Vertex-antimagic total labeling of the union of suns *

Andrea Parestu, Denny R. Silaban, Kiki A. Sugeng

Department of Mathematics,
Faculty of Mathematics and Sciences,
University of Indonesia
Depok 16424, Indonesia
{anpa50, denny-rs, kiki}@ui.ac.id

Abstract. Let $G = (V, E)$ be a graph with $V(G)$ as a set of vertices and $E(G)$ as a set of edges, where $n = |V(G)|$ and $e = |E(G)|$. A graph $G = (V, E)$ is said to be (a, d)-vertex antimagic total if there exist positive integers a, d and a bijection λ from $V(G) \cup E(G)$ to the set of consecutive integers $\{1, 2, \ldots, n+e\}$ such that the weight of vertices form arithmetical progression with initial term a and common difference d. In this paper we will give (a, d)-vertex antimagic total labeling of disconnected graph, which consists of the union of t suns for $d \in \{1, 2, 3, 4, 6\}$.

Key words: Vertex antimagic total labeling, sun graph.

1 Introduction

In this paper all graphs are finite, simple and undirected. Let G has vertex set $V(G)$ and edge set $E(G)$, and let $n = |V(G)|$ and $e = |E(G)|$.

A labeling λ of a graph G is a mapping that assigns elements of a graph to a set of numbers (usually positive integers). If the domain of the mapping is the set of vertices (respectively, the set of edges) then we call the labeling vertex labeling (respectively, edge labeling). If the domain is the set of vertices and edges, then we call the labeling total labeling. A general survey of graph labelings can be found in [2].

The vertex-weight $w(x)$ of a vertex x, under a labeling $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, n+e\}$ (or $w_\lambda(x)$), is the sum of values $\lambda(xy)$ assigned to all edges incident to given vertex x together with the value assigned to x itself.

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A bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, n+e\}$ is called an (a,d)-vertex antimagic total labeling (in short, (a,d)-VAT labeling or (a,d)-VATL) of G if the set of vertex-weights of all vertices in G is $\{a, a+d, \ldots, a+(n-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

If $d = 0$ then we call λ as a *vertex-magic total labeling.* The concept of the vertex-magic total labeling was introduced by MacDougall *et al.* [3].

Bača *et al.* [1] investigated basic properties of (a,d)-VAT labelings and constructed such labelings for some families of graphs. They also studied dual labeling and relationship between a SVMT labeling (or super $(a,0)$-VAT labeling) and an (a,d)-antimagic labeling.

A sun S_n is a cycle C_n with adding an edge connecting to each vertex v_i in a cycle with single vertex u_i, for $i = 1, 2, \ldots, n$. The sun S_n consists of vertex set $V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and edge set $E(S_n) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\}$, where v_i are inner vertices and u_i are outer vertices of S_n. Note that if $i = n$ then $i + 1 = 1$. The union of t suns, denoted by $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$, consists of vertex set $V(S_{n_j}) = \{v_i^j : 1 \leq i \leq n_j, 1 \leq j \leq t\} \cup \{u_i^j : 1 \leq i \leq n_j, 1 \leq j \leq t\}$ and edge set $E(S_{n_j}) = \{v_i^j v_{i+1}^j : 1 \leq i \leq n\} \cup \{u_i^j v_i^j : 1 \leq i \leq n\}$. Thus $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has $2\sum_{k=1}^{t} n_k$ vertices and $2\sum_{k=1}^{t} n_k$ edges.

Rahim *et al.* [5] proved that the union of t suns has a VMT labeling (or $(a,0)$-VAT labeling) with magic number $k = 6\sum_{k=1}^{t} n_k + 1$. Parestu *et al.* [4] proved that the union of t suns has an $(a,1)$-VAT labeling with $a = 4\sum_{k=1}^{t} n_k + 2$ (also given in this paper).

We note that if δ is the smallest degree in G, then the minimum possible weight on a vertex is at least $1 + 2 + \cdots + (\delta + 1)$, consequently

$$a \geq \frac{(\delta+1)(\delta+2)}{2}.$$

Similarly, if Δ is the largest degree, then the maximum vertex weight is less than the sum of the $\Delta + 1$ largest labels. Thus

$$a + (|V| - 1)d \leq \sum_{i=|V|+|E|-\Delta+1}^{|V|+|E|} i$$

$$= \frac{(\Delta+1)(2(|V|+|E|)-\Delta+1))}{2}.$$

Then we obtain the restriction of d as follows

$$d \leq \frac{(\Delta+1)(2(|V|+|E|)-\Delta))-(\delta+1)(\delta+2)}{2(|V|-1)}.$$

Since every sun has $\delta = 1$ and $\Delta = 3$, then we have

$$d \leq 7.$$

In this paper we will give (a, d)-VATL of graphs, which consists of the union of t suns for $d \in \{1, 2, 3, 4, 6\}$.

2 The Union of Suns

We start this section with a construction of $(a, 1)$-VATL of the union of t suns.

The following theorem presents $(a, 1)$-VATL of disconnected graph, which consist of the union of t suns. Example of union of t suns can be seen at Figure 1.

Theorem 1. *For $n_j \geq 3$ and $t \geq 1$, the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has a $(4\sum_{k=1}^{t} n_k + 2, 1)$-VATL.*

Proof. For $j = 1, 2, \ldots, t$, label the vertices and the edges of sun S_{n_j} in the following way:

$$\lambda_1(v_i^j) = \begin{cases} 3\sum_{k=1}^{t} n_k - \sum_{k=1}^{j} n_k + 1 & , i = 1, \\ \sum_{k=1}^{t} n_k - \sum_{k=1}^{j-1} n_k + 2 - i & , i = 2, 3, \ldots, n_j. \end{cases}$$

$$\lambda_1(u_i^j) = 3\sum_{k=1}^{t} n_k + \sum_{k=1}^{j-1} n_k + i, \; i = 1, 2, \ldots, n_j.$$

$$\lambda_1(v_i^j v_{i+1}^j) = \sum_{k=1}^{j-1} n_k + i, \; i = 1, 2, \ldots, n_j.$$

$$\lambda_1(u_i^j v_i^j) = \sum_{k=1}^{t} n_k + \sum_{k=1}^{j-1} n_k + i,\ i = 1, 2, \ldots, n_j.$$

The vertex and edge labels under the labeling λ_1 are

$$\lambda_1(V) = \{2\sum_{k=1}^{t} n_k + 1, 2\sum_{k=1}^{t} n_k + 2, \ldots, 4\sum_{k=1}^{t} n_k\}$$

and

$$\lambda_1(E) = \{1, 2, \ldots, 2\sum_{k=1}^{t} n_k\}.$$

It means that the labeling λ_1 is a bijection from the set $V(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}) \cup E(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t})$ onto the set $\{1, 2, \ldots, 4\sum_{k=1}^{t} n_k\}$.

The vertex-weights of S_{n_j} are

$$w_{\lambda_1}(v_i^j) \cup w_{\lambda_1}(u_i^j) = \{4\sum_{k=1}^{t} n_k + 2, 4\sum_{k=1}^{t} n_k + 3, \ldots, 6\sum_{k=1}^{t} n_k + 1\}.$$

Thus, the labeling λ_1 is $(4\sum_{k=1}^{t} n_k + 2, 1)$-VATL of the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$. $\square$

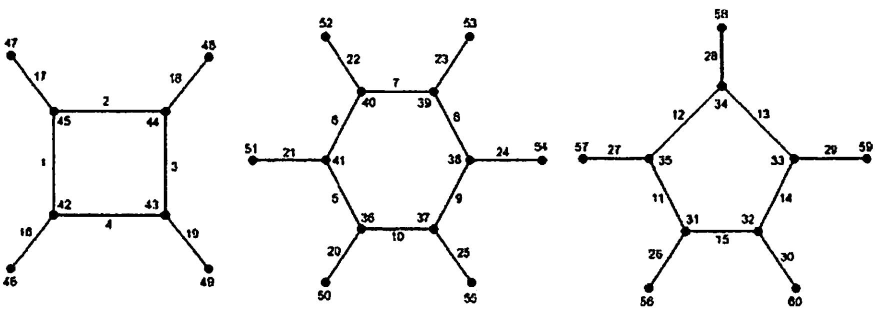

Fig. 1. (64, 1)-VATL on $S_4 \cup S_6 \cup S_5$.

The following theorem presents $(a, 2)$-VATL of disconnected graph, which consist of the union of t suns.

Theorem 2. *For $n_j \geq 3$ and $t \geq 1$, the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has a $(4\sum_{k=1}^{t} n_k + 3, 2)$-VATL.*

Proof. For $j = 1, 2, \ldots, t$, label the vertices and edges of sun S_{n_j} in the following:

$$\lambda_2(v_i^j) = \begin{cases} 2\sum_{k=1}^{t} n_k - \sum_{k=1}^{j} n_k + 1 & , i = 1, \\ 2\sum_{k=1}^{t} n_k - \sum_{k=1}^{j} n_k + 2n_j + 3 - 2i & , i = 2, 3, \ldots, n_j. \end{cases}$$

$$\lambda_2(u_i^j) = 2\sum_{k=1}^{t} n_k + 2\sum_{k=1}^{j-1} n_k - 1 + 2i,\ i = 1, 2, \ldots, n_j.$$

$$\lambda_2(v_i^j v_{i+1}^j) = 2\sum_{k=1}^{j-1} n_k + 2i,\ i = 1, 2, \ldots, n_j.$$

$$\lambda_2(u_i^j v_i^j) = 2\sum_{k=1}^{t} n_k + 2\sum_{k=1}^{j-1} n_k + 2i,\ i = 1, 2, \ldots, n_j.$$

The vertex and edge labels under the labeling λ_2 are

$$\lambda_2(V) = \{1, 3, \ldots, 4\sum_{k=1}^{t} n_k - 3, 4\sum_{k=1}^{t} n_k - 1\}$$

and

$$\lambda_2(E) = \{2, 4, \ldots, 4\sum_{k=1}^{t} n_k - 2, 4\sum_{k=1}^{t} n_k\}.$$

It means that the labeling λ_2 is a bijection from the set $V(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}) \cup E(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t})$ onto the set $\{1, 2, \ldots, 4\sum_{k=1}^{t} n_k\}$.

The vertex-weights of S_{n_j} are

$$w_{\lambda_2}(v_i^j) \cup w_{\lambda_2}(u_i^j) = \{4\sum_{k=1}^{t} n_k + 3, 4\sum_{k=1}^{t} n_k + 5, \ldots, 8\sum_{k=1}^{t} n_k + 1\}$$

Consequently, the labeling λ_2 is $(4\sum_{k=1}^{t} n_k + 3, 2)$-VATL of the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$. □

The following theorem presents $(a, 3)$-VATL of disconnected graph, which consists of the union of t suns.

Theorem 3. *For odd $n_j \geq 3$ and $t \geq 1$, the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has a $(4\sum_{k=1}^{t} n_k + 3, 3)$-VATL.*

Proof. For $j = 1, 2, \ldots, t$, label the vertices and edges of sun S_{n_j} in the following way:

$$\lambda_3(v_i^j) = 3\sum_{k=1}^{t} n_k + \sum_{k=1}^{j} n_k + 1 - i,\ i = 1, 2, \ldots, n_j.$$

$$\lambda_3(u_i^j) = 3\sum_{k=1}^{t} n_k - \sum_{k=1}^{j} n_k + i,\ i = 1, 2, \ldots, n_j.$$

$$\lambda_3(v_i^j v_{i+1}^j) = \begin{cases} 2\sum_{k=1}^{j-1} n_k + i & , i = 1, 3, \ldots, n_j, \\ 2\sum_{k=1}^{j-1} n_k + n_j + i & , i = 2, 4, \ldots, n_j - 1. \end{cases}$$

$$\lambda_3(u_i^j v_i^j) = 2\sum_{k=1}^{t} n_k - 2\sum_{k=1}^{j} n_k + 2i,\ i = 1, 2, \ldots, n_j.$$

The edge and vertex labels under the labeling λ_3 form the following set

$$\lambda_3(E) = \{1, 2, \ldots, 2\sum_{k=1}^{t} n_k - 1, 2\sum_{k=1}^{t} n_k\}$$

and

$$\lambda_3(V) = \{2\sum_{k=1}^{t} n_k + 1, 2\sum_{k=1}^{t} n_k + 2, \ldots, 4\sum_{k=1}^{t} n_k - 1, 4\sum_{k=1}^{t} n_k\}.$$

Then the labeling λ_3 is a bijection from the set $V(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}) \cup E(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t})$ onto the set $\{1, 2, \ldots, 4\sum_{k=1}^{t} n_k\}$.

The vertex-weights of S_{n_j} are

$$w_{\lambda_3}(v_i^j) \cup w_{\lambda_3}(u_i^j) = \{2\sum_{k=1}^{t} n_k + 3, 2\sum_{k=1}^{t} n_k + 6, \ldots, 8\sum_{k=1}^{t} n_k\}.$$

Thus we have an $(2\sum_{k=1}^{t} n_k + 3, 3)$-VAT labeling for the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$. □

In the following theorem, we present the case for $d = 4$.

Theorem 4. *For $n_j \geq 3$ and $t \geq 1$, the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has a $(2\sum_{k=1}^{t} n_k + 3, 4)$-VATL.*

Proof. For $j = 1, 2, \ldots, t$, label the vertices and edges of sun S_{n_j} in the following way:

$$\lambda_4(v_i^j) = \begin{cases} 4\sum_{k=1}^{t} n_k - 2\sum_{k=1}^{j} n_k + 2 & , i = 1, \\ 4\sum_{k=1}^{t} n_k - 2\sum_{k=1}^{j} n_k + 2n_j + 4 - 2i & , i = 2, 3, \ldots, n_j. \end{cases}$$

$$\lambda_4(u_i^j) = 2\sum_{k=1}^{j-1} n_k + 2i, \ i = 1, 2, \ldots, n_j.$$

$$\lambda_4(v_i^j v_{i+1}^j) = 2\sum_{k=1}^{t} n_k - 2\sum_{k=1}^{j} n_k - 1 + 2i, \ i = 1, 2, \ldots, n_j.$$

$$\lambda_4(u_i^j v_i^j) = 2\sum_{k=1}^{t} n_k + 2\sum_{k=1}^{j-1} n_k - 1 + 2i, \ i = 1, 2, \ldots, n_j.$$

The labeling λ_4 form the following set of labels

$$\lambda_4(V) = \{1, 3, \ldots, 4\sum_{k=1}^{t} n_k - 3, 4\sum_{k=1}^{t} n_k - 1\}$$

and

$$\lambda_4(E) = \{2, 4, \ldots, 4\sum_{k=1}^{t} n_k - 2, 4\sum_{k=1}^{t} n_k\}.$$

The vertex-weights of S_{n_j} are

$$w_{\lambda_4}(v_i^j) \cup w_{\lambda_4}(u_i^j) = \{2\sum_{k=1}^{t} n_k + 3, 2\sum_{k=1}^{t} n_k + 7, \ldots, 10\sum_{k=1}^{t} n_k - 1\}.$$

Thus the labeling λ_4 is $(2\sum_{k=1}^{t} n_k + 3, 4)$-VATL of the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$. $\square$

In following theorem we presents $(a, 6)$-VATL of the union of t suns. Example of this labeling can be found at Figure 2.

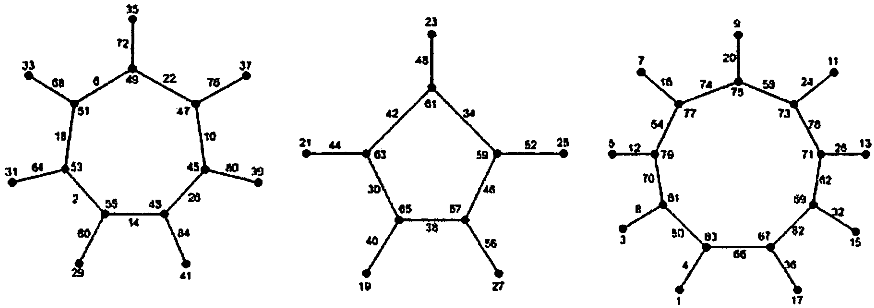

Fig. 2. $(5, 6)$-VATL on $S_4 \cup S_6 \cup S_5$.

Theorem 5. *For odd $n_j \geq 3$ and $t \geq 1$, the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has an $(5, 6)$-VATL.*

Proof. For $j = 1, 2, \ldots, t$, label the vertices and edges of sun S_{n_j} in the following way:

$$\lambda_5(v_i^j) = 2\sum_{k=1}^{t} n_k + 2\sum_{k=1}^{j} n_k + 1 - 2i,\ i = 1, 2, \ldots, n_j.$$

$$\lambda_5(u_i^j) = 2\sum_{k=1}^{t} n_k - 2\sum_{k=1}^{j} n_k - 1 + 2i,\ i = 1, 2, \ldots, n_j.$$

$$\lambda_5(v_i^j v_{i+1}^j) = \begin{cases} 4\sum_{k=1}^{j-1} n_k + 2i & , i = 1, 3, \ldots, n_j, \\ 4\sum_{k=1}^{j-1} n_k + 2n_j + 2i & , i = 2, 4, \ldots, n_j - 1. \end{cases}$$

$$\lambda_5(u_i^j v_i^j) = 4\sum_{k=1}^{t} n_k - 4\sum_{k=1}^{j} n_k + 4i,\ i = 1, 2, \ldots, n_j.$$

Then we have the following set

$$\lambda_5(E) = \{1, 3, \ldots, 4\sum_{k=1}^{t} n_k - 3, 4\sum_{k=1}^{t} n_k - 1\}$$

and

$$\lambda_5(V) = \{2, 4, \ldots, 4\sum_{k=1}^{t} n_k - 2, 4\sum_{k=1}^{t} n_k\}.$$

It means that the labeling λ_5 is a bijection from the set $V(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}) \cup E(S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t})$ onto the set $\{1, 2, \ldots, 4\sum_{k=1}^{t} n_k\}$.

The vertex-weights of S_{n_j} are

$$w_{\lambda_5}(v_i^j) \cup w_{\lambda_5}(u_i^j) = \{5, 11, , \ldots, 12\sum_{k=1}^{t} n_k - 1\}.$$

Thus the labeling λ_5 is (5, 6)-VATL of the union of t suns $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$. □

3 Open Problems

In this paper we studied an (a, d)-VAT labeling of the union of suns. We found that the graphs $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ has an (a, d)-VATL for $d \in \{1, 2, 3, 4, 6\}$. In the case when $d \in \{3, 6\}$, for even $n \geq 3$; and $d \in \{5, 7\}$, for $n \geq 3$, we do not have the complete answer. We list here two problems for further investigation.

Open Problem 1 *Find if there is an $(a, 3)$-VAT labeling and $(a, 6)$-VAT labeling for even $n \geq 3$.*

Open Problem 2 *Find if there is an $(a, 5)$-VAT labeling and $(a, 7)$-VAT labeling for all $n \geq 3$.*

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