Super (a, d)-vertex antimagic total labeling on a disjoint union of regular graphs *

Kiki A. Sugeng, Denny R. Silaban

Department of Mathematics,
Faculty of Mathematics and Sciences,
University of Indonesia
Depok 16424, Indonesia
{kiki, denny-rs}@ui.ac.id

Abstract. Let $G = (V, E)$ be a graph with order $|G|$ and size $|E|$. An (a, d)-vertex-antimagic total labeling is a bijection α from a set of all vertices and edges to the set of consecutive integers $\{1, 2, ..., |V| + |E|\}$, such that the weights of the vertices form an arithmetic progression with the initial term a and the common difference d. If $\alpha(V(G)) = \{1, 2, \ldots, |V|\}$ then we call the labeling super (a, d)-vertex antimagic total. In this paper we show some basic properties of such labelings on a disjoint union of regular graphs and show how to construct such labelings for some classes of graphs, such as cycles, generalised Pertersen graphs and circulant graphs, for $d = 1$.

1   Introduction

In this paper we consider simple and undirected graphs. The set of vertices of G will be denoted as $V = V(G)$ and the set of edges $E = E(G)$, while $n = |V(G)|$ and $e = |E(G)|$.

A *labeling* α of a graph G, basically is a mapping from elements (vertices, edges, faces) of a graph to set of numbers. A *vertex labeling* (respectively, *edge labeling*) is a labeling which its domain is the set of vertices (respectively, the set of edges). A *total labeling* is a labeling which its domain is $V(G) \cup E(G)$. For a further explanation of vertex, edge and total labelings, see [4].

The *vertex-weight* $wt(x)$ of a vertex $x \in V$, under a labeling $\alpha : V \cup E \to \{1, 2, ..., n + e\}$, is $\alpha(x) + \sum_{y \in N(x)} \alpha(xy)$, where $N(x)$ is neighbour of x.

An (a,d)-*vertex-antimagic total* (in short, (a,d)-VAT) *labeling of* G is a bijection $\alpha : V \cup E \rightarrow \{1, 2, ..., n + e\}$ which the set of vertex-weights of all vertices in G is $\{a, a + d, a + 2d, ..., a + (n - 1)d\}$, where for some integers $a > 0$ and $d \geq 0$. If $d = 0$ then we call α a *vertex-magic total* (in short, *VMT*) *labeling*.

An (a,d)-vertex-antimagic total labeling α is called a *super (a,d)-vertex-antimagic total* (in short, *super (a,d)-VAT*) labeling if $\alpha(V) = \{1, 2, ..., n\}$ and $\alpha(E) = \{n + 1, n + 2, ..., n + e\}$.

The concept of the vertex-magic total labeling was introduced by Mac-Dougall *et al.* [8], while Bača *et al.* [1] introduced basic properties of (a,d)-VAT labelings. Sugeng *et. al* [10] studied basic properties of super (a,d)-VAT labeling and showed the construction of such labelings for certain classes of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalised Petersen graphs. For other results in vertex magic and (a,d)-vertex-antimagic total labeling, see [?,3,4,6,8,9,11]. Most of the results on super (a,d)-VAT are on a connected graphs. However, lately there is a direction to study more on such labeling on disjoint union graphs. In the following section we discuss super (a,d)-VAT labeling on disjoint union of graphs when $d = 1$.

2 Basic properties

Suppose that graph G has a super (a,d)-VAT labeling. If δ is the smallest degree in G then the minimum possible vertex-weight is $1 + (n + 1) + (n + 2) + ... + (n + \delta)$. Then

$$a \geq 1 + n\delta + \frac{\delta(\delta + 1)}{2}. \tag{1}$$

Let Δ be the largest degree of G then

$$d \leq 1 + \frac{\Delta(2n + 2e - \Delta + 1) - \delta(2n + \delta + 1)}{2(n - 1)}. \tag{2}$$

and

$$a = \frac{1}{2}(n + 1 - (n - 1)d) + 2e + \frac{e(e + 1)}{n}. \tag{3}$$

3 Known results

The following theorems give the known results of a super (a,d)-VAT labeling on several classes of connected regular graphs. The classes of graphs that

will be discussed in this section and following sections are cycles, circulant graphs $C_n(1,2)$ and generalised Petersen graphs $P(n,1)$ and $(P(n,2)$, each of them representing respectively 2-regular, 3-regular and 4-regular graphs.

Theorem 4 *[10] The cycle C_n has a super (a,d)-VAT labeling if and only if either*

    (i) $d \in \{0,2\}$ and n is odd, $n \geq 3$, or

    (ii) $d = 1$ and $n \geq 3$.

Let n and m be positive integers, $n \geq 3$ and $1 \leq m < \lfloor \frac{n}{2} \rfloor$. The generalized Petersen graph $P(n,m)$ is a graph that consists of an outer-cycle $y_0, y_1, y_2, \ldots, y_{n-1}$, a set of n spokes $y_i x_i$, $0 \leq i \leq n-1$, and n edges $x_i x_{i+m}$, $0 \leq i \leq n-1$, where all the subscripts are taken modulo n. From (2), it follows that if $P(n,m)$, $n \geq 3$, $1 \leq m < \lfloor \frac{n}{2} \rfloor$, has a super (a,d)-VAT labeling then $d < 5$. If $m = 1$ then $P(n,1)$ is also known as a prism.

The following theorem shows that every Petersen Graph has a super $(a,1)$-VAT labeling.

Theorem 5 *[10] For n odd, $1 \leq m < \lfloor \frac{n}{2} \rfloor$, every generalised Petersen graph $P(n,m)$ has a super $(a,1)$-VAT labeling.*

Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \lfloor n/2 \rfloor$, where n and a_i ($i = 1, \ldots, k$) are positive integers. A circulant graph $C_n(a_1, a_2, \ldots, a_k)$ is a regular graph with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{(v_i v_{i+a_j})(mod\ n) : i = 0, 1, \ldots, n-1, j = 1, 2, \ldots, k\}$. For special circulant graph $C_n(1,m)$, Balbuena *et al.*[3] have the following results.

Theorem 6 *[3] For odd $n \geq 5$ and $m \in \{2, 3, \ldots, \lfloor (n-1)/2 \rfloor\}$, circulant graphs $C_n(1,m)$ have a super vertex magic total labeling with the magic constant $h = (17n+5)/2$.*

4   Disjoint union of regular graphs

If G is an r-regular graph, then from (3) we have the following lemma.

Lemma 1 *Let G be an r-regular graph whose a super (a,d)-VAT labeling, then*

$$a = \frac{n(2 + 4r + r^2) + 2r + 2 - 2(n-1)d}{4}. \tag{4}$$

Using the value of a from Equation (4) and considering the value of d (in this case we only see the case of $d = 1$ and $d = 2$) then we have the following observations

Observation 1 *Let G be a disjoint union of t isomorphic copies of an r-regular graph H. Let G has a super (a,d)-VAT labeling and $|V(G)| = n$, then*

1. *If $d = 1$ and r is odd, then $tn = 2(mod\ 4)$.*
2. *If $d = 2$ and*
   - *if r, t are odd then $n = 0(mod\ 4)$*
   - *if r is odd and t is even then $n = 0(mod\ 2)$ and*
   - *if r is even then t and n are odd.*

Observation 2 *Let G be a disjoint union of t isomorphic copies of an r-regular graph H. Let G has a super (a,d)-VAT labeling and $|V(G)| = n$, then*

1. *If $r = 2$ then $d \leq 2$, for $n \geq 3$.*
2. *If $r = 3$ then $d \leq 2$, for $n \geq 3$.*
3. *If $r = 4$ then $d \leq 5$, for $n \geq 10$.*

Theorem 7 *Let G be a disjoint union of t isomorphic copies of an r-regular graph H. Let H has an (a,d)-VAT labeling and $|V(H)| = n$. If G has a super (a,d)-VAT labeling then $a(G) = ta(H) + \frac{(t-1)(1+d+r)}{2}$.*

Proof: Let H has an (a,d)-VAT labeling and $|V(H)| = n$. Let $G = tH$. From (4), we have

$$a(H) = \frac{1}{2}(n+1-(n-1)d) + 2(\frac{rn}{2}) + \frac{\frac{rn}{2}(\frac{rn}{2}+1)}{n},$$

and

$$a(tH) = \frac{1}{2}(tn+1-(tn-1)d) + 2(\frac{rtn}{2}) + \frac{\frac{rtn}{2}(\frac{rtn}{2}+1)}{tn}.$$

By comparing the values of $a(H)$ and $a(tH)$ we obtain the following result.

$$a(G) = ta(H) + \frac{(t-1)(1+d+r)}{2}. \quad \square$$

5 Labeling construction for several classes of r-regular graphs

Gómez, in [5], proved that, under some circumstance, if the r-regular graph G has a super $(a,0)$-VAT labeling then graph tG has a super $(a,0)$-VAT labeling.

Theorem 8 *[5] Let t be a positive integer. If the graph G is an r-regular graph that admits a super VMT labeling and $(t-1)(r+1)/2$ is an integer, then the graph tG has a super VMT labeling.*

While Gómez proved the general case of super VMT labeling of t copies of graphs, in this section we give a construction of a super $(a, 1)$-VAT labeling for a disjoint union of several classes of graphs. The classes of graphs that we consider are cycles, generalised Petersen graphs and circulant graphs. Each class represents an r-regular graph with $r \in \{2, 3, 4\}$, respectively.

The only regular graph with $r = 2$ is a cycle. The following theorem shows that a disjoint union of t (not necessarily isomorphic) cycles has a super $(a, 1)$-VAT labeling. See Figure 1 for the example.

Theorem 9 *Let $\{C_{n_j} : j = 1, \ldots, t\}$ be a set of cycles with n_j vertices, where $n_j \geq 5$ are integers. Then, a disjoint union of cycles $\cup_{j=1}^{t} C_{n_j}$ has a super $(3 \sum_{k=1}^{t} n_k + 2, 1)$-VAT labeling.*

Proof: Let $\{C_{n_j} : j = 1, \ldots, t\}$ be a set of cycles with n_j vertices. Let $\{v_i^j : i = 1, \ldots, n_j\}$ be a set of vertices of cycle C_{n_j}. Label the vertices as follows

$$\alpha(v_i^j) = \sum_{k=1}^{j-1} n_k + i$$

for $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, t$ and label the edges as follows

$$\alpha(v_i^j v_{i+1}^j) = \sum_{k=1}^{t} n_k + \sum_{k=j}^{t} n_k - i + 1, \text{ for } i = 1, 2, \ldots, n_j.$$

Then the vertex weights form a sequence of consecutive integers starting with $a = 3 \sum_{k=1}^{t} n_k + 2$. Thus, $\cup_{j=1}^{t} C_{n_j}$ has a super $(3 \sum_{k=1}^{t} n_k + 2, 1)$-VAT labeling. $\square$

Now we show the construction of super $(a, 1)$-VAT labeling for t copies of generalised Petersen graphs. Let the number of vertices of generalised Petersen graph be equal to $2n$. Since generalised Petersen graphs are 3-regular graphs, then according to Observation (1), $2nt = 2 \pmod 4$. Obviously, both n and t must be odd.

Theorem 10 *Let $P(n, 1)$ be a prism with $2n$ vertices, n odd and $n \geq 3$. Let t be an odd number, then t-copies of prism $tP(n, 1)$ has a super $(a = (10t + 2)n - \lfloor \frac{n}{2} \rfloor + 2, 1)$-VAT labeling.*
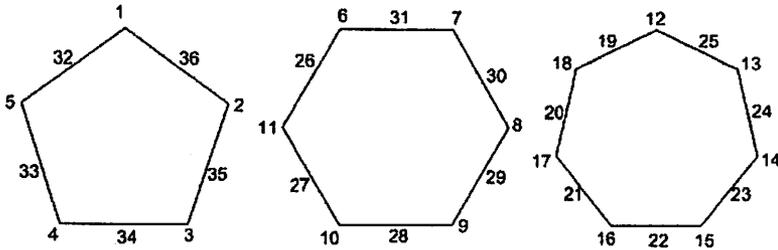
Fig. 1. Super $(56, 1)$-VAT labeling for $C_5 \cup C_6 \cup C_7$.

Proof: Let $P(n,1)$ be a prism with $2n$ vertices. Let $\{v_i^j : i = 1, ..., n\}$ be a set of outer vertices and $\{u_i^j : i = 1, ..., n\}$ be a set of inner vertices of prism $P(n,1)$.

For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, t$, label the vertices as follows

$$\alpha(u_i^j) = (j-1)n + i$$

and

$$\alpha(v_i^j) = (t+j-1)n + i$$

and label the edges as follows

$$\alpha(u_i^j u_{i+1}^j) = \begin{cases} (4t - 2j + 2)n - \frac{i-1}{2}, & \text{for } i \text{ odd} \\ (4t - 2j + 2)n - \frac{n+i-1}{2}, & \text{for } i \text{ even} \end{cases}$$

$$\alpha(v_i^j v_{i+1}^j) = \begin{cases} (4t - 2j + 1)n - \frac{i-1}{2}, & \text{for } i \text{ odd} \\ (4t - 2j + 1)n - \frac{n+i-1}{2}, & \text{for } i \text{ even} \end{cases}$$

and

$$\alpha(u_i^j v_i^j) = (4t + j - 1)n + i.$$

Then the vertex weights form consecutive integers starting with $a = (10t + 2)n - \lfloor \frac{n}{2} \rfloor + 2$. Thus, $tP(n,1)$ has a super $(a = (10t+2)n - \lfloor \frac{n}{2} \rfloor + 2, 1)$-VAT labeling. $\square$

Theorem 11 *Let $P(n,2)$ be the generalised Petersen graph with $2n$ vertices, n odd and $n \geq 3$. If t is odd integer then $tP(n,2)$ has a super $(a = (\frac{1}{2}(21tn + 5)), 1)$-VAT labeling.*

Proof: Let $\{v_i^j : i = 1, ..., n\}$ be a set of outer vertices and $\{u_i^j : i = 1, ..., n\}$ be a set of inner vertices of $P(n,2)$.

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For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, t$, label the vertices as follows

$$\alpha(u_i^j) = \begin{cases} (j-1)n - \frac{i+1}{2}, & \text{for } i \text{ odd} \\ (j-1)n - \frac{n+i+1}{2}, & \text{for } i \text{ even,} \end{cases}$$

and

$$\alpha(v_i^j) = (t+j-1)n + i,$$

and label the edges as follows

$$\alpha(v_i^j v_{i+1}^j) = \begin{cases} (3t-j+1)n - \frac{i-1}{2}, & \text{for } i \text{ odd} \\ (3t-j+1)n - \frac{n+i-1}{2}, & \text{for } i \text{ even.} \end{cases}$$

For $i = 1, 2, \ldots, n$, label the spokes as follows:

$$\alpha(u_i^j v_i^j) = \begin{cases} 4tn + \frac{i-1}{2}n + i, & \text{for } j \text{ odd} \\ 4tn + \frac{t+j-1}{2}n + i, & \text{for } j \text{ even.} \end{cases}$$

Finally, label the inner edges as follow,

$$\alpha(u_i^j u_{i+2}^j) = \begin{cases} (4t+j)n - \lfloor \frac{i+1}{4} \rfloor, & \text{for } i \equiv 1 (mod\ 4), \\ (4t+j)n - \lfloor \frac{n_j+i+1}{4} \rfloor, & \text{for } i \equiv 2 (mod\ 4); \\ & n \equiv 3 (mod\ 4) \text{ or} \\ & i \equiv 0 (mod\ 4); \\ & n \equiv 1 (mod\ 4), \\ (4t+j)n - \lfloor \frac{2n_j+i+1}{4} \rfloor, & \text{for } i \equiv 3 (mod\ 4), \\ (4t+j)n - \lfloor \frac{3n_j+i+1}{4} \rfloor, & \text{for } i \equiv 0 (mod\ 4); \\ & n \equiv 3 (mod\ 4) \text{ or} \\ & i \equiv 2 (mod\ 4); \\ & n \equiv 1 (mod\ 4). \end{cases}$$

Then, the vertex weights of this labeling form consecutive integers starting with $a = (\frac{1}{2}(21tn + 5), 1)$. Thus, $tP(n, 2)$ has a super $(a = (\frac{1}{2}(21tn + 5), 1)$-VAT labeling. □

Next, we show the construction of one family of 4-regular graphs. In this paper we choose circulant graphs $C_n(1, 2)$. It has been proved in [3] that circulant graphs $C_n(1, m)$ have a super vertex magic total labeling λ. Using modification of the labeling λ, we can show that $C_n(1, m)$ has a super $(a, 1)$-VAT labeling. Moreover, we prove that a disjoint union of $C_{n_j}(1, 2)$, for $j = 1, \ldots, t$, has a super $(a, 1)$-VAT labeling. Figure 2 gives an example of a super $(a, 1)$-VAT labeling on $2C_5(1, 2) \cup 2C_7(1, 2)$.

Theorem 12 *Let $\{C_{n_j}(1, 2) : j = 1, \ldots, t\}$ be a set of circulant graphs with n_j vertices, n_j odd integer. Then, the graph $\cup_{j=1}^t C_{n_j}(1, 2)$ has a super $(a = 8 \sum_{k=1}^t n_k + 3, 1)$-VAT labeling.*
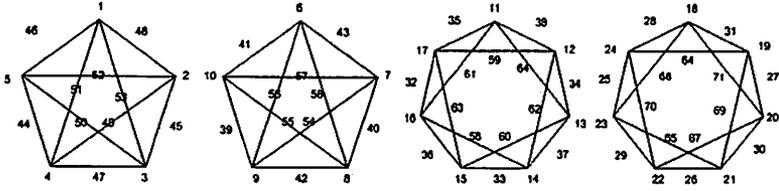
Fig. 2. Super $(a, 1)$-VAT labeling for $2C_5(1,2) \cup 2C_7(1,2)$.

Proof: Let $\{C_{n_j}(1,2) : j = 1, \ldots, t\}$ be a set of circulant graphs with n_j vertices, n_j odd integer. Let $\{v_i^j : i = 1, \ldots, n_j\}$ be a set of vertices of circulant graphs $C_{n_j}(1,2)$. Label the vertices as follows

$$\alpha(v_i^j) = \sum_{k=1}^{j-1} n_k + i$$

for $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, t$ and label the edges as follows

$$\alpha(v_i^j v_{i+1}^j) = \begin{cases} \sum_{k=1}^t n_k + \sum_{k=j}^t n_k - \lfloor \frac{i}{2} \rfloor, & \text{for } i \text{ odd} \\ \sum_{k=1}^t n_k + \sum_{k=j}^t n_k - \lfloor \frac{n_j+i}{2} \rfloor, & \text{for } i \text{ even} \\ \sum_{k=1}^t n_k + \sum_{k=j+1}^t n_k + n_j, & \text{for } i = n_j \end{cases}$$

and

$$\alpha(v_i^j v_{i+2}^j) = \begin{cases} 2\sum_{k=1}^t n_k + \sum_{k=1}^j n_k - \lfloor \frac{i+1}{4} \rfloor, & \text{for } i \equiv 1 (mod\ 4), \\ 2\sum_{k=1}^t n_k + \sum_{k=1}^j n_k - \lfloor \frac{n_j+i+1}{4} \rfloor, & \text{for } i \equiv 2 (mod\ 4); \\ & \quad n \equiv 3 (mod\ 4) \text{ or} \\ & \quad i \equiv 0 (mod\ 4); \\ & \quad n \equiv 1 (mod\ 4), \\ 2\sum_{k=1}^t n_k + \sum_{k=1}^j n_k - \lfloor \frac{2n_j+i+1}{4} \rfloor, & \text{for } i \equiv 3 (mod\ 4), \\ 2\sum_{k=1}^t n_k + \sum_{k=1}^j n_k - \lfloor \frac{3n_j+i+1}{4} \rfloor, & \text{for } i \equiv 0 (mod\ 4); \\ & \quad n \equiv 3 (mod\ 4) \text{ or} \\ & \quad i \equiv 2 (mod\ 4); \\ & \quad n \equiv 1 (mod\ 4). \end{cases}$$

Then, the vertex weights form consecutive integers starting with $a = 8\sum_{k=1}^t n_k + 3$. Thus, $\cup_{j=1}^t C_{n_j}(1,2)$ has a super $(a = 8\sum_{k=1}^t n_k + 3, 1)$-VAT labeling. $\square$

6 Open Problems

We list some problems for further investigation.

1. Find if there is a super (a, d)-VAT labeling of a disjoint union of r-regular graphs other than we have found in this paper for several values of d.
2. Find if we can generalise the property of t copies of graph G, when G has a super (a, d)-VAT labeling.

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