The partition dimension of a complete multipartite graph, a special caterpillar and a windmill

Darmaji*, S. Uttunggadewa, R. Simanjuntak, E.T. Baskoro

Combinatorial Mathematics Research Group
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung (ITB)
Jalan Ganesha No. 10 Bandung 40132, Indonesia.

darmaji@students.itb.ac.id,
{s_uttunggadewa, rino, ebaskoro}@math.itb.ac.id

Abstract. In this paper, we determine the partition dimension of a complete multipartite graph $K_{n_1,n_2,\cdots,n_r}$, namely $pd(K_{n_1,n_2,\cdots,n_r})$ is $r+n-1$ if $n_i = n$ for $1 \le i \le r$ and $pd(K_{n_1,n_2,\cdots,n_r})$ is $r+n-2$ for $n = n_1 \ge n_2 \ge \cdots \ge n_r$. We also show that the partition dimension of caterpillar graph C_n^m is m for $n \le m$ and $m+1$ for $n > m$, and the partition dimension of windmill graph W_2^m is k, where k is the smallest integer such that $\binom{k}{2} \ge m$.

Keywords and phrases: *caterpillar, complete multipartite graph, partition dimension, resolving partition, windmill graph.*

1 Introduction

For any two vertices u and v in a connected graph $G(V, E)$ and a subset S of $V(G)$, the *distance* $d(u, v)$ between u and v is the length of a shortest path between u and v, and the *distance* $d(u, S)$ between u and S is $\min\{d(u, x), x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$, the *representation* of vertex $v \in V(G)$ with respect to Π is the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$. The k-partition Π is called a *resolving partition* for G if every vertex of G has a unique representation. The *partition dimension* of G, denoted by $pd(G)$, is the least positive integer k for which there exists a resolving k-partition for G.

Trivially, $pd(G) = 1$ if and only if $G \approx K_1$. It is easy to see that $pd(G) \le n$ for any connected graph G of order $n \ge 2$. Chartrand et al.[2] also showed

* Permanent address: Jurusan Matematika FMIPA, Institut Teknologi Sepuluh Nopember (ITS), Jl. Arif Rahman Hakim No.10 Surabaya 60111, Indonesia. email address: darmaji@matematika.its.ac.id

that a path is the only graph with partition dimension 2. The complete graph G on n vertices is the only graph with $pd(G) = n$. For bipartite graphs, they showed that $pd(K_{m,n}) = \max\{m, n\}$ if $m \neq n$ and $pd(K_{m,n}) = m + 1$ if $m = n$. Furthermore, they showed that $pd(G)$ is $n - 1$ if and only if G is one of the graphs $K_{1,n-1}, K_n - e$ or $K_1 + (K_1 \bigcup K_{n-2})$.

Tomescu et al. [6] derived the lower and upper bounds of the partition dimension of a wheel graph W_n. They showed that $|\sqrt[3]{2n}| \leq pd(W_n) \leq p+1$ where p is the smallest prime number such that $p(p-1) \geq n$. In particular, they obtained the exact values of $pd(W_n)$ for $n \leq 19$. In [6] Tomescu also characterized all graphs G of n vertices whose $pd(G)$ equals $n - 2$.

In this paper, we consider the partition dimension of a complete multipartite graph $K_{n_1,n_2,\cdots,n_r}$. We show that $pd(K_{n_1,n_2,\cdots,n_r})$ is $r + n - 1$ if $n_i = n$ for $1 \leq i \leq r$ and $pd(K_{n_1,n_2,\cdots,n_r})$ is $r + n - 2$ for $n = n_1 \geq n_1 \geq n_2 \geq \cdots \geq n_r$. We also show that the partition dimension of caterpillar graph C_n^m is m for $n \leq m$ and $m + 1$ for $n > m$. At last, we prove that the partition dimension of windmill graph W_2^m is k, where k is the smallest integer such that $\binom{k}{2} \geq m$.

2 The Partition Dimension of Complete multipartite Graph

A *complete multipartite* graph $K_{n_1,n_2,\cdots,n_r}$ is a graph consisting of r-partite $V_1, V_2, \cdots, V_r$ where every two vertices in different partite are always adjacent. The cardinality of each partite is $n_1, n_2, \cdots n_r$, respectively. A multipartite graph whose $r-$partite is also called *r-partite* graph. If $n_1 = n_2, \cdots = n_r = n$, the r-partite graph is denoted by $K_{r \times n}$. A j-th vertex in i-th partite of r-partite graph is denoted by $v_{i,j}$ where $1 \leq i \leq r$ and $1 \leq j \leq n$. If $r = 2$ then an r-partite graph is called a *bipartite* graph and denoted by $K_{m,n}$.

Let $\Pi = \{S_1, S_2, \cdots, S_t\}$ be any resolving partition of $K_{n_1,n_2,\cdots,n_r}$. Each S_i, for $i = 1, 2, \cdots, t$, is called a *partition class* of Π. For each partite V_i of $K_{n_1,n_2,\cdots,n_r}$, define a *partite index set* I_i such that $j \in I_i$ if and only if $(V_i \bigcap S_j) \neq \emptyset$, where $1 \leq i \leq r$ and $1 \leq j \leq t$. It means, for $j \in I_i$, j represents a vertex $x \in V_i$ and $x \in S_j$.

Lemma 1. *If Π is a resolving partition of $K_{n_1,n_2,\cdots,n_r}$, then there is a resolving partition Π' such that (a) every partite index set has an element belongs to a singleton partition class, and (b) the cardinality of Π' equals the cardinality of Π.*

Proof. Let Π be any resolving partition of $K_{n_1,n_2,\cdots,n_r}$. If every partite index set I_i, where $1 \leq i \leq r$, already contains an element belongs to a singleton partition class then $\Pi' = \Pi$. Otherwise, there are at least one partite index set, say I_j, has no singleton partition class. For every partite index I_j, without loss of generality, take the last element of I_j, say m. Since I_j has no singleton partition class, then there are at least one partite index set contains m. Substitute $m \notin I_j$ with an index, say o, such that $o \neq m$ and o in a partite index set other than it's partite index set itself. We repeat the procedure for every partite of $K_{n_1,n_2,\cdots,n_r}$ that has no singleton partition class until there is a singleton partition class in every partite of $K_{n_1,n_2,\cdots,n_r}$.

Define the resolving partition after the procedure as Π'. From the procedure, it is clear that every partite index set I_i, where $1 \leq i \leq r$, has a singleton partition class. Since every substitution only uses index that already exist in index set, the substitution doesn't change the cardinality of the index set, i.e., $|\Pi'| = |\Pi|$. Every vertex in every partite index set has different representation with respect to Π', because they are distinguished by the singleton partition class. Thus, Π is a resolving partition. □

Lemma 2. *[1] Let G be a connected graph and let Π be a resolving partition for G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u,v\}$, then u and v belong to different partition classes of Π.*

The following result is a consequence of Lemma 2.

Corollary 1. *Let Π be a resolving partition of the complete r-partite graph $K_{n_1,n_2,\cdots,n_r}$. Every two distinct vertices x and y in the same partite of $K_{n_1,n_2,\cdots,n_r}$ must be in different partition classes of Π.*

Proof. If two distinct vertices x and y are in the same partite of $K_{n_1,n_2,\cdots,n_r}$ then $d(x,u) = d(y,u)$ for any vertex u other than x and y. Therefore, by Lemma 2, vertices x and y must be in different partition classes of Π. □

Theorem 1. *The partition dimension of the complete r-partite graph is:*

$$pd(K_{n_1,n_2,\cdots n_r}) = \begin{cases} n+r-1 \ , \ n_i = n \ for \ 1 \leq i \leq r \\ n+r-2 \ , \ n = n_1 \geq n_2 \geq \cdots \geq n_r. \end{cases}$$

Proof. Let $\Pi = \{S_1, S_2, \cdots, S_t\}$ be any resolving partition of $K_{n_1,n_2,\cdots,n_r}$. There are two cases to be considered.

Case 1: $n_i = n$ for $1 \leq i \leq r$.
We will show that $t \geq r+n-1$. By Lemma 1, there are r singleton partition classes of $K_{n_1,n_2,\cdots,n_r}$ and, by Corollary 1, the remaining vertices of G must be distributed into at least $n-1$ non-singleton partition classes. So, there are r singleton partition classes and $n-1$ non-singleton partition classes, hence $t \geq r+n-1$.

To show $t \le r+n-1$, take the partition $\Pi = \{S_1, S_2, \cdots, S_{r+n-1}\}$ such that

$$S_j = \begin{cases} \{v_{1,j}, v_{2,j}, v_{3,j} \cdots v_{r,j}\}, 1 \le j \le n-1 \\ \{v_{(j-n+1),n}\} \qquad\qquad , n \le j \le r+n-1 \end{cases}$$

where $v_{i,j}$ represents the j-th vertex in the i-th partite of $K_{n_1,n_2,\cdots n_r}$. Then for each vertex $v_{i,j}$, $r(v_{i,j}|\Pi)$ has '0' in the j-th entry, '2' in the $(n+i-1)$-th entry and '1' in the remaining entries. Therefore each $r(v_{i,j}|\Pi)$ is unique. So, Π is a resolving partition and $t \le r+n-1$.

Case 2: $n = n_1 \ge n_2 \ge \cdots \ge n_r$.
Let $|I_1| = n$ be the maximum cardinality index set of $K_{n_1,n_2,\cdots,n_r}$ and $|I_r|$ be a minimum one. We will show that $t \ge r+n-2$. By Lemma 1, there are r singleton partition classes in Π. By Corollary 1, the remaining vertices in partite index set I_1 can be distributed to $n-1$ non-singleton partition classes. So, Π consists of r singleton partition classes and $n-1$ non-singleton partition classes. Now, by replacing I_r with $I'_r \subseteq I_j \setminus \{c\}$ for some $j \in \{1, 2, \cdots, r-1\}$, and c is an element belongs to a singleton partition, we can still distinguish all vertices. Therefore, $t \ge (r+n-1)-1 = r+n-2$.

To show $t \le r+n-2$, take the partition $\Pi = \{S_1, S_2, \cdots, S_{r+n-2}\}$ where $S_i = \{v_{i,n_i}\}$, for $i = 1, 2, \cdots, r-1$; and $S_{(r-1)+j} \supseteq \{v_{1,j}\}$ for $j = 1, 2, \cdots, n-1$ and all other vertices in $V_2 \bigcup V_3 \bigcup \cdots \bigcup V_n$ are distributed into $S_{(r-1)+j}$ such that Corollary 1 is satisfied. It is clear that every vertex in every partition class has distinct representation with respect to Π because they are distinguished by the singleton partitions. Thus, $t \le r+n-2$. □

In this section we have given partition dimension for all complete multipartite graphs. In the next two sections, we will consider two families of multipartite graphs which are not complete. The first graph under consideration is a special case of caterpillar which is bipartite and the last graph is windmill graph which is tripartite.

3 The Partition Dimension of Caterpillar Graph

Let us define a (specific) *caterpillar* C_n^m as a graph on $(m+1)n$ vertices constructed by adding m vertices at each vertex x of a path P_n, and connecting them to vertex x. Every vertex in P_n is labeled by $x_1, x_2, \cdots, x_n$ and m vertices on x_i are labeled respectively by $u_{1i}, u_{2i}, \cdots, u_{mi}$. A caterpillar is a tree, and therefore it is bipartite.

Theorem 2.

$$pd(C_n^m) = \begin{cases} m, \text{ for } n \leq m, \\ m+1, \text{ for } n > m. \end{cases}$$

Proof. There are two cases to be considered.

Case 1: for $n \leq m$.
It will be showed that $pd(C_n^m) = m$ for $n \leq m$. By Lemma 2, each of m leaves at vertex x_i belong to m partition classes of Π. Therefore, $pd(C_n^m) \geq m$. Now, let $\Pi = \{S_1, S_2, \cdots, S_m\}$ where

$$S_k = \begin{cases} \{u_{k1}, u_{k2}, \cdots, u_{kn}, x_k\} & \text{, for } 1 \leq k \leq n \\ \{u_{k1}, u_{k2}, \cdots, u_{kn}\} & \text{, for } n+1 \leq k \leq m \end{cases}$$

For $1 \leq k \leq n$, the representation $r(x_k|\Pi) = (1, 1, \cdots, 0, \cdots, 1, 1)$, where the k-th entry is '0'. The representation $r(u_{ki}|\Pi)$ for every leave $u_{ki} \in V(C_n^m)$, for $k = 1, 2, \cdots, m$ and $i = 1, 2, \cdots, n$, is different because they have different vertex x_k that belongs to a partition class S_k for $k = 1, 2, \cdots, n$. Therefore, Π is a resolving partition for C_n^m, and $pd(C_n^m) \leq m$.

Case 2: for $n > m$.
It will be showed that $pd(C_n^m) = m + 1$ for $n > m$. Lemma 2 insures that there are at least m partition classes for m leaves. By a contradiction, we will show $pd(C_n^m) \geq m + 1$. Assume $\Pi_1 = \{S_1, S_2, \cdots, S_m\}$ and $S_k = \{u_{k1}, u_{k2}, \cdots, u_{kn}, x_k\}$ for $1 \leq k \leq m$. Since $n \geq m$, at least one of $S_1, S_2, \cdots S_m$ have at least two vertices $x_i \in P_n$ for $i = 1, 2, \cdots, n$. Without loss of generality, if $x_1, x_2 \in S_1$ then $r(x_1|\Pi_1) = r(x_2|\Pi_1) = (0, 1, 1, \cdots)$. There is a contradiction for Π_1 as a resolving partition for C_n^m. Therefore, $pd(C_n^m) \geq m+1$. Now, let $\Pi_2 = \{S_1, S_2, \cdots, S_{m+1}\}$ be a resolving partition where

$$S_k = \begin{cases} \{u_{11}, u_{12}, \cdots, u_{1n}, x_1, x_2, \cdots, x_{n-1}\}, & \text{for } k = 1 \\ \{u_{k1}, u_{k2}, \cdots, u_{kn}\} & \text{, for } 2 \leq k \leq m \\ \{x_n\} & \text{, for } k = m+1 \end{cases}$$

For $j = 1, 2, \cdots, n$, the representation leaves u_{1j} with respect to Π_2, $r(u_{1j}|\Pi_2) = (0, 2, 2, \cdots, n+1-j)$, and for $j = 1, 2, \cdots, n-1$, the representation x_j with respect to Π_2, $r(x_j|\Pi_2) = (0, 1, 1, \cdots, n-j)$. Every leave $u_{ki} \in V(C_n^m)$, where $2 \leq k \leq m$ and $1 \leq i \leq n$, the representation $r(u_{ki}|\Pi_2)$ is different because we have different distance $d(u_{ki}, S_{m+1})$. Therefore, for every vertex $x \in V(C_n^m)$ has a different representation with respect to Π_2. So, $pd(C_n^m) \leq m + 1$. □

4 The Partition Dimension of Windmill Graph

A windmill graph W_2^m can be obtained from connecting a center vertex c to all vertices of graph mK_2. A windmill graph W_2^m has m blades and every blade has two rim vertices a and b that denoted by a_i and b_i for the i-th blade. A windmill graph W_2^m is a tripartite graph $G((V_1, V_2, V_3), E)$ where $V_1 = \{a_1, a_2, \cdots, a_m\}$, $V_2 = \{b_1, b_2, \cdots, b_m\}$ and $V_3 = \{c\}$. The edges of graph W_2^m are defined by $E(W_2^m) = \{ca_i, a_ib_i, c_ib_i | i = 1, 2, \cdots, m\}$.

Lemma 3. *If Π is a resolving partition of a windmill graph W_2^m with $V(W_2^m) = \{c, a_i, b_i | i = 1, 2, \cdots, m\}$ and $E(W_2^m) = \{ca_i, a_ib_i, c_ib_i | i = 1, 2, \cdots, m\}$, then a_i and b_i must be in different partitions of Π for each i.*

Proof. Since $d(a_i, v) = d(b_i, v)$ for any $v \in V(W_2^m) - \{a_i, b_i\}$ then a_i and b_i must be in different partitions of Π. □

Lemma 4. *Let Π be a resolving partition of a windmill graph W_2^m with $V(W_2^m) = \{c, a_i, b_i | i = 1, 2, \cdots, m\}$, $E(W_2^m) = \{ca_i, a_ib_i, c_ib_i | i = 1, 2, \cdots, m\}$ and let S_x be a partition of Π containing x. For each i, define $L_i = \{u, v\}$ if $a_i \in S_u$ and $b_i \in S_v$. Then, $L_i \neq L_j$ for $i \neq j$.*

Proof. For a contradiction, suppose there are some i and j such that $L_i = L_j = \{u, v\}$. Without loss of generality, assume $\{a_i, a_j\} \subseteq S_u$ and $\{b_i, b_j\} \subseteq S_v$. Then, $r(a_i|\Pi) = r(b_i|\Pi)$ since $d(a_i|S) = d(a_j|S)$ for any $S \in \Pi$, a contradiction to Π being a resolving partition. □

Theorem 3. *The partition dimension of a windmill graph $V(W_2^m)$ is k, where k is the smallest integer such that $\binom{k}{2} \geq m$.*

Proof. Let Π be a resolving partition for a windmill graph W_2^m. By Lemmas 3 and 4, a_i and b_i must be in different partition of Π, and $L_i \neq L_j$ for $i \neq j$. The definition of L_i refers to the one in Lemma 4. Therefore, the number of partition in Π is at least k, where k is the smallest integer satisfying $\binom{k}{2} \geq m$. Thus, $pd(W_2^m) \geq k$. Now, we will show that $pd(W_2^m) \leq k$. Consider $\Pi = \{S_1, S_2, \cdots, S_k\}$ obtained by assigning $L_i (i = 1, 2, \cdots, m)$ with a subset of two elements from $\{1, 2, ..., k\}$ such that $L_i \neq L_j$ for $i \neq j$, and letting $c \in S_1$. Since $r(v|\Pi)$ is unique for every $v \in V(W_2^m)$ then Π is a resolving partition of Π. Therefore, $pd(W_2^m) \leq k$, where k is the smallest integer such that $\binom{k}{2} \geq m$. □

References

1. G. Chartrand and P. Zhang, The theory and application of resolvability in graphs, *Congressus Numerantium* 160 (2003), 47-68.

2. G. Chartrand, E. Salehi and P. Zhang, The partition dimension of a graph, *Aequationes Math* 59 (2000), 45-54.
3. G. Chartrand, E. Salehi, and P. Zhang, On the partition dimension of a graph, *Congressus Numerantium* 130 (1998), 157-168.
4. G. Chappell, J. Gimbel, and C. Hartman, Bounds on the metric and partition dimension of a graph, preprint.
5. I. Tomescu, Discrepancies between metric dimension and partition dimension of a connected graph, *Discrete Mathematics* 308 (2008), 5026-5031.
6. I. Tomescu, I. Javaid, and Slamin, On the partition dimension and connected partition dimension of wheels, *Ars Combinatoria* 84 (2007), 311-317.