	244	250	262	274	280	292	304
310	322	334	340	352	364	370	382	394	400
412	424	430	442	454	460	472	484	490	502

Table 1: The first 50 values of d for which a $(d, 2, 2)$-graph might still exist for even d.

Let A be the adjacency matrix of G, and let B be the adjacency matrix of H_G, called *the defect matrix*, in which the main diagonals consist entirely of 0's, and the row and column sums are equal to 2. With a suitable labeling of H_G, B becomes a direct sum of symmetric a^{th}-order circulants of the form,

$$D_a = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad a \geq 4$$

This matrix has been studied in [3] and [7] in the context of regular graphs of girth 5.

Let n be the order of G, let b be the number of cycles in H_G, and let a_i, $i = 1, \dots, b$, be the lengths of these cycles. We consider the following equation

$$A^2 + A - (d-1)I = J + B \tag{1}$$

where J is a matrix all of whose entries are 1, and I is the identity matrix of order n.

The special case when there is just one repeat cycle, that is, $b = 1$ and $H_G = C_n$, was studied by Fajtlowicz in [5]. In that paper, Fajtlowicz proved the following,

Theorem 5 [5] *If B is the adjacency matrix of the n-cycle then $d = 3$.*

As A, B and J are symmetric matrices, they are diagonalizable. Since J commutes with A and B, B commutes with A, and hence all the three matrices are simultaneously diagonalizable, that is, there is an orthogonal matrix P for which $P^{-1}AP$, $P^{-1}BP$ and $P^{-1}JP$ are diagonal, and the columns of P are the corresponding eigenvectors for each of these matrices.

Furthermore, it is well known that the eigenvalues and their corresponding multiplicities of a matrix representing a m-cycle are

$$\begin{bmatrix} 2 & 2\cos\frac{2\pi}{m}\times 1 & 2\cos\frac{2\pi}{m}\times 2 & \dots & 2\cos\frac{2\pi}{m}\times(\frac{m}{2}-1) & -2 \\ 1 & 2 & 2 & \dots & 2 & 1 \end{bmatrix} (m \text{ even})$$

$$\begin{bmatrix} 2 & 2\cos\frac{2\pi}{m}\times 1 & 2\cos\frac{2\pi}{m}\times 2 & \dots & 2\cos\frac{2\pi}{m}\times(\frac{m-1}{2}) \\ 1 & 2 & 2 & \dots & 2 \end{bmatrix} (m \text{ odd})$$

The first row displays the eigenvalues, and the second row their corresponding multiplicities.

Therefore, the eigenvalues of $J+B$ are $n+2$, 2 and $2\cos\frac{2\pi c_i}{a_i}$ with $c_i = 1\dots a_i - 1$ and $i = 1\dots b$, of multiplicities 1, $b-1$ and 1, respectively.

Thus, the spectrum of A in the general case $b \geq 1$ is:

(*i*) The eigenvalue d with multiplicity 1,

(*ii*) $b-1$ roots of the equation

$$\alpha^2 + \alpha - (d-1) = 2, \tag{2}$$

(*iii*) one root of each of the equations

$$\alpha^2 + \alpha - (d-1) = 2\cos\frac{2\pi c_i}{a_i} \tag{3}$$

where $i = 1\dots b$ and $c_i = 1\dots a_i - 1$.

We denote by $m(\alpha)$ the multiplicity of an eigenvalue α of A. The solutions of Equation (2) are $\beta_1 = \frac{-1+\sqrt{4d+5}}{2}$ and $\beta_2 = \frac{-1-\sqrt{4d+5}}{2}$, and $m(\beta_1)+m(\beta_2) = b-1$. The general solution of Equation (3) is

$$\beta = \frac{-1 \pm \sqrt{4d + 8\cos(\frac{2\pi c_i}{a_i}) - 3}}{2}$$

For each even a_i, $i = 1, \dots, b$, and when $c_i = \frac{a_i}{2}$, there is exactly one eigenvalue β of A with multiplicity 1 satisfying Equation (3). In other words, corresponding to the special case when $\cos(\frac{2\pi c_i}{a_i}) = -1$, there are eigenvalues $\beta_3 = \frac{-1+\sqrt{4d-11}}{2}$, $\beta_4 = \frac{-1-\sqrt{4d-11}}{2}$. Let $m_e = m(\beta_3) + m(\beta_4)$. Then m_e is exactly the number of even cycles in H_G.

Observation 7 *For odd d, $n = d^2 - 1$ is even. Therefore, if d is odd then* $m_e \equiv b \pmod 2$.

We now need to state (without proof) the following known result; see, for instance, [1, 7].

Lemma 3 *Let α_i and α_j be eigenvalues of a real square matrix A with rational entries, and let $m(\alpha_i)$ and $m(\alpha_j)$ be their multiplicities. If α_i and α_j are algebraic conjugates over $\mathbb{Q}$ then $m(\alpha_i)=m(\alpha_j)$.*

Using a similar method to that described in [7], we next show that

Theorem 6 *G does not exist for any odd $d \geq 7$ such that $d \neq l^2 + l + 3$ and $d \neq l^2 + l - 1$, for each nonnegative integer l.*

Proof. If all the four eigenvalues $\beta_1, \ldots, \beta_4$ are irrational then, by Lemma 3, $m(\beta_1) = m(\beta_2) = \frac{b-1}{2}$ and $m(\beta_3) = m(\beta_4) = \frac{m_c}{2}$. But, by Observation 7, these equations cannot hold at the same time. We can then see that corresponding to those odd values of d such that $d \neq l^2 + l + 3$ and $d \neq l^2 + l - 1$, for each nonnegative integer l, the four eigenvalues $\beta_1, \ldots, \beta_4$ are irrational, thus the proof follows. □

By counting the total number (N_5) of 5-cycles in G, we derive some further necessary conditions for the existence of G.

Theorem 7 *G does not exist for any odd $d \geq 7$ such that N_5 is not an integer, where*

$$N_5 = \frac{(d^2-1)(d^3-2d^2+3d-8)}{10}.$$

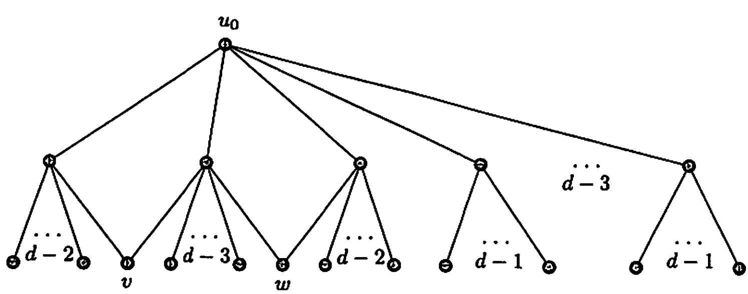


Figure 5: Auxiliary figure for Theorem 7.

Proof. As said earlier, the number of 5-cycles in G is closely related to the number of edges involving vertices at level 2 from a particular vertex in a $(d,2,2)$-graph G. The number of vertices at level 2 is $d(d-1)-2$, and the number E of edges involving only vertices at level two is $\frac{2(d-2)+[d(d-1)-4](d-1)}{2}$.

From Theorem 2, we know that $n_{2b} = d^2 - 1$. Let u_0 be a vertex of Type $2b$. Let v, w be the two repeats of u_0; see Figure 5.

We partition the number E into E_1, E_2 and E_3, where E_1 and E_2 are the number of edges adjacent to vertices v and w, respectively, and E_3 are the remaining edges. Then $E = E_1 + E_2 + E_3$.

Each edge counted in either E_1 or E_2 determines two 5-cycles, while each edge counted in E_3 determines only one 5-cycle. Therefore, given that $E_1 = E_2 = d - 2$, we have that the number of 5-cycles passing through the vertex u_0 is the following.

$$2E_1+2E_2+E_3 = 2(d-2)+\frac{2(d-2)+[d(d-1)-4](d-1)}{2} = \frac{d(d-1)^2+2d-8}{2}$$

Thus, the total number of 5-cycles is given by the following expression.

$$N_5 = \frac{(d^2-1)}{5}\left[\frac{d(d-1)^2+2d-8}{2}\right] = \frac{(d^2-1)(d^3-2d^2+3d-8)}{10}$$

□

The results of Theorems 6 and 7 improve the upper bound for the order of $(d,2,2)$-graphs so that $n_{d,2} \leq d^2 - 3$ for infinitely many odd degrees d. Indeed, if a value of d is ruled out by Theorems 6 and/or 7 then $n_{d,2} \leq d^2-3$. For $d \geq 7$, the first 50 values of d for which G might still exist are shown in Table 2.

9	11	19	23	29	33	41	59	71	89
93	109	113	131	159	181	209	213	239	243
271	309	341	379	383	419	423	461	509	551
599	603	649	653	701	759	811	869	873	929
933	991	1059	1121	1189	1193	1259	1263	1331	1409

Table 2: The first 50 values of d for which a $(d,2,2)$-graph might still exist for odd d.

5 Conclusions

Let Γ be a $(d,2,2)$-graph for $d \geq 3$. In this paper, we presented the following new results.

(i) We proved that for even $d \geq 4$, if $d \not\equiv 1 \pmod 3$ then there is no $(d,2,2)$-graph.

(ii) We proved that G does not exist for any odd $d \geq 7$ such that $d \neq l^2+l+3$ and $d \neq l^2+l-1$, for each nonnegative integer l.

(iii) By counting the total number N_5 of 5-cycles in G, we derived some further necessary conditions for the existence of G.

- For even $d \geq 4$, if $N_5 = \frac{(d-2)(d^4+2d^2-2d-25)}{10}$ is not an integer then there is no $(d,2,2)$-graph.
- For odd $d \geq 7$, if $N_5 = \frac{(d^2-1)(d^3-2d^2+3d-8)}{10}$ is not an integer then there is no $(d,2,2)$-graph.

Finally, we conjecture the following.

Conjecture 1 *For $d \geq 6$, $(d,2,2)$-graphs do not exist.*

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