On the Ramsey number for a linear forest versus a cocktail party graph

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Abstract. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the least natural number n such that for every graph F of order n the following condition holds: either F contains G or the complement of F contains H. In this paper we determine the Ramsey number for a disjoint union of paths versus the cocktail party graph.

Keywords : Ramsey number, path, cocktail party graph.

1 Introduction

For graphs G and H, the Ramsey number $R(G, H)$ is the smallest positive integer n such that every graph F of order n contains G or the complement of F contains H. The study of Ramsey numbers for general (not necessarily complete) graphs have received tremendous efforts in the last two decades. The first result was given by Geréncser and Gyárfás [10], in proving the Ramsey number for paths. They determined that $R(P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1$ for $2 \leq n \leq m$. To find the Ramsey number for any combination of graphs, Chvátal and Harary [8] gave some lower bound: $R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of G and

$c(H)$ is the number of vertices of the largest component of H. Unfortunately, many Ramsey numbers are much higher than this lower bound. This makes the problem of finding Ramsey numbers difficult.

In particular, for trees versus complete graphs, Chvátal [7] showed that $R(T_n, K_m) = (n-1)(m-1)+1$. However, up to now, the Ramsey number of trees versus wheels $R(T_n, W_m)$ is not completely solved. Some result concerning this problem was given in [2]. It showed that the Ramsey number $R(T_n, W_4) = 2n-1$ for $n \geq 4$ and $R(T_n, W_5) = 3n-2$ for $n \geq 3$. Furthermore, Chen *et. al* [6] determined the Ramsey number $R(T_n, W_6)$ and $R(T_n, W_7)$.

For special type of trees, Baskoro and Surahmat in [1] studied the Ramsey numbers of paths versus wheels. They proved that $R(P_n, W_m) = 2n-1$ if $m \geq 4$ is even and $n \geq \frac{m}{2}(m-2)$. They also showed that $R(P_n, W_m) = 3n-2$ for $m \geq 5$ is odd and $n \geq \frac{m-1}{2}(m-3)$. This result was improved by Chen *et. al* [5].

Some upper bounds for the path-tree Ramsey numbers were given by Faudree, Schelp and Simonovits [9]. They showed that Ramsey numbers of path versus tree are $R(P_n, T_m) \leq m+n-2$ for $n \geq m$ or $m \geq 432n^6 - n^2$ and $R(P_n, T_m) \leq m + 6n^2 - 2n$ for other values of m and n. For further detail see a nice survey paper [11].

Let $G(V, E)$ be a graph. For any set $S \subset V(G)$, the *induced subgraph* $G[S]$ of G by S is the maximal subgraph of G with vertex-set S. Let P_n be a path on n vertices. For $m \geq 3$, let H_m be a *cocktail party graph*, namely a graph obtained by removing perfect matching from a complete graph K_{2m}. In this paper, we determine the Ramsey numbers $R(P_n, H_m)$, and $R(tP_n, H_m)$. The main results are given in the following three theorems.

Theorem 1. $R(P_n, H_m) = \begin{cases} 2m & \text{if } n = 2 \text{ and } m \geq 3, \\ (n-1)(m-1)+2 & \text{if } n \geq 3 \text{ and } m \geq 3. \end{cases}$

Theorem 2. *For $t \geq 2$ and $m \geq 3$, $R(tP_2, H_m) = (2t-1)+(2m-3)+1$.*

Theorem 3. *$R(tP_n, H_m) = (n-1)(m-1)+(t-1)n+2$, for $n \geq 3$, $m \geq 3$ and $t \geq 1$.*

2 The Proof of Theorems

The proof of Theorem 1.
For first case of Theorem 1, graph $G = (2m-1)K_1$ shows that $R(P_2, H_m) \geq$

$2m$ as G does not contain path P_2 and $\overline{G}$ does not contain H_m. For the reverse inequality, consider graph F on $2m$ vertices such that $P_2 \not\subseteq F$, we will show that $\overline{F} \supseteq H_m$. Since F does not contain P_2, so $F \cong 2mK_1$ and $\overline{F} \cong K_{2m} \supset H_m$. Hence $R(P_n, H_m) \leq 2m$ and proof is finished for case 1.

For second case of Theorem 1, we will use induction on $n+m$. First, we show that $R(P_n, H_3) = 2(n-1)+2$. Consider graph $G \cong 2K_{n-1} \cup K_1$. Clearly, G contains no P_n and $\overline{G}$ contains no H_3. Thus, $R(P_n, H_3) \geq 2n$. Now, we will show that $R(P_n, H_3) \leq 2n$. Let F be a graph of $2n$ vertices containing no P_n. If $F \cong \overline{K}_{2n}$ then $\overline{F}$ contains H_m. Otherwise, let P be a longest path in F with endpoints p_1 and p_2. Obviously, $|V(P)| \leq n-1$ and $zp_1, zp_2 \notin E(F)$ for each $z \in V(F)\backslash V(P)$. Now, consider the subgraph $F[X]$ induced by $X = V(F)\backslash V(P)$. If $F[X]$ contains isolated vertices only then its complement must contain H_3. Otherwise, let Q be a longest path in $F[X]$. Let q_1 and q_2 be its endpoints. Since $|V(Q)| \leq |V(P)| \leq n-1$ then there exists at least two vertices $v, w \in Y = V(F)\backslash(V(P) \cup V(Q))$ which are not adjacent to all endpoints p_1, p_2, q_1 and q_2. Then, the graph induced by $\{v, w, p_1, p_2, q_1, q_2\}$ will contain a H_3.

To show the lower bound for the general case, consider the graph $F \cong (m-1)K_{n-1} \cup K_1$. It is clear that F contain no P_n and its complement contains no H_m. To show the upper bound, we use induction on $n+m$. Now, assume the assertion true for $k = n+m-1$, namely $R(P_{n-1}, H_m) = (n-2)(m-1)+2$ and $R(P_n, H_{m-1}) = (n-1)(m-2)+2$. Consider any graph G with $(n-1)(m-1)+2$ vertices. Assume that $\overline{G}$ contains no H_m. Then, by induction hypothesis, G must contain P_{n-1}. Let a, b be its end-vertices. Now, let $Y = V(G)\backslash V(P_{n-1})$ and consider the graph $G[Y]$. Since $|G[Y]| = (n-1)(m-2)+2$ then by induction hypothesis $G[Y]$ contains a P_n or its complement contains a H_{m-1}. If $G[Y]$ contains a P_n then the proof is finished. If its complement contains a H_{m-1}, then the graph formed by H_{m-1} together with $\{a, b\}$ will contain a H_m in $\overline{G}$. □

The proof of Theorem 2.

Firstly we prove the result for $t = 2$. We will show that $R(2P_2, H_m) = 2m+1$. Consider $G = K_3 \cup (2m-3)K_1$, clearly G does not contain $2P_2$ and $\overline{G}$ does not contain H_m. Thus, $R(2P_2, H_m) \geq 2m+1$. For the upper bound, consider graph F on $2m+1$ vertices such that $2P_2 \not\subseteq F$, we will show that $H_m \subseteq \overline{F}$. Since F does not contain $2P_2$, then $F \cong (2m+1)K_1$ or $F \cong S_q \cup rK_1$ where $q+r = 2m+1$, $q \geq 2$ and $\overline{F} \supseteq K_{2m} \supseteq H_m$ and result is true for $t = 2$.

To show the lower bound for general case, graph $G = K_{2t-1} \cup (2m-3)K_1$

shows that $R(tP_2, H_m) \geq (2t-1) + (2m-3) + 1$ as G does not contain paths tP_2 and $\overline{G}$ does not contain H_m. For the reverse inequality, we will use induction on t. For $t = 2$, it is true. Assume the assertion is true for $t = k-2$. Now, take any graph G on $(2t-1)+(2m-3)+1$ and its complement contains no H_m. By induction hypothesis, G contains $(k-2)P_2$. Now, consider the graph $G_1 \cong G \backslash (k-2)P_2$. Then, $|G_1| = 2m+1 = R(2P_2, H_m)$. Since $\overline{G_1} \not\supseteq H_m$ then $G_1 \supseteq 2P_2$. Therefore, altogether $G \supseteq tP_2$. □

The proof of Theorem 3.
To show the lower bound, let us consider graph $F \cong (m-2)K_{n-1} \cup K_{tn-1} \cup K_1$. Clearly, $|V(F)| = (n-1)(m-1) + (t-1)n + 2$. Graph F contains no tP_n and $\overline{F}$ contains no H_m. To show the Ramsey number we use induction on t. For $t = 1$, it is true by Theorem 1. Assume the assertion is true for $t = k-1$. Now, take any graph G on $(n-1)(m-1) + (k-1)n + 2$ and its complement contains no H_m. By induction hypothesis, G contains $(k-1)P_n$. Now, consider the graph $G_1 \cong G \backslash (k-1)P_n$. Then, $|G_1| = (n-1)(m-1)+2$. By Theorem 1, if $\overline{G_1} \not\supseteq H_m$ then $G_1 \supseteq P_n$. Therefore, altogether $G \supseteq kP_n$. □

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