Vertex-magic total labelings of union of generalized Petersen graphs and union of special circulant graphs *

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Abstract. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let $n = |V(G)|$ and $e = |E(G)|$. A *vertex-magic total labeling* (VMTL) of a graph is defined as a one-to-one mapping taking the vertices and edges onto the set of integers $\{1, 2, \ldots, n + e\}$, with the property that the sum of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex. In this paper, we present the vertex magic total labeling of disjoint union of t generalized Petersen graphs $\cup_{j=1}^{t} P(n_j, m_j)$, and disjoint union of t special circulant graphs $\cup_{j=1}^{t} C_n(1, m_j)$.

Key words: Vertex magic total labeling, regular graph, generalized Petersen graph, circulant graph.

1 Introduction

In this paper all graphs are finite, simple, and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$, and let $n = |V(G)|$ and $e = |E(G)|$.

MacDougall *et al.* [4] introduced the notion of a vertex-magic total labeling. The *vertex-magic total labeling* of a graph G is a one -to-one mapping from

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$V \cup E$ onto the integers $\{1, 2, \ldots, n+e\}$ such that for every vertex $x \in V$ there is a constant k so that

$$w_\lambda(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy) = k,$$

where $N(x)$ is the set of all vertices y that adjacent to x. The constant k is called the magic constant for λ and $w_\lambda(x)$is called the weight of x under labeling λ.

The *generalized Petersen* graph $P(n, m), n \geq 3, 1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, is a regular graph with an outer n-cycle $u_1, u_2, \ldots, u_n$, a set of n spokes $u_i v_i, 1 \leq i \leq n$, and n inner edges $v_i v_{i+m}$ with indices taken modulo n. The standard Petersen graph is P(5,2). Generalized Petersen graphs were first defined by Watkins [7].

Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \lfloor n/2 \rfloor$, where n and a_i ($i = 1, \ldots, k$) are positive integers. A *circulant* graph $C_n(a_1, a_2, \ldots, a_k)$ is a regular graph with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{(v_i v_{i+a_j})(mod\ (n-1)) | i = 0, 1, \ldots, n-1, j = 1, 2, \ldots, k\}$.

If a regular graph G possesses a vertex-magic total labeling, we can create a new labeling λ' from λ by setting

$$\lambda'(x) = n + e + 1 - \lambda(x)$$

for every vertex x, and

$$\lambda'(xy) = n + e + 1 - \lambda(xy)$$

for every edge xy. Clearly, λ' is also a one-to-one mapping from the set $V \cup E$ to $1, 2, \ldots, n+e$ and we call λ' as the *dual* of λ. If r is the degree of each vertex of the regular graph G, then

$$k' = (r+1)(n+e+1) - k$$

is the new magic constant for λ'.

Since the introduction of labeling, there have been several results on vertex magic total labeling of particular classes of graphs. For example, MacDougall *et al.*[4] proved that the cycle C_n for $n = 3$, path P_n for $n = 2$, complete graph K_n for odd n, and complete bipartite graph $K_{n,n}$ for $n > 1$, have vertex-magic total labeling. Baca, Miller, and Slamin [?] proved that for $n = 3, 1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, every generalized Petersen graph $P(n, m)$ has vertex-magic total labelings with the magic constant $k = 9n + 2, k =$

$10\,n+2$, and $k = 11\,n+2$. Balbuena *et al* [2] proved that for odd $n \geq 5$ and $m \in \{2, 3, \ldots, (n-1)/2\}$, circulant graphs $C_n(1, m)$ have a super vertex-magic total labeling with $k = (17n + 5)/2$. The complete survey of the known results on vertex-magic total labeling of graphs can be found in [3].

Most of the known results are concerning on vertex-magic total labeling of connected graphs. For the case of disconnected graph, Wallis [6] proved Theorem 1. Slamin *et al.* [5] proved that the 2 copies of generalized Petersen graphs $2P(n, m)$ has a vertex-magic total labeling with the magic constant $k = 19\,n + 2$ and $k = 21\,n + 2$.

Theorem 1 *[6] Suppose G is a regular graph of degree r which has a vertex-magic total labeling. (i) If r is even, then tG is vertex-magic whenever t is an odd positive integer. (ii) If r is odd, then tG is vertex-magic for every positive integer t.*

2 Main Result

As mentioned in the introduction, Slamin *et al.* [5] had given the construction of vertex-magic total labeling for the 2 copies of generalized Petersen graphs $2P(n, m)$. We found that this label can be extended to t-copies of generalized Petersen graphs $tP(n, m)$.

In this section, we present a construction of a vertex-magic total labeling for a more general case. We construct the vertex-magic total labeling for the union of t generalized Petersen graphs, $\bigcup_{j=1}^{t} P(n_j, m_j)$. The union of t generalized Petersen graphs $\bigcup_{j=1}^{t} P(n_j, m_j)$ has a vertex set $V(\bigcup_{j=1}^{t} P(n_j, m_j)) = \{u_i^j \,|\, 1 \leq i \leq n_j, 1 \leq j \leq t\} \cup \{v_i^j \,|\, 1 \leq i \leq n_j, 1 \leq j \leq t\}$ and the edge set $E(\bigcup_{j=1}^{t} P(n_j, m_j)) = \{u_i^j\, u_{i+1}^j \,|\, 1 \leq i \leq n_j,\ 1 \leq j \leq t\} \cup \{u_i^j\, v_i^j \,|\, 1 \leq i \leq n_j,\ 1 \leq j \leq t\} \cup \{v_i^j\, v_{i+m_j}^j \,|\, 1 \leq i \leq n_j,\ 1 \leq j \leq t\}$. In the next theorem we prove that union of t generalized Petersen graphs has a vertex-magic total labeling.

Theorem 2 *For $n_j \geq 3, 1 \leq m_j \leq \lfloor \frac{n_j-1}{2} \rfloor$, the union of t generalized Petersen graphs $P(n_j, m_j), j = 1, 2, \ldots, t$, has a vertex-magic total labeling with the magic constant $k = 10\sum_{l=1}^{t} n_l + 2$.*

Proof.

For all $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, t$, label the vertices and edges of $\bigcup_{j=1}^{t} P(n_j, m_j)$ as follows.

$$\begin{aligned}
\lambda(u_i^j) &= (n_j + 1 - i)\alpha(1, i-1) + 1 + \sum_{l=1}^{j-1} n_l, \\
\lambda(v_i^j) &= (m_j + 1 - i)\,\alpha(i, m_j) + 4\sum_{l=1}^{t} n_l + \sum_{l=1}^{j-1} n_l + \\
&\quad (n_j + m_j + 1 - i)\,\alpha(m_j + 1, i), \\
\lambda(u_i^j u_{i+1}^j) &= i + 3\sum_{l=1}^{t} n_l + \sum_{l=j+1}^{t} n_l, \\
\lambda(u_i^j v_i^j) &= (n_j + 1 - i) + 2\sum_{l=1}^{t} n_l + \sum_{l=1}^{j-1} n_l,
\end{aligned}$$

$$\lambda(v_i^j v_{i+m_j}^j) = i + \sum_{l=1}^{t} n_l + \sum_{l=j+1}^{t} n_l,$$

where

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{if } x > y. \end{cases}$$

It is easy to verify that the labeling λ is a bijection from the set $V(\bigcup_{j=1}^{t} P(n_j, m_j))$ $\cup\, E(\bigcup_{j=1}^{t} P(n_j, m_j))$ onto the set $\{1, 2, \ldots, 5\sum_{l=1}^{t} n_l\}$.

Let us denote the weight (under labeling λ) of vertex u_i^j of $\bigcup_{j=1}^{t} P(n_j, m_j)$ by

$$w_\lambda(u_i^j) = \lambda(u_i^j) + \lambda(u_{i-1}^j u_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(u_i^j v_i^j)$$

and the weights of vertex v_i^j by

$$w_\lambda(v_i^j) = \lambda(v_i^j) + \lambda(v_{i-m_j}^j v_i^j) + \lambda(v_i^j u_{i+m_j}^j) + \lambda(u_i^j v_i^j).$$

Then, for $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, t$, the labeling λ gives a vertex magic total labeling for $\bigcup_{j=1}^{t} P(n_j, m_j)$ with the magic constant $k = 10\sum_{l=1}^{t} n_l + 2$. □

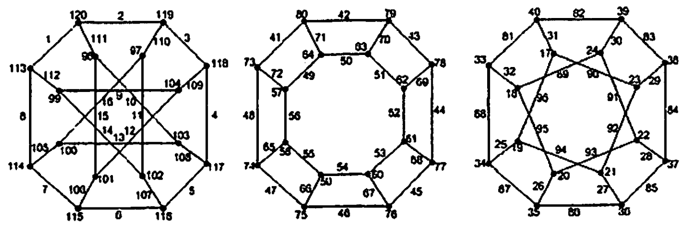

Fig. 1. VMTL of $P(8,3) \cup P(8,1) \cup P(8,2)$ with $k = 234$

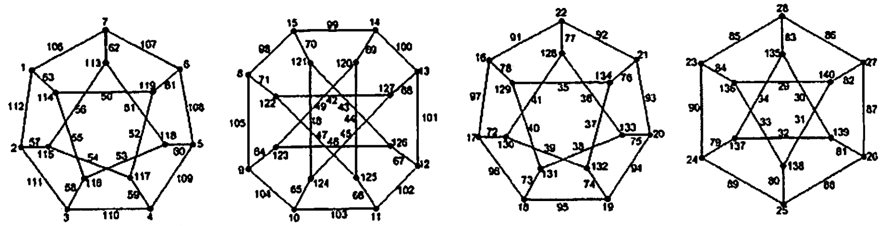

Fig. 2. VMTL of $P(7,2) \cup P(8,3) \cup P(7,2) \cup P(6,2)$ with $k = 282$

Figures 1 and 2 show examples of vertex-magic total labeling on the union of 3 non-ismorphic and 4 non-ismorphic generalized Petersen graphs with the magic constant $k = 234$ and $k = 282$ respectively.

From duality, we can show that the construction of vertex-magic total labeling for union of t generalized Petersen graph is not unique. However, both labels have the same magic number $k = 10\sum_{l=1}^{t} n_l + 2$.

The union of t special circulant graphs $\bigcup_{j=1}^{t} C_n(1, m_j)$ has a vertex set $V(\bigcup_{j=1}^{t} C_n(1, m_j)) = \{v_i^j \,|\, 0 \leq i \leq n-1, 1 \leq j \leq t\}$ and the edge set $E(\bigcup_{j=1}^{t} C_n(1, m_j)) = \{v_i^j v_{i+1}^j \,|\, 0 \leq i \leq n-1,\ 1 \leq j \leq t\} \cup \{v_i^j v_{i+m_j}^j \,|\, 0 \leq i \leq n-1,\ 1 \leq j \leq t\}$. Special circulant graph $C_n(1, m)$ is a regular graph with $r = 4$. Theorem 1 states that if G is a regular graph with r is even, then tG is vertex-magic whenever t is an odd positive integer. Moreover, we construct that the union of t special circulant graphs, $\bigcup_{j=1}^{t} C_n(1, m_j)$, for not only odd t, but also for even t. For the case of even t, our result is an example for G regular and even r that tG can have vertex magic labeling.

Theorem 3 *For odd $n \geq 5$ and $m_j \in \{2, 3, \ldots, (n-1)/2\}$, the disjoint union of t circulant graphs $\cup_{j=1}^{t} C_n(1, m_j)$ has a vertex magic total labeling with $k = 8tn + \frac{n-1}{2} + 3$.*

Proof.

Let $\{C_n(1, m_j) : j = 1, \ldots, t\}$ be a set of circulant graphs with n vertices. Let $\{v_i^j | i = 0, \ldots, n-1\}$ and $\{v_i^j v_{i+1}^j | i = 0, \ldots, n-1\} \cup \{v_i^j v_{i+m_j}^j | i = 0, \ldots, n-1\}$ be the sets of vertices and edges of j^{th} circulant graphs $C_n(1, m_j)$, $j = 1, \ldots, t$, where $i+1$ and $i+m_j$ are taken modulo $(n-1)$. Label the vertices and edges as follows

$$\begin{aligned}
\lambda(v_i^j) &= n(j - \alpha(i, m_j - 1))\ odd(j) + n(2t - j + \alpha(m_j, i))\ even(j) \\
&\quad + m_j - i, \\
\lambda(v_i^j v_{i+1}^j) &= \frac{1}{2}(4tn + n\ j\ odd(j) + n(2t - j + 1)\ even(j) + n\ even(i) - i), \\
\lambda(v_i^j v_{i+m_j}^j) &= n(2t - j)\ odd(j) + n(j-1)\ even(j) + i + 1,
\end{aligned}$$

where

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y, \end{cases}$$

$$odd(x) = \begin{cases} 1 & \text{if odd } x \\ 0 & \text{if another,} \end{cases}$$

$$even(x) = \begin{cases} 1 & \text{if even } x \\ 0 & \text{if another.} \end{cases}$$

Let us denote the weight of vertex v_i^j of $\bigcup_{j=1}^{t} C_j(1, m_j)$ by

$$w_\lambda(v_i^j) = \lambda(v_i^j) + \lambda(v_{i-1}^j v_i^j) + \lambda(v_i^j v_{i+1}^j) + \lambda(v_{i-m_j}^j v_i^j) + \lambda(v_i^j v_{i+m_j}^j)$$

Then, for $i = 0, 2, \ldots, n_j - 1$ and $j = 1, 2, \ldots, t$, the labeling λ gives a vertex magic total labeling for $\bigcup_{j=1}^{t} C_n(1, m_j)$ with the magic constant $k = 8tn + \frac{n-1}{2} + 3$. □

Figure 3 shows an example of vertex-magic total labeling on the union of 4 non-ismorphic circulant graphs with the magic constant $k = 295$.

From duality property we can obtain Corollary 1.

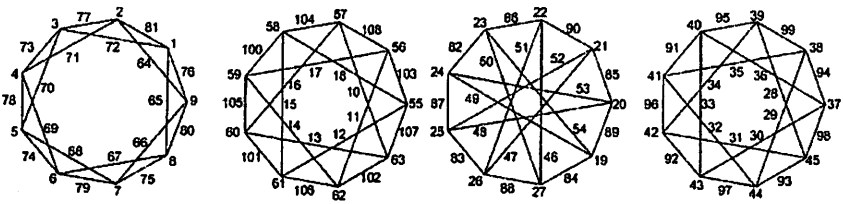

Fig. 3. VMTL of $C_9(1,2) \cup C_9(1,3) \cup C_9(1,4) \cup C_9(1,3)$ with $k = 295$

Corollary 1 *For odd $n \geq 5$ and $m_j \in \{2, 3, \ldots, (n-1)/2\}$, the disjoint union of circulants $\cup_{j=1}^{t} C_n(1, m_j)$ has a vertex magic total labeling with $k = 7tn - \frac{n-1}{2} + 2$.*

3 Conclusion

We conclude this paper with an open problem for further research direction in this area.

Open Problem 3 *Find if there is a vertex magic labeling for disjoint union of t (non)-isomorphic regular graphs other then $\bigcup_{j=1}^{t} P(n_j, m_j)$ and $\cup_{j=1}^{t} C_n(1, m_j)$.*

References

1. M. Bača, M. Miller and Slamin, Vertex-magic total labelings of generalized Petersen graphs, *Int. J. of Computer Mathematics* **79**, Issue **12**, (2002) 1259-1264.
2. C. Balbuena, E. Barker, K. C. Das, Y. Lin, M. Miller, J. Ryan, Slamin, K. Sugeng, M. Tkac, On the degrees of a super vertex-magic graph, *Discrete Mathematics* **306** (2006) 539-559.
3. J. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics* **6**(2008), DS6.
4. J. MacDougall, M. Miller, Slamin and W. D.Wallis, Vertex-magic total labelings, Utilitas Math., **61** (2002) 3-21.
5. Slamin, A.C. Prihandoko, T.B. Setiawan, V. Rosita and B. Shaleh, Vertex-magic total labeling of disconnected graphs, *Journal of Prime Research in Mathematics*, to appear.
6. W. D. Wallis, *Magic Graph*, Birkhuser, (2001).
7. M. E. Watkins, A Theorem on Tait Colorings with an Application to the Generalized Petersen Graphs, *J. Combin. Theory* **6** (1969) 152-164.